# MELLIN TRANSFORMS AND CORRELATION DIMENSIONS

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We consider the Mellin transform of the correlation integrals and show that the divergence abscissa is the correlation dimension. The analytic structure of the Mellin transform is explicitly described for some Julia and Cantor sets. The existence of oscillations in the correlation integral for the Cantor sets is proved. Extensions of the results to the order $d$ correlation integrals are discussed.

The analysis of the geometrical properties of attractors for the flow of differential equations and maps or fractals of any physical origin invariably involves the computation of a generalized dimension [1].

The Hausdorff dimension $D$ in general escapes numerical investigations while bounds to it like the fractal and the correlation dimensions [2] can be computed even though the algorithms are always quite time consuming [3].

The correlation dimension $\nu$ is defined as the exponent of the power law behavior $C(l) \simeq l^{\nu}$ of the correlation function: deviations from linearity in the log $C(l)$ versus log $l$ were observed and can seriously affect the extrapolation procedure [4].

In this letter we first show how the correlation integral is related to the "energy integral", widely used in the mathematical literature [5], by a Mellin transform [6] and that the correlation dimension corresponds to the abscissa of divergence of the Mellin transform. For some simple models we have worked out the exact analytic structure of the Mellin transform and we find that for the analytic curves we considered it has only poles on the real axis so that $C(l) \simeq l$ as $l \to 0$. For fractal sets like the Cantor with one scale $\lambda$, there is a sequence of poles on a line parallel to the imaginary axis whose abscissa is precisely $\nu$. As a consequence $C(l)$ behaves like $l^{\nu}$ but exhibits oscillations.

The Mellin transform of the "order $d$ correlation integrals" [7,8] defines the "order $d$ energy integrals" which have the same singularity structure as the ordinary energy integrals with a residue at the first pole on the real axis which is proportional to $\exp(-dK_2)$, asymptotically in $d$, where $K_2$ is the second Renyi entropy [7] (a lower bound to the Kolmogorov entropy).

Let $E \subset \mathbb{R}^n$ be a compact set and $\mu$ a probability measure with support on E; the energy integral is defined by

$$\Phi(\alpha; \mu) = \int_{E \times E} \frac{d\mu(x)\, d\mu(y)}{\|x-y\|^{\alpha}}, \tag{1}$$

where $\| \ \|$ denotes the euclidean norm in $\mathbb{R}^n$. The equilibrium measure $\mu_{eq}(x; \alpha)$ is the one which minimizes $\Phi$ for a given $\alpha$. The Hausdorff dimension $D$ is defined by

$$D = \inf_{\alpha} \{\alpha: \Phi(\alpha, \mu_{eq}) = \infty\}. \tag{2}$$

For any other measure we define a divergence abscissa

$$\nu(\mu = \inf_{\alpha} \{\alpha: \Phi(\alpha, \mu) = \infty\} \tag{3}$$

and one can show that

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$$\nu(\mu) \leqslant D . \tag{4}$$

The correlation integral is defined by

$$C(l;\mu) = \int_{E\times E} \vartheta(l - \|x-y\|)\, d\mu(x)\, d\mu(y) \tag{5}$$

and $C(l;\mu)$ is a positive Stieltjes measure with supprt on $[0, \delta]$ where $\delta$ is the diameter of the set. The Mellin transform of $C(l;\mu)$ gives the energy integral

$$\Phi(\alpha;\mu) = \int_0^{\delta} l^{-\alpha}\, dC(l;\mu) . \tag{6}$$

The correlation dimension defined by Procaccia is the limit (if it exists)

$$\lim_{l\to 0} \frac{\log C(l;\mu)}{\log l} . \tag{7}$$

When the limit (7) exists it is, under suitable conditions, equal to $\nu(\mu)$ defined by (3) which turns out to be the correlation dimension.

For a certain number of models $\Phi(\alpha;\mu)$ has been computed exactly. We have considered the Julia sets for the quadratic map $z' = z^2 - p$ with $p=0$ (circle) and $p=2$ (segment) with $\mu$ equal to the balanced measure [6] [11]. For $p=0$ $\mu$ is uniform and

$$\Phi(\alpha) = \frac{\Gamma(1-\alpha)}{\Gamma^2(1-\frac{1}{2}\alpha)} , \tag{8}$$

while for $p=2$ we have

$$d\mu(x) = \frac{1}{\pi} \frac{dx}{\sqrt{4-x^2}} ,$$

and

$$\Phi(\alpha) = \left( \frac{\Gamma(1-\alpha)}{\Gamma^2(1-\frac{1}{2}\alpha)} \right)^2 . \tag{9}$$

In both cases $\Phi(\alpha)$ appears to be meromorphic, the lowest pole is at $\alpha = 1$ and the correlation integral can be exactly computed: in the first case it vanishes as $l$, in the second as $-l \log l$ (which makes the extrapolation very hard). The difference is that the pole is simple in the former case, double in the second.

The second type of model we consider, still a repeller, is the one-scale Cantor set; let $T$ be the map

[11] In general the equilibrium measure and the balanced measure are not the same.

$[0, 1] \to [0, q/2]$, $q > 2$,

$$T(x) = qx , \quad 0 \leqslant x < \tfrac{1}{2} , \qquad = q(1-x) , \quad \tfrac{1}{2} \leqslant x \leqslant 1 , \tag{10}$$

and denote by $T_1^{-1}(x)$, $T_2^{-1}(x)$ the two inverses defined on $[0, q/2]$

$$T_1^{-1}(x) = \lambda x , \quad T_2^{-1}(x) = 1 - \lambda x , \quad \lambda = 1/q . \tag{11}$$

On the Cantor set $E = \lim_{n\to\infty} T_{-n}(I)$, where I is any interval in $[0, 1]$, we can define an invariant balanced measure according to Zygmund [9]. It can then be shown, exploiting the balance properties of the measure, that

$$\Phi(\alpha) = (1 - \tfrac{1}{2}\lambda^{-\alpha})^{-1} R(\alpha) , \tag{12}$$

where $R(\alpha)$ is an entire function of $\alpha$. As a consequence $\Phi(\alpha)$ is meromorphic with poles at

$$\alpha = \frac{\log 2}{\log q} + i \frac{2\pi}{\log q} n , \quad n = 0, \pm 1, \ldots . \tag{13}$$

The correlation integral turns out to be

$$C(l) = l^{\log 2/\log q} \times \sum_{n=0}^{\infty} B_n \cos\left( 2\pi n \frac{\log l}{\log q} + \beta_n \right) \tag{14}$$

and exhibits oscillations since the $B_n$, which are proportional to the residue of the poles, can be shown to be non-zero at least for $n \leqslant \frac{1}{8} q \log q$. These results can also be extended to the Cantor with $s$ scales.

When no analytic results are available one can use a numerical procedure to compute $\Phi(\alpha)$ and to approximate its singularities if $\mu$ is the "physical measure" (Sinai–Bowen–Ruelle measure) $\mu_{ph}$ [1] such that

$$\int_E f(x)\, d\mu_{ph}(x) = \lim_{n\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x_0) ,$$

$$f \in C(\mathbb{R}^n) , \tag{15}$$

where $x_0$ is any point in the basin of attraction of E (up to a set of zero Lebesgue measure). In such a case both $\Phi(\alpha;\mu_{ph})$ and $C(l;\mu_{ph}$ can be approximated by a finite sum and are still related by the Mellin transform. A rational interpolation of Padé type can then be used to extract the singularities of

Table 1
Smallest real pole, with non-vanishing residue, of the $[m/n]$ Padé interpolations to $\Phi(\alpha)$ for the Hénon map. $N$ iterations are considered with initial point $x_0=(0.1, 0.1)$.

| $N$ | [3/2] | [3/3] | [4/3] | [4/4] | [5/4] | [5/5] |
|---|---|---|---|---|---|---|
| 2048 | 1.203 | 1.211 | 1.228 | 1.215 | 1.227 | 1.222 |
| 8192 | 1.196 | 1.200 | 1.200 | 1.200 | 1.200 | 1.201 |
| 32768 | 1.196 | 1.200 | 1.199 | 1.200 | 1.201 | 1.200 |

$$\Phi(\alpha, \mu_{\rm ph}; N)=\frac{2}{N(N-1)}\sum_{k=0}^{N-1}\sum_{j=k+1}^{N-1}\|T^k x-T^j x\|^{-\alpha}\,. \tag{16}$$

For the Hénon map, for which the existence of a physical measure is only conjectured, the values of $\nu(\mu_{\rm ph})$ obtained with the interpolation of $\Phi(1/n, \mu_{\rm eq}; N)$ for $1\leqslant n\leqslant 13$, $N\leqslant 2^{15}$ with the $[m/m]$, $[m-1/m]$ Padé approximants for $m\leqslant 5$, see table 1, are stable and in good agreement with the values quoted in the literature (we remark that the singularities of $\Phi(\alpha)\sigma^\alpha$ are the same as $\Phi(\alpha)$; the independence on $\sigma$ was numerically checked and suggests that the results are reliable).

Finally we observe that the "order $d$ energy integral"

$$\Phi_d(\alpha;\mu)=\int_{E\times E}\|x-y\|_{\rm d}^{-\alpha}\,{\rm d}\mu(x)\,{\rm d}\mu(y)\,, \tag{17}$$

where

$$\|x-y\|_d=\left(\sum_{j=0}^{d-1}\|T^j x-{\rm T}^j y\|^2\right)^{1/2} \tag{18}$$

is the Mellin transform of the "order $d$ correlation integral" $C_d(\alpha;\mu)$ defined by (5) where $\|\ \|$ is replaced by $\|\ \|_d$.

The analytic structure of $\Phi_d(\alpha;\mu)$ is expected to be the same as $\Phi(\alpha;\mu)$ and the residue at the pole $\alpha=\nu$ to be given by $\nu e^{-dK_2}$ asymptotically in $d$.

For the Cantor set with one scale we were able to prove that

$$\Phi_d(\alpha)=\frac{1}{2^{d-1}}\left(\sum_{k=0}^{d-1}\lambda^{2k}\right)^{-\alpha}\Phi(\alpha)+S(\alpha)\,, \tag{19}$$

where $S(\alpha)$ is an entire function of $\alpha$ so that the residue at $\alpha=\nu=\log 2/\log q$ is

$$r_d=2r\left(\frac{1-\lambda^{2(d-1)}}{1-\lambda^2}\right)^{-\nu}{\rm e}^{-d\log 2}\,, \tag{20}$$

$r$ being the residue at the pole $\alpha=\nu$ of $\Phi(\alpha)$. These results do extend to the Cantor with two scales. For some hyperbolic totally disconnected Julia sets (Fatou's dust) a result similar to (20) can be obtained.

*Noted added.* For the Cantor set with $s$ scales $\lambda_1, \ldots \lambda_s$ the Hausdorff dimension $D$, given by $\lambda_1^D+\ldots+\lambda_8^D=1$, is obtained for a balanced mesure with weights $p_j=\lambda_j^D$ and the second Renyi entropy can be explicitly computed and reads $K_2=-\log(\lambda_1^{2D}+\ldots+\lambda_8^{2D})$.

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## References

[1] J.P. Eckmann and D. Ruelle, Rev. Mod. Phys. 57 (1985) 617.
[2] P. Grassberger and I. Procaccia, Physica D 9 (1983) 189.
[3] A. Giorgilli, D. Casati, L. Sironi and L. Galgani, Phys. Lett. A 115 (1986) 5.
[4] R. Badii and A. Politi, Phys. Lett. A 107 (1984) 303.
[5] K.J. Falconer, The geometry of fractal sets (Cambridge Univ. Press, Cambridge, 1985).
[6] D. Bessis, J.S. Geronimo and P. Moussa, J. Stat. Phys. 34 (1984) 75.
[7] P. Grassberger and I. Procaccia, Phys. Rev. A 28 (1983) 2591.
[8] A. Cohen and I. Procaccia, Phys. Rev. A 31 (1985) 1872.
[9] A. Zygmund, Trigonometrical series, 2nd Ed. (Cambridge Univ. Press, Cambridge, 1968) p. 194.