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Quantum intermittency in almost-periodic lattice systems derived from their spectral properties

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Abstract

Hamiltonian tridiagonal matrices characterized by multi-fractal spectral measures in the family of iterated function systems can be constructed by a recursive technique described here. We prove that these Hamiltonians are almost-periodic. They are suited to describe quantum lattice systems with nearest neighbours coupling, as well as chains of linear classical oscillators, and electrical transmission lines.

We investigate numerically and theoretically the time dynamics of the systems so constructed. We derive a relation linking the long-time, power-law behaviour of the moments of the position operator, expressed by a scaling function β of the moment order α , and spectral multi-fractal dimensions, D_q , via $\beta(\alpha) = D_{1-\alpha}$. We show cases in which this relation is exact, and cases where it is only approximate, unveiling the reasons for the discrepancies.

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Dedicated to the memory of Professor Joseph Ford, teacher, original researcher, and founder of Physica D

1. Introduction

Usually, the study of almost/quasi-periodic systems starts by assigning a suitable rule for building a quantum Hamiltonian operator, and then proceeds to the determination of its spectral quantities [1] (which is frequently a hard task) and of the time dynamics it generates. In doing so, *multi-fractal* energy spectra have been frequently observed, and anomalous characteristics of the time evolution have been exhibited [2–6].

These findings raise the question if multi-fractal spectra are typical in almost/quasi-periodic systems [7], and, vice versa, if almost/quasi-periodicity is always associated with singular continuous spectral measures. The relations between this pair (spectral multi-fractality and Hamiltonian almost-periodicity) and the time dynamics generated via Schrödinger's equation are also interesting, and intricate: do the former always imply anomalous scaling relations of

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physical quantities like, for instance, the expectation value of the position operator? Can we make any quantitative statement to this effect?

In this paper, we employ a new algorithm for deriving a Hamiltonian operator (called a *Jacobi matrix* because of its mathematical nature) with a pre-assigned spectral measure in the vast class of *iterated function systems* (IFSs). This technique provides us with an ideal patient for our surgical table, who can be fully dissected and analysed. In particular, we provide evidence that Jacobi matrices associated with IFSs are almost periodic, and we argue that this is likely to be the typical case in a large class of measures with fractal support.

The analysis which can be carried out in this example permits us to compute exactly the asymptotic behaviour of the wave function projections, for short and long times. By introducing a renormalization approach in the theory of orthogonal polynomials, we also derive a relation linking the asymptotic power-law growth of the moments of the position operator and multi-fractal generalized dimensions. This theory explains the phenomenon that we have termed *quantum intermittency*.

The specific properties of IFSs are crucial for our theory; Yet, since particular IFSs can be found so to approximate arbitrarily well (in a technical sense) any “fractal” measure [8–11], the results obtained in the IFS class may have a much wider generality.

We shall present our results as follows: in Section 2 we introduce the general formalism of IFS and of Jacobi matrices, employed to solve the *inverse problem* of finding a Hamiltonian with a given spectrum. This formalism is then applied in Section 3 to derive a stable solution algorithm. The almost-periodic properties of the Hamiltonian so determined are studied numerically in Section 4, and the intermittent quantum dynamics it generates is then discussed in Sections 5 and 6. The conclusions summarize the work and present some previews on further investigations.

2. IFS and their Jacobi matrices

Systems of linear iterated functions [12–15] are finite collections of maps

$$\phi_i(x) := \delta_i x + \beta_i, \quad i = 1, \dots, M, \quad (1)$$

where δ_i, β_i are real constants, and the contraction rates δ_i have modulus less than one. Without loss of generality, we may assume that each ϕ_i maps $[0, 1]$ into itself, and that $\phi_1(0) = 0$.

A probability, π_i , is associated with each map: $\pi_i > 0$, $\sum_i \pi_i = 1$. Employing these probabilities, a measure over $[0, 1]$ can be defined as the unique positive measure satisfying the balance property

$$\int_0^1 f \, d\mu = \sum_{i=1}^M \pi_i \int_0^1 (f \circ \phi_i) \, d\mu \quad (2)$$

for any continuous function f . This measure is supported on A , the subset of $[0, 1]$ which solves the equation

$$A = \bigcup_{i=1, \dots, M} \phi_i(A). \quad (3)$$

The set A is invariant under the action of shrinking it to smaller copies of itself, and glueing them together. Because of Eq. (3), the geometry of this set is typically fractal (except for special choices of the map parameters); in turn, the balance relation (2) is responsible for the multi-fractal properties of the measure μ . In fact, let us consider a *disconnected* IFS, that is to say, one for which the sets $\phi_i(A)$ do not intersect each other. Under these circumstances,

the multi-fractal properties of the balanced measure are easily computable: the spectrum of generalized dimensions D_q follows from the equation

$$\sum_{j=1}^M \pi_j^q \delta_j^{-\tau} = 1, \quad (4)$$

whose unique real solution defines τ as a function of q , and leads to $D_q = \tau(q)/(q - 1)$. In virtue of this relation, one can tune the map parameters to obtain various multi-fractal spectra.

The problem of determining a Hamiltonian possessing μ as spectral measure can be solved [16] considering the set of associated orthonormal polynomials $\{p_n\}$:

$$\int p_i(x) p_k(x) d\mu(x) = \delta_{i,k}. \quad (5)$$

In fact, any such set of polynomials is characterized by a three-term recurrence relation which can be written

$$xp_j(x) = r_{j+1}p_{j+1}(x) + A_j p_j(x) + r_j p_{j-1}(x), \quad (6)$$

or, in matrix form

$$Hp(x) = xp(x). \quad (7)$$

In the above, $p(x)$ is the vector whose components are the orthonormal polynomials evaluated at site x , and H is the Jacobi matrix, which is constructed as the real, symmetric, tridiagonal matrix whose diagonal and outer diagonals are the vectors A_j and r_j , respectively:

$$H_{i,i} = A_i, \quad H_{i+1,i} = H_{i,i+1} = r_{i+1}, \quad i = 0, 1, \dots \quad (8)$$

H defines a nearest neighbours lattice system, with site energies A_i and hopping constants r_i . Similarly, H can describe a linear array of masses coupled by springs, and also an electrical transmission line, whose characteristics vary from one element to the next.

Standard theory proves that, letting the Jacobi matrix H act in l_2 (the space of square summable sequences, whose canonical basis will be indicated by $\{e_0, e_1, \dots\}$) the spectral measure of H with respect to the vector e_0 (the *local density of states* of physical jargon) is precisely μ : in fact, one has

$$(e_0, g(H)e_0) = \int g(x) d\mu(x) \quad (9)$$

for well-behaved functions g . This is the theoretical solution of the inverse problem we have proposed. In order to translate it into a *practical* solution, we need to compute the Jacobi matrix coefficients starting from the measure μ , i.e. from the map parameters defining the IFS.

3. A stable technique for computing IFS Jacobi matrices

The problem of constructing the Jacobi matrix associated with IFS measures is hard, and the usual techniques of *polynomial sampling* [17–19] are plagued by exponentially increasing errors which allow only computation of very few Jacobi matrix coefficients [20]. Alternatively, the sole technique available so far has been an *algebraic* procedure programmed in *MAPLE* by Vrscay [21]. Yet, it is severely limited by memory and time requirements. To overcome these difficulties we have devised a direct algorithm applicable to IFS measures.

We first observe that, for any n ,

$$p_n(\phi_i(x)) = \sum_{l=0}^n \Gamma_{i,l}^n p_l(x), \quad i = 1, \dots, M. \tag{10}$$

This is immediate, since $p_n(\phi_i(x))$ is an n th degree polynomials which can be expanded on the first n orthogonal polynomials. Less immediate is to derive a recursive rule for the coefficients $\Gamma_{i,l}^n, l = 0, \dots, n$. It turns out that, at fixed n , they can be determined from the map parameters, and from the Jacobi matrix entries A_j , for $j = 0, 1, \dots, n - 1$, and r_m , for $m = 0, 1, \dots, n$. In fact (dropping for simplicity the map index i), we have that $p_0(\delta x + \beta) = p_0(x)$, and hence $\Gamma_0^0 = 1$. Suppose now that Γ_l^k is known for $k = 0, \dots, n - 1$ and all relative l 's: from Eq. (6) we obtain the complete decomposition of $p_n(\delta x + \beta)$ over $p_l, l = 1, \dots, n$:

$$r_n p_n(\delta x + \beta) = (\beta - A_{n-1}) \sum_{l=0}^{n-1} \Gamma_l^{n-1} p_l(x) + \delta \sum_{l=0}^{n-1} \Gamma_l^{n-1} (r_{l+1} p_{l+1}(x) + A_l p_l(x) + r_l p_{l-1}(x)) - r_{n-1} \sum_{l=0}^{n-2} \Gamma_l^{n-2} p_l(x). \tag{11}$$

Eq. (11) allows now the determination of the coefficients Γ_l^n .

We observe that the highest-order polynomial, p_n , appears twice in the above equation, always in the form of the product $r_n p_n$: hence, the coefficients in the expansion of the polynomial $r_n p_n$ can be determined *without* knowing r_n . Therefore, if we let $\tilde{p}_n(x) = r_n p_n(x)$, a second decomposition can be written as

$$\tilde{p}_n(\phi_i(x)) = \tilde{\Gamma}_{i,n}^n \tilde{p}_n(x) + \sum_{l=0}^{n-1} \tilde{\Gamma}_{i,l}^n p_l(x), \tag{12}$$

where the coefficients $\tilde{\Gamma}$ can be computed recursively from Eq. (11), on the basis of the knowledge of only A_j, r_j , for $j = 0, 1, \dots, n - 1$.

We can now compute the non-diagonal entries of the Jacobi matrix: from Eq. (6) we write

$$r_n^2 = \int \tilde{p}_n(x) x p_{n-1}(x) d\mu. \tag{13}$$

Hence, using the balance property (2) and Eqs. (10) and (12) this becomes

$$r_n^2 = \sum_{i=1}^M \pi_i \int (\delta_i x + \beta_i) \left[\sum_{m=0}^{n-1} \sum_{l=0}^{n-1} \tilde{\Gamma}_{i,m}^n \Gamma_{i,l}^{n-1} p_m(x) p_l(x) + \sum_{l=0}^{n-1} \tilde{\Gamma}_{i,n}^n \Gamma_{i,l}^{n-1} \tilde{p}_n(x) p_l(x) \right] d\mu. \tag{14}$$

Again, we can use the recurrence relations (6), to get

$$r_n^2 = \sum_{i=1}^M \pi_i (B_i + C_i + D_i), \tag{15}$$

where we have put

$$B_i = \sum_{l=0}^{n-1} (\beta_i + \delta_i A_l) \tilde{\Gamma}_{i,l}^n \Gamma_{i,l}^{n-1}, \tag{16}$$

$$C_i = \delta_i \sum_{l=0}^{n-2} r_{l+1} (\tilde{\Gamma}_{i,l}^n \Gamma_{i,l+1}^{n-1} + \tilde{\Gamma}_{i,l+1}^n \Gamma_{i,l}^{n-1}), \tag{17}$$

and

$$D_i = \delta_i \tilde{\Gamma}_{i,n}^n \Gamma_{i,n-1}^{n-1} r_n^2. \tag{18}$$

Because of contractivity of the maps, $|D_i| r_n^{-2} < 1$. Therefore, r_n^2 (and hence $r_n > 0$) can be computed from Eq. (15), on the basis of the knowledge of the coefficients in expansions (10) of order $n - 1$, of order n in (12), of the map parameters, and of the matrix entries A_j, r_j , for $j = 0, 1, \dots, n - 1$.

A similar trick allows the computation of the diagonal entries A_n ; We use Eqs. (2) and (6) (integrals are taken with respect to μ):

$$A_n = \int x p_n^2(x) = \sum_{i=1}^M \pi_i \int (\delta_i x + \beta_i) p_n^2(\delta_i x + \beta_i) = \sum_{i=1}^M \pi_i \int (\delta_i x + \beta_i) \sum_{m,l=0}^n \Gamma_{i,l}^n \Gamma_{i,m}^n p_l(x) p_m(x). \tag{19}$$

Using the orthonormality properties of the sequence p_n , and the recurrence relation, Eq. (6), we get

$$A_n = \sum_{i=1}^M \pi_i \left[\sum_{m=0}^n (\Gamma_{i,m}^n)^2 (\beta_i + \delta_i A_m) + \sum_{m=0}^{n-1} \Gamma_{i,m}^n \Gamma_{i,m+1}^n \delta_i (r_m + r_{m+1}) \right], \tag{20}$$

thereby determining A_n as a function of the coefficients in Eq. (10) of order n fixed, of the map parameters, and of the matrix entries A_j , for $j = 0, 1, \dots, n - 1$, and r_m , for $m = 0, 1, \dots, n$.

These results can be properly chained into an iterative construction of the Jacobi matrix H : The algorithm is structured as follows:

- *Initialization.* At the first step, we have $A_0 = \mu_1, r_0 = 0, \Gamma_0^0 = 1$. The first-order moment of μ, μ_1 , can be simply computed from Eq. (2).
- *Iteration.* Suppose that A_l, r_l , and Γ^l are known for $l = 0, 1, \dots, n - 1$. Then we:
 - Compute $\tilde{\Gamma}^n$. We use Eqs. (11) and (12).
 - Compute r_n . We use Eqs. (13)–(18).
 - Compute Γ^n . This is immediate at this stage.
 - Compute A_n . We use Eqs. (19) and (20). Then we iterate the procedure.

Graphically:

$$\left(\begin{matrix} \Gamma^{n-1}, r_0, \dots, r_{n-1} \\ A_0, \dots, A_{n-1} \end{matrix} \right) \Rightarrow \tilde{\Gamma}^n \Rightarrow r_n \Rightarrow \Gamma^n \Rightarrow A_n \Rightarrow \left(\begin{matrix} \Gamma^n, r_0, \dots, r_n \\ A_0, \dots, A_n \end{matrix} \right). \tag{21}$$

In a separate work [20] we have analysed the reasons of the failure of classical polynomial sampling [17,19] when applied to singular measures, and assessed the numerical stability of the recursive algorithm presented above. We have observed a polynomial error propagation with respect to matrix order for the recursive algorithm, while using the classical algorithms the error growth was found to be exponential.

4. Almost periodicity of IFS Jacobi matrices

Having devised a stable solution of the Hamiltonian inverse problem, we can study the properties of large Jacobi matrices. Fig. 1 shows an IFS measure, one of its orthogonal polynomials, and the beginning of the sequence of r_n coefficients. Let us focus our attention on the last.

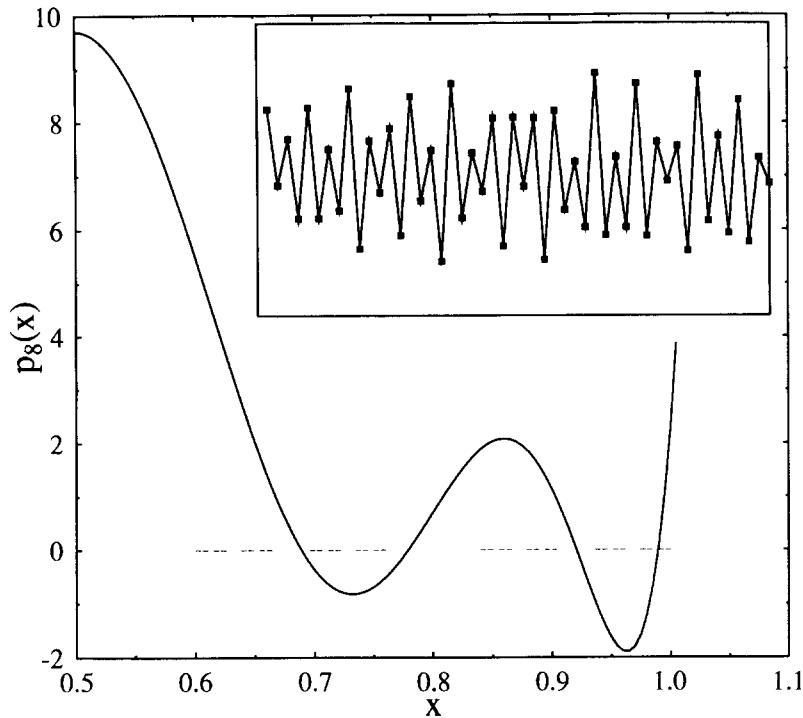


Fig. 1. Orthogonal polynomial $p_8(x)$ of the IFS measure with maps $(\delta_i, \beta_i, \pi_i) = (\frac{2}{5}, 0, \frac{1}{2}), (\frac{2}{5}, \frac{3}{5}, \frac{1}{2})$, with a finite-resolution representation of the support of the measure obtained by plotting a large number of points on the attractor. Because of the finite size of points, this latter appears as a sequence of dashes. Only the symmetrical half is shown; in the inset, the beginning of the sequence of r_n . The vertical scale ranges from zero to $\frac{1}{2}$. Lines are merely to guide the eye.

We can clearly observe a zero frequency (the average value), a π frequency (flipping up and down), and clearly other frequencies are present in the sequence. A Fourier analysis is simply effected writing

$$r_n = \sum_k F_k e^{in\omega_k}. \quad (22)$$

This sum may not converge in the usual sense, and it might have to be replaced by an integral in the case of a continuous component in the “spectrum” of the sequence r_n . If the continuous component is absent, the system is almost-periodic. Within this case, if the set of frequencies ω_k can be derived from a finite set of periods, the sequence r_n is *quasi-periodic*: that is, this is the case if there exist suitable $\Omega_1, \dots, \Omega_p$ such that for all k the frequency ω_k can be written $\omega_k = n_1\Omega_1 + \dots + n_p\Omega_p$ for integer n_1, \dots, n_p .

A numerical, fast Fourier analysis of the sequence r_n is presented in Fig. 2, where peaks in the distribution of $|F_k|^2$ with a clear hierarchical structure are observed. These peaks seem to suggest the presence of a point component in the spectrum of this sequence. Yet, care has always to be exerted to assess this fact numerically. To obtain a further piece of evidence we performed an analysis of the phase of F_k around these peaks, like that shown in Fig. 3, and found a π discontinuity, which indicates [22] that they are indeed related to a point component. The sequence r_n is therefore almost-periodic.

Since no simple rational relation among the peak sequences seems to hold, numerical evidence seems to suggest that the sequence is not quasi-periodic. Our numerical investigations have shown that these characteristics are typical in the class of Hamiltonian associated with IFS measures, supported on Cantor sets. In view of the approximation properties of IFS measures, this result is likely to be much more general: indeed, in

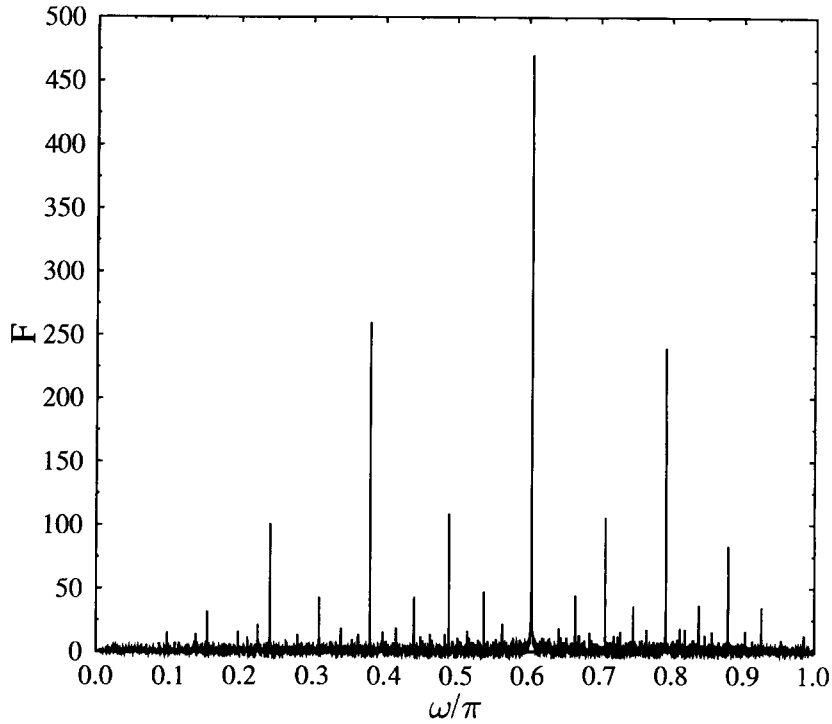


Fig. 2. Discrete Fourier transform of the r_n sequence ($n = 1, \dots, 2^{13}$) associated with the IFS of Fig. 1. The constant and π frequencies exceed the vertical scale, and are not reported.

the family of Jacobi matrices associated with real Julia sets [23,24], which can be well approximated by IFS, limit periodicity of the sequence r_n has been proven directly [25]. The problem of a formal proof is therefore open.

5. Quantum dynamics of almost-periodic lattice systems

Jacobi matrices generate a quantum dynamics in l_2 via Schrödinger's equation,

$$i \frac{d\psi}{dt} = H\psi, \quad \psi(0) = e_0 := (1, 0, \dots). \tag{23}$$

The initial state of the evolution, e_0 , is the zeroth lattice state. In oscillator terms, this corresponds to a situation where the first mass is displaced from its equilibrium position, while all the other masses are at rest in their equilibria. In electrical terms, the current (or the voltage) is non-zero only in the first element of the transmission line described by the Jacobi matrix H .

The solution of Schrödinger equation can be formally obtained as [26,27]

$$c_n(t) := (e_n, e^{-itH} e_0) = \int e^{-itx} p_n(x) d\mu(x), \tag{24}$$

where $c_n(t)$ is the component of $\psi(t)$ at the n th lattice state. Eq. (24) shows that this component is the Fourier transform of the orthogonal polynomial p_n with respect to the spectral measure μ . This fact allows us to derive important results.

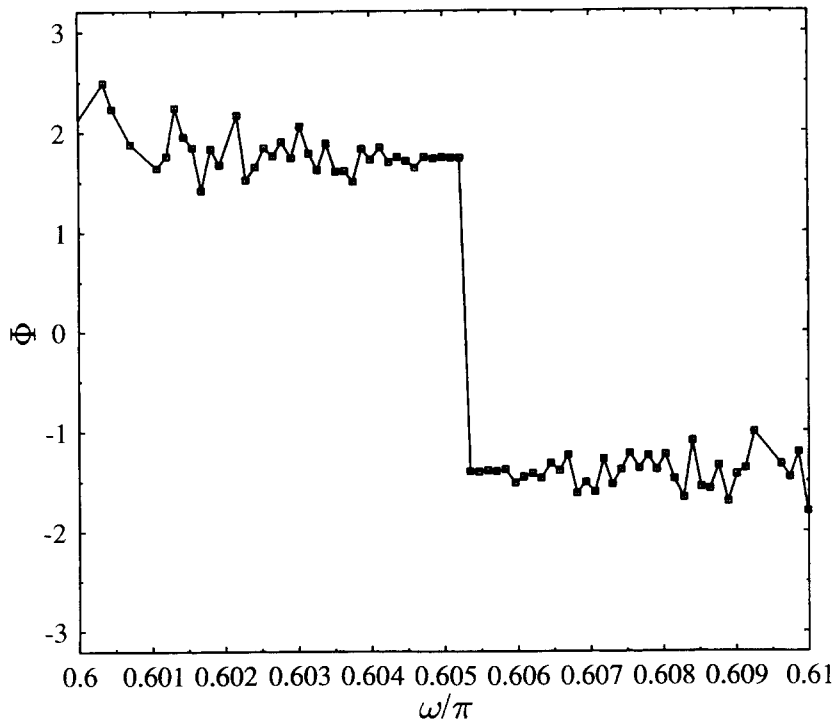


Fig. 3. Plot of the phase Φ of the discrete Fourier transform of the sequence r_n associated with the IFS of Figs. 1 and 2, to show the π discontinuity close to the value of the main peak of Fig. 2.

Firstly, the asymptotic behaviour for small t can be controlled as follows: $|c_n(t)|^2 \sim t^{2n}$. In fact,

$$c_n(t) = \int d\mu(x) p_n(x) \sum_{l=0}^{\infty} \frac{(-it)^l}{l!} x^l = \sum_{l=n}^{\infty} \frac{(-it)^l}{l!} \int d\mu(x) p_n(x) x^l. \tag{25}$$

Because of the orthogonality properties of the set p_n this expansion begins with $l = n$, which proves the result.

Secondly, in the infinite time limit, denoting by $\bar{S}_n(T)$ the time average of $|c_n|^2$ up to time T ,

$$\bar{S}_n(T) = \frac{1}{2T} \int_{-T}^T |c_n|^2(t) dt,$$

we have that

$$\bar{S}_n(T) \sim T^{-D_2} \tag{26}$$

for all n , a result which involves the correlation dimension D_2 of the fractal measure μ . The case with $n = 0$ is implicitly contained in [28], and was originally proposed in the present context by Ketzmerick et al. [4]. Successively, it has attracted a lot of attention, mainly from cultors of mathematical rigour. Our generalization has the advantage of requiring a simple proof, via the usage of the Mellin transform, as in [28]. In fact, we write

$$\bar{S}_n(T) = \int d\mu(x) \int d\mu(y) \frac{\sin(x-y)T}{(x-y)T} p_n(x) p_n(y). \tag{27}$$

To find the asymptotic behaviour of Eq. (27), we take the Mellin transform, $M_n(z)$, of $\bar{S}_n(T)$:

$$M_n(z) = \int T^{z-1} \bar{S}_n(T) dT = G(z) \times \int d\mu(x) \int d\mu(y) \frac{p_n(x)p_n(y)}{|x-y|^z} = G(z) \times E_n(z), \tag{28}$$

where $G(z) = \Gamma(z-1) \sin(\frac{1}{2}\pi(z-1))$ and $E_n(z)$ is defined implicitly by the last equality. The dominating power law in the long time behaviour of S_n is determined by the divergence abscissa of $M_n(z)$: that is to say, $\bar{S}_n(T) \sim T^{-w}$, where w is the largest real z for which $M_n(z)$, hence $E_n(z)$, converges. It is apparent from Eq. (28) that the divergence of E_n is piloted by the small scale structure of the measure μ . Since the polynomials p_n are smooth functions, with bounded derivatives on the support of μ , the divergence abscissa of E_n is the same for all n , and, in particular, it coincides with that of E_0 . $E_0(z)$ is known as the *generalized electrostatic energy* of the measure μ and its divergence abscissa is known to be D_2 [28], the correlation dimension of the measure μ .

It is important to remark that the domains of validity of the asymptotic expansions just derived are not uniform in n . This adds to the difficulty of the problem to be discussed in the following section.

6. Renormalization theory of quantum intermittency

An important characteristics of the quantum motion introduced in Section 5, is the way it spreads over the l_2 lattice basis, $\{e_n\}$. In fact, in oscillator terms, spreading corresponds to energy transmission along the linear chain, be it mechanical or electrical. In quantum mechanical terms, it corresponds to unbounded motion of the lattice particle, of the kind treated only qualitatively by RAGE theorems. To gauge this phenomenon, we define the moments of the position operator \hat{n} :

$$\nu_\alpha(t) := (\psi(t), \hat{n}^\alpha \psi(t)) = \sum_n n^\alpha |c_n(t)|^2. \tag{29}$$

Their asymptotic behaviour follows a power law,

$$\nu_\alpha(t) \sim t^{\alpha\beta}, \tag{30}$$

where β is a non-trivial function of the moment order α . In [6,27] we found that β is convex, non-decreasing, and non-constant even in the case of a one-scale Cantor set, characterized by trivial thermodynamics: this is what we call *quantum intermittency*. Corrections to Eq. (30) can also be observed in the form of log-scaled oscillations of $\nu_\alpha(t)$, super-imposed to its leading behaviour. They can be explained by the Mellin-type analysis presented in Section 5.

We can estimate the function $\beta(\alpha)$ on the basis of simple renormalization group considerations. For simplicity, let us consider an IFS with M maps, of equal probability $\pi_i = 1/M$. Let this IFS be non-overlapping. Then, let I be the smallest interval containing A , the IFS attractor, and let I_l be the image of I under the map ϕ_l . Clearly, $I_l \cap I_m = \emptyset$ if $l \neq m$, and the measure μ restricted to I_l is a linearly rescaled copy of the original. Then, as a first approximation, we can assume that the orthogonal polynomials of the restricted measure are also obtained by linear rescaling of the original polynomials:

$$p_{Mn}(\phi_l(x)) = \sum_{k=0}^{Mn} \Gamma_{l,k}^{Mn} p_k(x) \simeq \sigma_l^n p_n(x), \tag{31}$$

where $\sigma_l^n = \pm 1$. In other words, we assume a very simple form for the coefficients Γ^{Mn} , which amounts to making a renormalization ansatz.

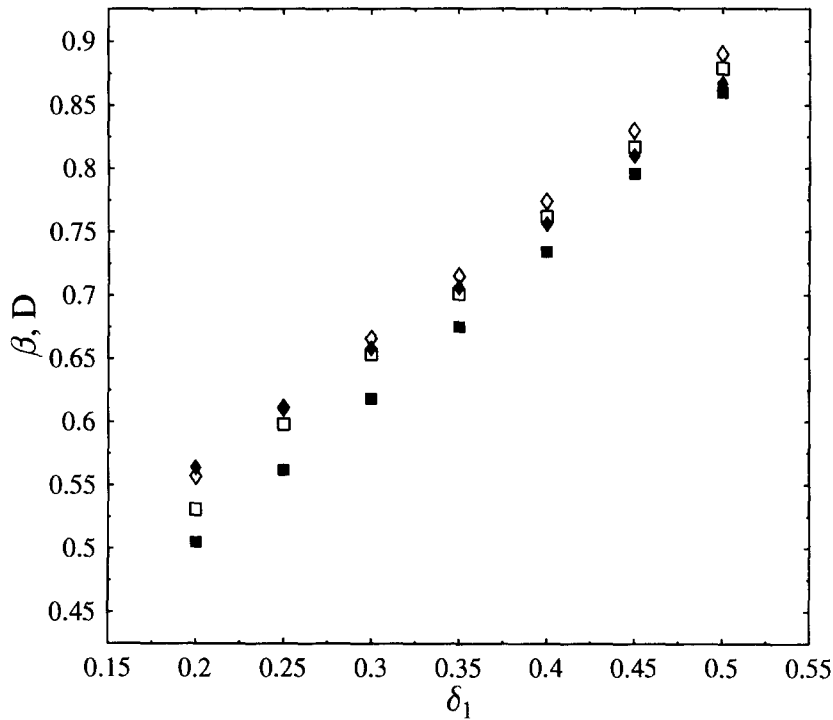


Fig. 4. Multi-fractal dimensions D_0 (full diamonds), D_1 (full squares), dynamical exponents $\beta(0)$ (open squares) and $\beta(1)$ (open diamonds) versus contraction rate δ_1 , for the family of IFS described in the text.

Let us now consider $\bar{S}_{Mn}(T)$, as defined above. Because of the balance property (2), it can be written

$$\bar{S}_{Mn}(T) = \sum_{l,m=1}^M \pi_l \pi_m \int d\mu(x) \int d\mu(y) \frac{\sin T(\phi_l(x) - \phi_m(y))}{T(\phi_l(x) - \phi_m(y))} P_{Mn}(\phi_l(x)) P_{Mn}(\phi_m(y)). \quad (32)$$

In the previous equation, $\phi_l(x)$ and $\phi_m(y)$ belong to I_l and I_m , respectively. If $l \neq m$, these intervals are separated by a finite gap. As T tends to infinity, these contributions tend to zero as T^{-1} . We can therefore retain only the diagonal terms in Eq. (32).

If we now employ the approximate estimate (31) in the RHS of Eq. (32) we can write

$$\bar{S}_{Mn}(T) = \sum_{l=1}^M \pi_l^2 \bar{S}_n(\delta_l T). \quad (33)$$

This too is a sort of renormalization equation which links the wave function component at site Mn and time T to the component at site n and at shorter times $\delta_l T$. When inserted in Eqs. (29) and (33) implies that the growth exponent β associated with the averaged moments \bar{v}_α via Eq. (30) must satisfy the relation

$$1 = M^{\alpha-1} \sum_{l=1}^M \delta_l^{\alpha\beta}. \quad (34)$$

Comparing this result with Eq. (4) we obtain the crucial equation

$$\beta(\alpha) = D_{1-\alpha}, \quad (35)$$

which links multi-fractal properties and time dynamics. In particular, Eq. (35) implies that $\beta(0) = D_1$, which is consistent with the rigorous result $\beta(0) \geq D_1$ [5]. Notice that $\beta(0)$ can be defined by a limiting procedure on $\beta(\alpha)$, or by the evolution of the logarithmic moment. We also have $\beta(1) = D_0$.

Because of the rough approximation involved in Eq. (31), and for the validity of Eq. (33) both c_{Mn} and c_n need to be in their asymptotic regimes, we do not expect Eq. (35) to be always exact. Indeed, in Fig. 4 we have considered a family of IFS measures, characterized by $M = 2$, $\delta_2 = \frac{2}{5}$, $\beta_1 = 0$, $\beta_2 = \frac{3}{5}$, $\pi_1 = \frac{3}{5}$, and $\pi_2 = \frac{2}{5}$. The contraction rate δ_1 is allowed to vary in the range $[0.2, 0.5]$, which implies a significant variation both in the structure of the support of the balanced measure and in its multi-fractal properties. Plotted in Fig. 4 are the scaling exponents $\beta(0)$ and $\beta(1)$, compared with the multi-fractal dimensions D_1 and D_0 , respectively. We observe a substantial agreement between the two data sets, dynamical and multi-fractal. Numerically, the discrepancy is always less than 5%. We can therefore conclude that relation (35) catches some essential part of the physics. Yet, the situation is more complicated, as the following pair of examples show.

The first is a magnificent counter-example. Let us consider a new class of IFS measures (and related Hamiltonians) characterized by $M = 2$ and by a particular choice of the weights:

$$\pi_j = \delta_j^D, \quad j = 1, 2, \tag{36}$$

where D is the (constant) value $\log 2 / (\log 5 - \log 2)$. This choice originates what is called a *uniform Gibbs measure*. The first of such IFS is that of Figs. 1–3, and D is its fractal dimension. Indeed, all IFS with property (36) are characterized by the same flat thermodynamic function $D_q = D$. Clearly, because of Eq. (36) and $\pi_1 + \pi_2 = 1$, only one parameter among the map weights and contraction rates is left free. By varying this parameter we can construct different IFS measures, with the same flat thermodynamics. What are then the corresponding dynamical exponents $\beta(\alpha)$? The approximate relation (35) predicts $\beta(\alpha) \simeq D$ for all α .

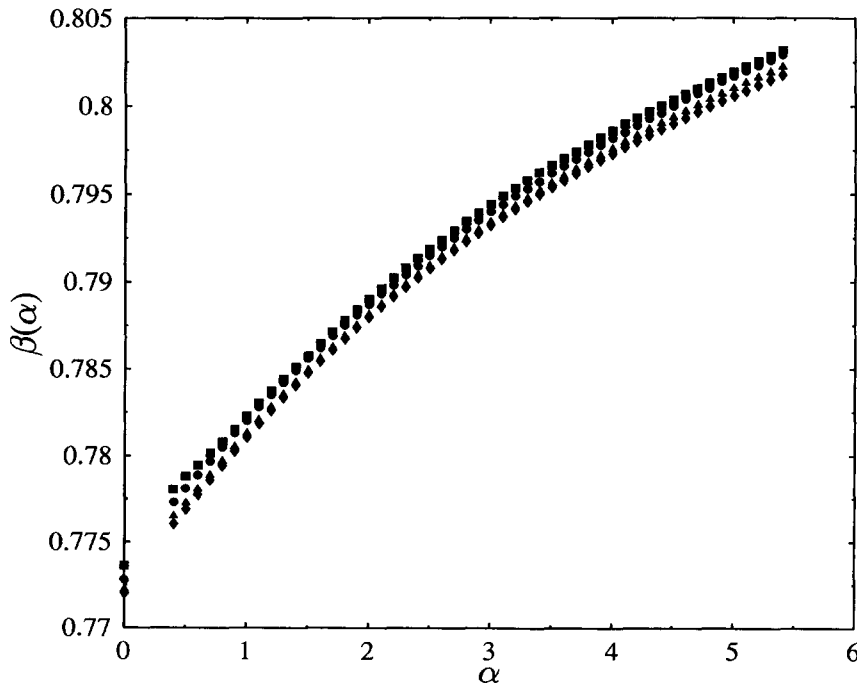


Fig. 5. Scaling functions $\beta(\alpha)$ for the four IFSs (a)–(d) described in the text: (a) circles; (b) squares; (c) triangles; (d) diamonds.

In Fig. 5 we have considered the IFS with: (a) $\delta_1 = \delta_2 = \frac{2}{5}$, $\pi_1 = \pi_2 = \frac{1}{2}$, which is a “pure” Cantor set; (b) $\delta_1 = 0.5090$, $\delta_2 = 0.2978$, and $\pi_1 = \frac{3}{5}$; (c) $\delta_1 = 0.5293$, $\delta_2 = 0.2802$, and $\pi_1 = 0.6180$. and (d) $\delta_1 = 0.6033$, $\delta_2 = 0.2196$, and $\pi_1 = 0.6823$. The first observation we can draw from this figure is that β is not flat, as shown in [27], even if the *intermittency range* in the $[0, 5]$ interval is very narrow. The second is that the prediction $\beta = D = 0.7565$ is correct within 2% at $\alpha = 0$ and about 5% at $\alpha = 5$. The third, and most important, is that the scaling function β is roughly invariant from case to case.

These results are intriguing – the coincidence of the curves in Fig. 5 suggests that the spectrum of generalized dimensions D_q must play some role in determining $\beta(\alpha)$: the fractal measures (a)–(d) seem to have little in common beyond having the same flat thermodynamics. Nevertheless, precisely because in these cases D_q is flat, neither Eq. (35), nor any general relation of the kind $\beta(\alpha) = D_{q(\alpha)}$, with q as an as yet unknown function of α can hold rigorously.

Let us now come to a favourable example: we can construct a class of measures for which the renormalization Eq. (31) is *exact*: these are the equilibrium measures of the Julia sets generated by the polynomials

$$P(z) = z^2 - \lambda, \tag{37}$$

where $\lambda \geq 2$ is a real constant. As we have already remarked, the Jacobi matrices for these problems can be constructed by a stable recursion algorithm [23,24]. Non-linearity of the IFS maps stemming from Eq. (37) as inverse branches of $P(z)$ can be treated by considering sufficiently high iterations $P^{(l)}$, and a theory perfectly analogous to (30)–(35) can be carried out, with the same result.

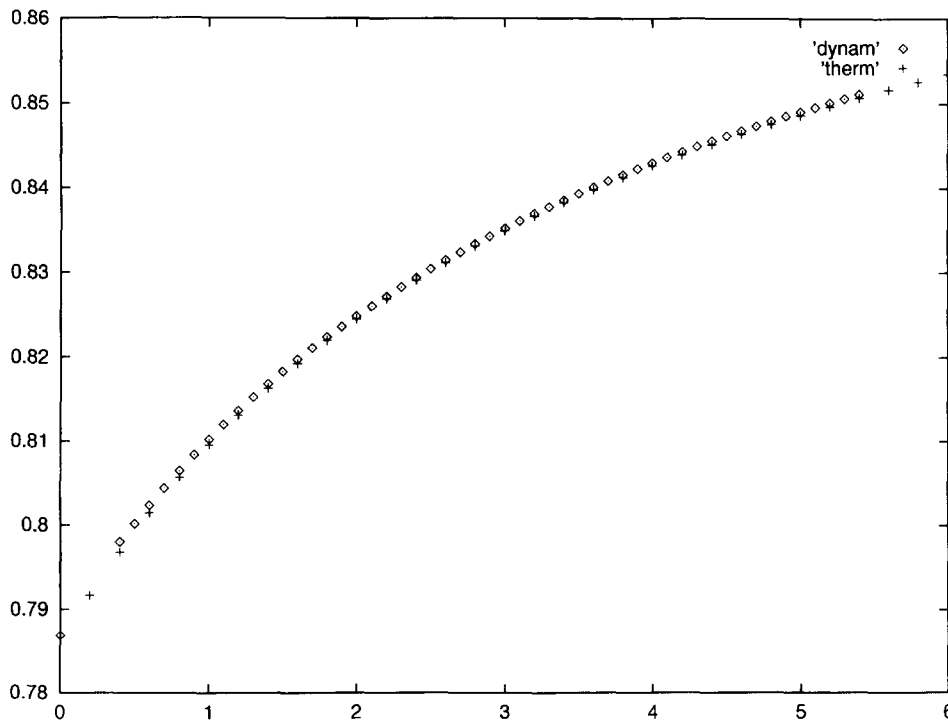


Fig. 6. Scaling function $\beta(\alpha)$ for the Julia set measure with $\lambda = 2.2$ (diamonds) and thermodynamical dimensions $D_{1-\alpha}$ (crosses).

In Fig. 6 we make the usual comparison between the moment scaling function, $\beta(\alpha)$, and thermodynamics [29], $D_{1-\alpha}$: the curves coincide within numerical precision! Therefore, one can conclude that discrepancies from Eq. (35) are due to the non-exactness of Eq. (31) when a spectral measure is approximated by IFS, except for the case of Julia measures, which are known to have strong algebraic properties.

Incidentally, we note that for Julia sets, the invariant measure coincides with the measure of the asymptotic distribution of the zeros of the associated orthogonal polynomials, the latter being also the physicists' global density of states. Might it be that the correct quantity entering Eq. (35) is this second measure? The analysis of the IFS data presented here seems to exclude this case, although we cannot exclude that this role is played by yet another spectral measure still to be determined.

7. Conclusions

We have presented a stable algorithm for the determination of lattice Hamiltonian operators possessing a given spectral measure, in the class of linear IFS. This algorithm consists of a recursive determination of the associated Jacobi matrix, in the framework of the theory of orthogonal polynomials.

The Hamiltonian operators determined in this way are characterized by almost-periodic coefficients: since IFS measures approximate arbitrarily well any measure supported on a Cantor set, this fact might lead to a proof that almost periodicity is always associated with this kind of spectra.

In a quantum mechanical context, the Jacobi matrices studied here can be employed as models of almost-periodic systems: the dynamical properties of such systems can be studied in their essence, having extracted the crucial information on the related spectral measures. We have shown that connections between spectral properties and dynamics go far beyond the conventional RAGE theorems: in particular, delocalization of particle's position along the lattice basis can be described by a scaling function β governing the moments of order α of the position operator. Non-constancy of this function translates mathematically the phenomenon of quantum interference.

We have derived an intriguing relation, $\beta(\alpha) = D_{1-\alpha}$, linking dynamics and the thermodynamical properties of the spectral measure: considering the Jacobi matrices associated with Julia sets we have constructed a family of quantum systems for which the relation is exact, and we have discussed the reasons for the discrepancies present in the general case. We believe that a further refinement of the results presented in this paper will lead to a profound understanding of the mathematical and physical properties of almost-periodic quantum systems.

Finally, we remark that the Jacobi Hamiltonians considered in this paper are not simple exotic curiosities, but can also describe time-resolved energy absorption in externally perturbed quantum systems, as well as electron dynamics in solid-state electrostructures like super-lattices [30], where by varying an alloy concentration along a deposition axis different spectral structures can be found [31]. Here, our results may become relevant in several problems, like – for instance – the design of lasers and radiation detectors.

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