SINGULARITY SPECTRUM OF GENERALIZED ENERGY INTEGRALS

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Both the multifractality and multilacunarity spectrum are embodied in the analytic structure of a new generalized energy integral. A first application is made to non-trivial linear Cantor sets and exact results are obtained for dimensions, for periodicity of oscillations in order-q correlation and for entropies.

1. Introduction

In the theory of dynamical systems the central role of characterizing the metric and ergodic properties of the sets which are invariant under the iteration of maps, is taken by the invariant normalized measures they support. It is remarkable that these measures often give information on the geometry of the set and, in particular, on its fractal properties. For example, given a strange set J and a finite measure μ, consider for each positive τ, the *energy integral*

$$\Phi(\tau)=\int_{\mathrm{J}}\int_{\mathrm{J}}\|x-y\|^{-\tau}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)\,, \tag{1}$$

where $\|x-y\|$ is a suitable norm on J, and define the *divergence abscissa* $\bar{\tau}$ as

$$\bar{\tau}=\inf\{\tau;\,\Phi(\tau)=+\infty\}\,. \tag{2}$$

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It is a well known result of fractal geometry (Frostman's theorem) [1] that $\bar{\tau}$ is a lower bound to the Hausdorff dimension of J.

In this Letter we extend this result by introducing a new *generalized energy integral* (GEI) whose singularities give the *generalized dimensions* (GD) of the set J. The definition of the GD we adopt is the following [2]. Consider the minimal covering $\mathscr{C}$ of the set J with balls of radius l and, denoting with $\mathrm{B}(x;\,l)$ the ball whose center is at $x\in\mathrm{J}$, compute the cover function $D_q(l)$ according to

$$D_q(l)=\frac{1}{q-1}\log\left(\sum_{\mathscr{C}}\mu^q(\mathrm{B}(x;\,l))\right)\log^{-1}l\,. \tag{3}$$

If the function $D_q(l)$ has a limit for $l\to 0$ then we call this limit D_q the generalized dimension of order q of the set J with respect to the measure μ. We also recall that the Legendre transform of the GD gives the $f(\alpha)$ spectrum of singularities of the measure μ and this is one of the reasons for investigating the GD [3].

The covering procedure illustrated above has a continuous counterpart [2]. In fact if we define the

q-correlation integral $C_q(l)$ according to

$$C_q(l)=\int_J \mu^{q-1}(B(x;l))\,d\mu(x)\,, \tag{4}$$

then

$$D_q=\lim_{l\to 0}\frac{1}{q-1}\frac{\log C_q(l)}{\log l}\,. \tag{5}$$

2. Generalized energy integral

We define the *generalized energy integral* $\Phi_q(\tau)$ as

$$\Phi_q(\tau)=\int_J\int_J \|x-y\|^{-\tau} \times\mu^{q-2}(B(x;\|x-y\|))\,d\mu(x)\,d\mu(y)\,. \tag{6}$$

We observe that $\Phi_q(\tau)$ can be written as the Mellin transform of the Stieltjes measure $d\tilde{C}_q(l)$,

$$\Phi_q(\tau)=\int_0^\infty l^{-\tau}\,d\tilde{C}_q(l)\,, \tag{7}$$

where

$$\tilde{C}_q(l)=\int_J\int_J \theta(l-\|x-y\|) \times\mu^{q-2}(B(x,\|x-y\|))\,d\mu(x)\,d\mu(y)\,. \tag{8}$$

It is easy to check that $\tilde{C}_q(l)$ is proportional to the correlation function $C_q(l)$ previously defined. Indeed using a standard argument in fractal geometry we observe that $\mu(B(x,r))$ defines itself a measure and

$$\begin{aligned}\tilde{C}_q(l)&=\int_J d\mu(x)\int_{B(x,l)}\mu^{q-2}(B(x;\|x-y\|)\,d\mu(y)\\&=\int_J d\mu(x)\int_0^l \mu^{q-2}(B(x;r))\,d\mu(B(x,r))\,.\\&=\frac{1}{q-1}\int_J \mu^{q-1}(B(x;l))\,d\mu(x)\\&=\frac{1}{q-1}\,C_q(l)\,,\end{aligned} \tag{9}$$

where on the second line the inner integration is meant to be on r. Now, according to (5) the q-correlation integral scales as $C_q(l)\sim l^{D_q(q-1)}$ for l small. As a consequence if we take this into account in the Stieltjes integral (7) becomes

$$\Phi_q(\tau)\sim\int_0^\Delta l^{D_q(q-1)-\tau-1}\,dl\,, \tag{10}$$

where Δ is the diameter of J. The smallest real τ for which the GEI diverges, is $\tau=(q-1)D_q$. This suggests extending Frostman's theorem by defining the spectrum of the generalized dimensions directly from the divergence abscissa of the GEI:

$$D_q=\frac{1}{q-1}\inf\{\tau;\,\Phi_q(\tau)=+\infty\}\,. \tag{11}$$

Taking (11) as a definition for the GD, one can invert the procedure in the following way. First compute the GEI through (6), take its inverse Mellin to obtain $C_q(l)$ and finally look at the power law scaling of $C_q(l)$.

We applied [4] this technique successfully to some dynamical systems in the case $q=2$, where the GEI (6) reduces to the ordinary energy integral (1). Additional results were obtained for the linear Cantors. In particular the meromorphic structure of $\Phi(\tau)$ was proven, which implies the appearance of oscillations in $\log l$ for $C(l)$, in agreement with numerical experiments [6,7]. Moreover, if J arises as the invariant set under the smooth map T and the norm is replaced by the dynamic norm

$$\|x-y\|_k=\max_{0\leqslant j\leqslant k-1}\|T^jx-T^jy\|\,,\quad k\in\mathbb{Z}^+\,, \tag{12}$$

the residue of the pole of $\Phi^k(\tau)$ at $\tau=D_2$ scales for large k as $\exp[-kh_2(\mu)]$, where $h_2(\mu)$ is the 2-Renyi entropy [8] of the measure μ (which is supposed to be invariant and ergodic [9] under the mapping T). This allows one to grasp the 2-entropy with a new exact formula.

Using the GEI $\Phi_q(\tau)$ with q a parameter and τ a complex number, these and similar results can be extended to any q. For instance oscillatory corrections, which have been numerically observed for some $q\neq 2$, are exhibited by *all* correlation functions $C_q(l)$; they are not in general trivially related to each other and we propose to call them the multilacunarity spectrum [6b,16]. They all have their roots in the com-

plex poles of the GEI. The positions of the real poles give the dimension spectrum. If the dynamic norm is used, defining $\Phi_q^k(\tau)$, the residues of the poles at $\tau=(q-1)D_q$ scale for large k as $\exp[-k(q-1)h_q(\mu)]$, where $h_q(\mu)$ is the q-Renyi entropy of the measure μ.

We have started this program on a variety of examples, both analytically and numerically. In this Letter we restrict the analysis to be particular family of Cantors for which the singularities of the GEI can be relatively simply and explicitly determined.

3. Multifractal linear Cantor sets

Let L: $[0,1]\to\mathbb{R}$ be a piecewise linear map of the unit interval such that $L^{-1}[0,1]$ is the disjoint union of s sets:

$$L^{-1}[0,1]=\bigcup_{j=1}^{s} \mathrm{I}_j, \quad \mathrm{I}_j\cap\mathrm{I}_l=\emptyset, \quad \text{for } l\neq j .$$

The length of each I_j is the scale $\lambda_j<1$. The invariant set for this map is the Cantor set:

$$\mathrm{J}=\bigcap_{n=0}^{\infty} L^{-n}[0,1] . \tag{13}$$

We can put on J a set of L-ergodic measures μ characterized by the fact that each I_j has a weight: $p_j=\mu(\mathrm{I}_j\cap\mathrm{J})$ and $\sum_{j=1}^{s} p_j=1$. Each of these measures is called a *balanced measure* with weights $(p_1,\ldots,p_s)$. These measures have two other interesting properties [10]. Letting L_j^{-1} be the jth inverse band of L defined on $[0,1]$, A a measurable set of J and $f(x)$ a continuous scalar function on J, we have

$$\mu(L_j^{-1}\mathrm{A})=p_j\mu(\mathrm{A}) , \tag{14a}$$

$$\int_{\mathrm{J}} f(x)\,\mathrm{d}\mu(x)=\sum_{j=1}^{s} p_j\int_{\mathrm{J}} f(L_j^{-1}x)\,\mathrm{d}\mu(x) . \tag{14b}$$

Using the property (14b) it is possible to find a functional equation for $\Phi_q(\tau)$. Since we are interested in the singular behaviour of $\Phi_q(\tau)$, we apply the balance property (14b) to each variable in (6) but collecting the terms with the same L_j^{-1}. Indeed the terms like $\|L_i^{-1}-L_j^{-1}y\|$ with $i\neq j$ give a finite contribution to the integral since the points $L_i^{-1}x$ and $L_i^{-1}y$ are at a positive distance. We obtain

$$\Phi_q(\tau)=\sum_{i=1}^{s} p_i^2\int_{\mathrm{J}}\int_{\mathrm{J}} \|L_i^{-1}x-L_i^{-1}y\|^{-\tau}$$
$$\times\mu^{q-2}(\mathrm{B}(L_i^{-1}x,\|L_i^{-1}x-L_i^{-1}y\|))\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)$$
$$+\Psi_q(\tau) , \tag{15}$$

where $\Psi_q(\tau)$ is an entire function of τ.

To go further we need a scaling condition on the measure of the ball $\mathrm{B}(x,\|x-y\|)$. Using again the balance property we have

$$\mu(\mathrm{B}(L_i^{-1}x,\|L_i^{-1}x-L_i^{-1}y\|)$$
$$=p_i\int_{\mathrm{J}}\theta(\|L_i^{-1}x-L_i^{-1}y\|$$
$$-\|L_i^{-1}z-L_i^{-1}x\|)\,\mathrm{d}\mu(z)$$
$$-\sum_{j\neq i} p_j\int_{\mathrm{J}}\theta(\|L_i^{-1}x-L_i^{-1}y\|$$
$$-\|L_j^{-1}z-L_i^{-1}x\|)\,\mathrm{d}\mu(z) . \tag{16}$$

If we require that

$$\max_{1\leqslant i\leqslant s}\text{diameter}(L_i^{-1}[0,1])$$
$$\leqslant\min_{1\leqslant i\leqslant s-1}\text{distance}(L_i^{-1}[0,1],L_{i+1}^{-1}[0,1]) , \tag{17}$$

then the second term in the right hand side of (16) vanishes and we obtain (using the linearity of the L_i^{-1}):

$$\mu(\mathrm{B}(L_i^{-1}x,\|L_i^{-1}x-L_i^{-1}y\|))$$
$$=p_i\mu(\mathrm{B}(x,\|x-y\|)) . \tag{18}$$

The technical condition (17) is introduced here for simplicity. It means that the non-empty intervals of the Cantor set are smaller than the empty intervals. Equality is reached for the ordinary ternary Cantor set, for which of course (18) holds. The condition could probably be always relaxed at the cost of a more refined asymptotic analysis. In section 4 we also propose an alternative. Here we continue using (18), which allows to rewrite (15) as a functional equation

$$\Phi_q(\tau)\left(1-\sum_{i=1}^{s} p_i^q\lambda_i^{-\tau}\right)=\Psi_q(\tau) . \tag{19}$$

Therefore $\Phi_q(\tau)$ is a meromorphic function of τ and the poles are located at the solutions of

$$\sum_{i=1}^{s} p_i^q \lambda_i^{-\tau} = 1 \,. \tag{20}$$

For a given q, the real solution $\tau_q^{(0)} = (q-1)D_q$ of (20) is exactly the same one would obtain using the thermodynamic formalism [11]. We point out that D_0 is just the fractal dimension of the set [12] and is independent of the balanced measure which appears in the integral (6). We expect that $\Phi_0(\tau)$ does not depend on the measure at least for systems such as the mixing repellers which can be nicely approximated with sequences of linear Cantors [13–15].

When the scales are equal ($\lambda_j = \lambda$) but not the weights, we get explicitly all the poles of $\Phi_q(\tau)$,

$$\tau_q^{(n)} = \log\left(\sum_{j=1}^{s} p_j^q\right) \log^{-1}\lambda + 2\pi i n \log^{-1}\lambda \,,$$

$$n = 0, \pm 1, \pm 2, \ldots \,, \tag{21}$$

providing a description of the multifractal multilacunar measure μ. The generalized spectrum of dimension is given by

$$D_q = \frac{\tau_q^{(0)}}{q-1}$$

$$= \frac{1}{q-1} \log\left(\sum_{j=1}^{s} p_j^q\right) \log^{-1}\lambda \,. \tag{22}$$

Although D_q is not defined for $q=1$, we can obtain D_1 by a limiting process. This is the case for the above Cantor sets, and the first term in the right hand side of (22) gives

$$D_1 = \left(\sum_{j=1}^{s} p_j \log p_0\right) \log^{-1}\lambda \,, \tag{23}$$

which is just the *information dimension* [12], given by the ratio of the Shannon–Kolmogorov entropy of the measure $-\sum_{j=1}^{s} p_j \log p_j$ and its Lyapunov exponent $-\log\lambda$ [9,11,13]. If we suppose (as it happens for $q=2$) that the residues $r_q^{(n)}$ of the poles $\rho_q^{(n)}$ of $\Phi_q(\tau)$ decrease sufficiently fast for n large and do not all vanish, then we can take the inverse Mellin transform of the GEI, obtaining

$$\tilde{C}_q(l) = 2 l^{(q-1)D_q}$$

$$\times \sum_{n=1}^{\infty} \frac{|r_q^{(n)}|}{|\tau_q^{(n)}|} \cos(2\pi n \log^{-1}\lambda \log l - \gamma_q^{(n)} + \beta_q^{(n)}) \,, \tag{24}$$

where $\gamma_q^{(n)} = \arg(\tau_q^{(n)}) = \arg(r_q^{(n)})$.

The correlation functions exhibit oscillations in $\log l$ with the same period for any q. We have already noticed that these features have been observed numerically for the ternary Cantor set [7] and are related to the *lacunarity*. Here, for a class of Cantor sets, we explicitly establish the connection between the multilacunarity spectrum and the meromorphic structure of the GEI.

Finally to obtain the q-Renyi entropies for the Cantor sets considered above, we consider the limit

$$\lim_{k\to\infty} \frac{1}{k} \frac{1}{1-q} \log r_{q,k}^{(0)} \,, \tag{25}$$

where $r_{q,k}^{(0)}$ is the residue of the GEI (computed with the dynamic norm (12)) at the pole D_q. Using again the method illustrated above and the technique in section IV of ref. [5] one easily obtains

$$h_q(\mu) = \frac{1}{1-q} \log\left(\sum_{j=1}^{s} p_j^q\right) = h_q(\mu) \,, \tag{26}$$

which is precisely the expression for the q-Renyi entropy one obtains using the thermodynamical formalism.

4. Discussion and conclusion

First a possible alternative. The EI can be generalized as

$$\hat{\Phi}_1(\tau) = \int_{J\times J} \|x-y\|^{-\tau}$$

$$\times \mu^{q-2}(\mathrm{B}(\tfrac{1}{2}(x+y); \tfrac{1}{2}\|x-y\|)) \, \mathrm{d}\mu(x) \, \mathrm{d}\mu(y) \,, \tag{27}$$

a more symmetric definition. It would make straightforward the above analysis for any linear Cantor set with arbitrary scales, because the analog of (18) for the scaling of the ball holds without any restriction (such as (17)) on the structure of the set.

As a consequence eqs. (21) and (22) which give the dimension spectra, in agreement with the results of the thermodynamical formalism and the lacunarity structure, hold in full generality.

Moreover the correlation integral corresponding to (27), and given by the r.h.s. of (8) where $B(x, \|x-y\|)$ is replaced by $B(\frac{1}{2}(x+y), \frac{1}{2}\|x-y\|)$, turns out to be much more convenient for numerical computations. Indeed some preliminary numerical results, obtained with this definition, are in excellent agreement with our theoretical results and will be reported elsewhere.

The only inconvenience is that the equivalence with the standard definition of the correlation integral $C_q(l)$ has not yet been proved, even though we expect it to hold asymptotically.

We conclude by observing that the linear Cantor sets are a basic tool to understand more general systems. For example, any non-linear expanding map with an invariant Cantor set can be approximated in a *dynamic* and "*fractal*" sense by a sequence of linear Cantor systems [13–15]. Moreover every hyperbolic map, with more complicated invariant sets, behaves topologically as the linear Cantor systems, the role of the sets $L^{-n}[0,1]$ being taken by the Markov partitions. All this suggests that the energy integral formalism could be applied to a larger class of dynamical systems.

We finally point out that for smooth sets endowed with the Lebesgue measure, the GEI is easily computable. This and the extension to more general non-hyperbolic systems will be the object of future work. One can also observe that the definition (11) of the GD can be used for actual computations. For a given set J and the physical measure one can indeed approximate the GEI by using a time series and detect the divergence abscissa by any analytic continuation procedure. This procedure was already used in ref. [4] to compute D_2 for the Henon attractor.

To summarize, in our opinion the generalized energy integral has the advantage of collecting a variety of information on the possibly multifractal and/or multilacunar set J and measure μ into a single object. It enjoys natural relations to other fractal concepts and is reasonably manageable mathematically and numerically.

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