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Orthogonal polynomials associated to almost periodic Schrödinger operators. A trend towards random orthogonal polynomials

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Abstract

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We introduce a special class of Schrödinger type H -operators in l^2 as $(\phi, H\psi) = \sum_{n=0}^{\infty} \phi_n^* [\sqrt{R_{n+1}}\psi_{n+1} + \sqrt{R_n}\psi_{n-1}]$, R_n being a nonnegative real number. H satisfies the renormalization equation $HD = D(H^2 - \lambda)$, with λ real, $\lambda \geq 2$. D is the decimation operator defined by $(\phi, D\psi) = \sum_{n=0}^{\infty} \phi_n^* \psi_{2n}$. A consequence of the renormalization equation is that the R_n fulfil the recursion relation $R_0 = 0$, $R_{2n}R_{2n-1} = R_n$, $R_{2n} + R_{2n+1} = \lambda$. From the above relations, it can be shown that the R_n are quasi-periodic functions of their index n .

The components of the eigenfunctions of H corresponding to the eigenvalue x are the orthonormalized polynomials $P_n(x)$ satisfying $\sqrt{R_{n+1}}P_{n+1}(x) + \sqrt{R_n}P_{n-1}(x) = xP_n(x)$. The spectrum of H is the support of the measure associated to the polynomials. In the present case it is a compact perfect set of Lebesgue measure zero (Cantor set). It is therefore purely singular continuous.

We are led to study classes of orthogonal polynomials whose three-terms recursive relations are quasi periodic functions of their index. We will present several results, conjectures and open questions which may have relevant physical applications. We study the randomness of the eigenfunctions, and we discuss their algorithmic complexity.

Keywords: Orthogonal polynomials; Schrödinger equation; singular spectrum; quasi periodicity; algorithmic complexity; chaotic motion.

1. Introduction

The spectral theory of Schrödinger operators is a crucial tool to understand important physical properties of various materials, ranging from crystalline lattices (with their electronic and vibrational

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motions), to amorphous materials (showing Anderson localization [2]), and to quasi periodic arrays (characterized by singular continuous spectra [16]). Exact results are nevertheless rare in this field, and can be obtained only at the price of sophisticated mathematics, such as functional analysis. To obviate this fact, we present in this paper a more modest point of view, introducing a class of Hamiltonian models which can be exactly solved, and offer highly nontrivial dynamical behaviours, to be related to those of quasi periodic media.

The systems we will study have also relevance from a different viewpoint. It is known that many physical properties can be described by entering a random element into the definition of a quantum Hamiltonian, for instance, considering Schrödinger operators with random potentials. The randomness of the potentials originates here the phenomenon of Anderson localization [2]. On the other hand, the quantization of classically chaotic systems (e.g., systems displaying a chaotic diffusion in phase space) usually leads to the so-called *suppression of classical chaos*, that is, to a quantum motion which is more stable than the classical analogue [10,12,14]. The class of systems that we will explicitly solve lies in-between these extremal cases, and hence can shed some light on the “chaoticity” of quantum evolution. For our models, we will explicitly determine the “randomness” of these objects (quantum spectral properties) showing that, although nontrivial, it is not sufficient to yield “quantum chaos”. Although this absence of chaos might be regarded as a negative result, we believe that it is particularly instructive, and might point the direction towards more complex generalizations of this procedure.

The key feature of the models we consider here is their renormalizability: our class of Hamiltonians will be defined as satisfying an exact renormalization equation.

Without loss of generality, we will work in l^2 , the space of square summable sequences. Basis vectors in this space will be labelled by e_n , $n = 0, 1, \dots$, where—as usual— e_n has a unity in the n th component, and zeros otherwise: $e_n = (0, \dots, 0, 1, 0, \dots)$. Given any element ψ in l^2 , its n th component will be simply defined as $\psi_n = (e_n, \psi)$. Thanks to this notation, we can introduce the *decimation operator* D :

$$(D\psi)_n = \psi_{2n}. \quad (1)$$

The meaning of the decimation operator is transparent: any vector in Hilbert space is deprived of its odd components. If we want to think of these components as physical frequencies—of instance—the decimation operator produces a halving of all even frequencies, and a deletion of the odd.

The basic renormalization equation for the Hamiltonian H reads

$$HD = DT(H), \quad (2)$$

where as a first approximation we will let T be a polynomial in the Hamiltonian H . In the next section, we investigate the consequences of the renormalization condition (2). We will show that they lead to a Hamiltonian which is quasi periodic. We then proceed to determine the spectral properties of H , and its eigenfunctions, which are further discussed in Sections 3 and 4.

2. Schrödinger–Jacobi Hamiltonians

In this section, we specialize (2) to a quadratic polynomial $T(H) = H^2 - \lambda$. We remark that most results will extend to general polynomials [3], and the present choice is dictated by simplicity

of exposition. The presentation will follow without substantial modifications the original references [3,6].

A particular class of Hamiltonians satisfying the renormalization condition is given by Schrödinger-like discrete operators of the form

$$(H\psi)_n = \sqrt{R_{n+1}}\psi_{n+1} + \sqrt{R_n}\psi_{n-1}, \tag{3}$$

where R_n are constants to be determined, in order to satisfy (2). This form of the Hamiltonian is a generalization of the discrete form of the Laplace operator ($R_n = \text{constant}$), and allows for important “quasi periodicity” properties of the sequence R_n , as we shall see.

In matrix form, we see that (3) corresponds to a tridiagonal Jacobi matrix

$$\begin{pmatrix} 0 & \sqrt{R_1} & 0 & 0 & \cdots \\ \sqrt{R_1} & 0 & \sqrt{R_2} & 0 & \cdots \\ 0 & \sqrt{R_2} & 0 & \sqrt{R_3} & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{4}$$

To satisfy renormalizability, we use (1) and (3) to compute the matrix elements of HD and $D(H^2 - \lambda)$:

$$(HD)_{i,j} = \sqrt{R_{i+1}}\delta_{2i+2,j} + \sqrt{R_i}\delta_{2i-2,j} \tag{5}$$

and

$$(DH^2)_{i,j} = \sqrt{R_{2i+1}}\sqrt{R_{2i+2}}\delta_{2i+2,j} + (R_{2i+1} + R_{2i})\delta_{2i,j} + \sqrt{R_{2i-1}}\sqrt{R_{2i}}\delta_{2i-2,j}. \tag{6}$$

Using these expressions into (2), and identifying the matrix coefficients, one obtains a set of recursion relations to be satisfied by the sequence R_n :

$$R_0 = 0, \quad R_{2n} + R_{2n+1} = \lambda, \quad R_{2n}R_{2n-1} = R_n. \tag{7}$$

In all this, λ is a real parameter, which can take any value larger than or equal to two. When $\lambda = 2$, the solution of the recursion equations (7) is $R = (0, 2, 1, 1, \dots)$, which corresponds exactly to a discrete Laplace operator with a given boundary condition at the origin. It is then to be expected that the spectrum possesses in this case an absolutely continuous component.

Let us now examine more closely the recursion relation (7). The first terms read

$$R_1 = \lambda, \quad R_2 = 1, \quad R_3 = \lambda - 1, \quad R_4 = \frac{1}{\lambda - 1}, \quad R_5 = \frac{\lambda^2 - \lambda - 1}{\lambda - 1}, \tag{8}$$

showing that R_n are rational functions of λ . Self-adjointness of the Hamiltonian (3) is insured when these coefficients are positive. It is therefore crucial to determine for which values of the parameter λ this fact occurs. This is established by the following lemma.

Lemma 1. *If $\lambda \geq 2$, we have for all integer n ,*

- (i) $R_{2n} \leq R_n$,
- (ii) $R_{2n} \leq 1$,
- (iii) $0 \leq R_0, R_1, \dots, R_{2n}$.

Proof. We prove the lemma by induction. Let us suppose that the relations (i)–(iii) hold for $n = 0, \dots, p-1$. Then, we have

$$R_{2p} = \frac{R_p}{\lambda - R_{2p-2}} \leq \frac{R_p}{\lambda - 1} \leq R_p, \quad (9)$$

where we have used the recursion relations and the induction hypothesis (ii). This proves induction on (i).

For (ii), we consider two cases: p even and p odd. If $p = 2s$,

$$R_{2p} \leq R_p \leq 1, \quad (10)$$

where the first inequality follows from (9), and the second from the induction hypothesis (ii). If $p = 2s + 1$,

$$R_{2p} = \frac{\lambda - R_{2s}}{\lambda - R_{4s}} \leq \frac{\lambda - R_{4s}}{\lambda - R_{4s}} = 1, \quad (11)$$

where the inequality is obtained by (i). Finally, we prove positivity. To verify induction, we must prove that $R_{2p-1}, R_{2p} \geq 0$:

$$R_{2p-1} = \lambda - R_{2p-2} \geq \lambda - 1 \quad (12)$$

proves the first part, and

$$R_{2p} = \frac{R_p}{R_{2p-1}} \geq 0 \quad (13)$$

proves the second, since both R_p and R_{2p-1} are positive. \square

From the above lemma it follows easily that

$$\lambda - 1 \leq R_{2n+1} \leq \lambda. \quad (14)$$

Together with Lemma 1, this shows that the respective ranges of variation of even and odd R_n are separated, if $\lambda > 2$. From (8) we see that the extrema are also realized:

$$0 = R_0 \leq R_{2n} \leq R_2 = 1, \quad \lambda - 1 = R_3 \leq R_{2n+1} \leq R_1 = \lambda. \quad (15)$$

Equation (15) seems to imply a hierarchical, number-theoretical splitting of the ranges of variation of the coefficient R_n . Such a hierarchical structure can be conjectured to have the following form:

$$(-1)^{\sigma(m)} R_m \leq (-1)^{\sigma(m)} R_{m+k2^r} \leq (-1)^{\sigma(m)} R_{m+2^r}, \quad (16)$$

where $r = 1, 2, \dots$, $m = 0, 1, \dots, 2^r - 1$, and k takes all integer values $k = 1, 2, \dots$. The integer $\sigma(m)$ is defined as the sum of the digits in the binary expansion of m (e.g., $5 = 101$, $\sigma(5) = 2$). Equation (15) is a special case of (16), for $r = 1$. The case $r = 2$ has been proven [15] by explicit calculation.

It is known that conjecture (16) is false for values of r higher than two if λ is close to the critical value $\lambda = 2$ [13]. On the other hand, one might expect that for increasing λ the inequalities (16) are verified for larger values of r . It might also happen that there exists a critical value of λ after which all of the above inequalities are verified. No proof of this fact is known at the moment.

The structure of the set of possible values of R_n might therefore have a fractal structure, in the light of the above conjecture. An additional piece of information is given by the following lemma.

Lemma 2. For all integer $k = 1, 2, \dots$, we have

$$\lim_{r \rightarrow \infty} R_{m+k2^r} = R_m,$$

m, r being also integers.

Proof. We first prove the lemma for $m = 0$. We have that, for all n , $R_{2n} = R_n / (\lambda - R_{2n-2})$, and Lemma 1 then implies

$$\frac{R_n}{\lambda - 1} \geq R_{2n} \geq \frac{R_n}{\lambda}.$$

Iterating this relation, we obtain

$$\frac{R_n}{(\lambda - 1)^r} \geq R_{2^r n} \geq \frac{R_n}{\lambda^r},$$

which proves the lemma, if $\lambda > 2$. The other values of m can be obtained by simple induction. \square

We notice that this lemma implies that the set of possible values of R_n is a perfect set. In fact, it is compact (because of Lemma 2), and each point is an accumulation point. For the values of λ satisfying the conjecture, this set is also a Cantor set.

A more sophisticated result linking the values of R_n can be found.

Lemma 3. If $\lambda > 2$,

$$|R_{p2^k+s} - R_s| \leq B(\lambda, k),$$

where $B(\lambda, k)$ is a function independent of both p and s . One can choose

$$B(\lambda, k) = \frac{\lambda}{(\lambda - 2)^k}. \quad (17)$$

Proof. For $s = 0$ we have

$$|R_{p2^k} - R_0| = R_{p2^k} \leq \frac{R_p}{(\lambda - 1)^k} \leq \frac{\lambda}{(\lambda - 1)^k}, \quad (18)$$

where the first inequality has been previously derived. Therefore, for $s = 0$ a suitable $B(\lambda, k)$ exists.

Let us now consider an even value of s . We have that

$$|R_{p2^k+s+1} - R_{s+1}| = |R_{p2^k+s} - R_s|, \quad (19)$$

and hence if the bound (3) is verified up to s even, it is also verified for $s + 1$. The case s odd is more complicated. Since $s + 1$ is even, we can write

$$|R_{p2^k+s+1} - R_{s+1}| = \left| \frac{R_a}{R_b} - \frac{R_c}{R_d} \right|, \quad (20)$$

where $a = p2^{k-1} + \frac{1}{2}(s+1)$, $b = p2^k + s$, $c = \frac{1}{2}(s+1)$ and $d = s$. Now, we use the following inequality, valid for all positive R :

$$\left| \frac{R_a}{R_b} - \frac{R_c}{R_d} \right| \leq \frac{|R_a - R_c|}{R_b} + \frac{R_c}{R_d} \frac{|R_b - R_d|}{R_b}. \quad (21)$$

Because of the induction hypothesis,

$$|R_a - R_c| \leq B(\lambda, k-1) \quad \text{and} \quad |R_b - R_d| \leq B(\lambda, k).$$

Moreover, since b is odd, we have $R_b \geq \lambda - 1$. Finally, we remember that

$$\frac{R_c}{R_d} = R_{s+1} \leq 1,$$

because s is odd. Using these inequalities in (21), we obtain

$$\left| \frac{R_a}{R_b} - \frac{R_c}{R_d} \right| \leq \frac{B(\lambda, k-1)}{\lambda-1} + \frac{B(\lambda, k)}{\lambda-1}. \quad (22)$$

We have therefore that $B(\lambda, k)$ must satisfy the relation

$$\frac{B(\lambda, k-1)}{\lambda-1} + \frac{B(\lambda, k)}{\lambda-1} \leq B(\lambda, k), \quad (23)$$

that is,

$$B(\lambda, k) \geq \frac{B(\lambda, k-1)}{\lambda-2}, \quad (24)$$

and a possible choice of this bound is therefore

$$B(\lambda, k) = \frac{\lambda}{(\lambda-2)^k}. \quad \square$$

It must be remarked that this bound tends to zero only when $\lambda > 3$. In this case, the sequence R_n is an almost periodic function of the index n . In fact, by definition, $f(x)$ is almost periodic iff [9] for all ε , and for all $T_0 > 0$, there exists a $T > T_0$ such that

$$|f(x+T) - f(x)| \leq \varepsilon,$$

uniformly in x . This is clearly the case for the sequence R_n . For the reader's convenience, we note that an almost periodic function has Fourier coefficients

$$\hat{f}(\omega) = \lim_{L \rightarrow \infty} \int_{-L}^L e^{-i\omega x} f(x) dx,$$

which are identically null except for a countable set $\{\omega_n\}$ of frequencies. This is also an equivalent definition of almost periodic function [9]. We can then formally write

$$f(x) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} \hat{f}(\omega_m) e^{i\omega_m x}. \quad (25)$$

There may be cases where this series does not converge in the usual sense.

A common subclass of almost periodic functions is that of quasi periodic functions, which are defined by a property of the sequence of frequencies $\{\omega_n\}$. Precisely, any element of the algebra generated by $f(x)$ has a set of periods of the form

$$\omega_{n_1, \dots, n_k} = n_1 \omega_1 + \dots + n_k \omega_k,$$

where n_i are integers, and k is finite.

We are interested in still another class, that of limit periodic functions. These are defined by requiring that any finite truncation of the series (25) is a periodic function.

Lemma 4. *The sequence of R_n is limit periodic.*

Proof. We write

$$R_n = R_n^Q + \varepsilon_n,$$

where R^Q is a periodic sequence of period 2^Q , and can be expressed as

$$R_n^Q = \sum_{q=0}^Q \sum_{p=0}^{2^Q-1} f_{p,q} e^{2i\pi n(2p+1)/2^q}. \quad (26)$$

Because of the estimate (17), the remainder ε_n goes uniformly to zero, when Q tends to infinity. Hence R_n has a Fourier series of the form (26), which is limit periodic. \square

To end this section, we briefly comment on the properties of the set of poles and zeros of the rational fractions $R_n(\lambda)$. It is possible to prove that these sets are identical, but very little more is known about. The closure of this set could be related to the Mandelbrot set, but no rigorous result is available. Only outside such set the coefficients R_n can be quasi periodic functions [3].

3. The spectral problem associated to H

We can now solve the spectral problem of H , which we denote by

$$(H\psi)_n = x\psi_n = \sqrt{R_{n+1}}\psi_{n+1} + \sqrt{R_n}\psi_{n-1}, \quad (27)$$

with the boundary conditions $\psi_{-1} = 0, \psi_0 = 1$. Notice that we are employing the letter x to denote the quantum eigenvalue, not to be confused with λ , which is an external parameter.

In general, we will say that x belongs to the spectrum of H if and only if the associated eigenfunction ψ_n has at most polynomial growth: $|\psi_n| \leq n^P$, for n large.

Because H is a tridiagonal operator, we will employ the similarity of the spectral equation (27) with the three-terms recursion relation typical of orthogonal polynomials. In fact, (27) is also the definition of the orthonormalized polynomials associated to the invariant measure of the Julia set of $T(z)$ [8]. This fact will have a crucial role in the following. From now on, we will use the notation $\psi(x)$ to denote the solution of the spectral equation (29) with eigenvalue x . By posing

$$\psi_n = \prod_{j=1}^n R_j^{-1/2} P_n, \quad (28)$$

we obtain the set of monic orthogonal polynomials, which obey the recursion relation

$$P_{n+1}(x) = xP_n(x) - R_nP_{n-1}(x). \quad (29)$$

Thanks to the renormalizability of the Hamiltonian, eigenfunctions corresponding to different eigenvalues can be simply related, as in the following lemma.

Lemma 5.

$$\psi_{2n}(x) = \psi_n(x^2 - \lambda), \quad P_{2n}(x) = P_n(x^2 - \lambda).$$

Proof. Using the renormalization equation (2) on $H\psi(x) = x\psi(x)$, and taking the n th component, $(D(H^2 - \lambda)\psi)_n = (HD\psi)_n$, one obtains

$$(x^2 - \lambda)\psi_{2n}(x) = \sqrt{R_{n+1}}\psi_{2n+2}(x) + \sqrt{R_n}\psi_{2n-2}(x). \quad (30)$$

At the same time, the eigenvalue equation (27) written for $\psi(x^2 - \lambda)$, solution with eigenvalue $x^2 - \lambda$, reads

$$(x^2 - \lambda)\psi_n(x^2 - \lambda) = \sqrt{R_{n+1}}\psi_{n+1}(x^2 - \lambda) + \sqrt{R_n}\psi_{n-1}(x^2 - \lambda). \quad (31)$$

Since these two relations have identical coefficients, and are identically initialized, we get the relation

$$\psi_n(x^2 - \lambda) = \psi_{2n}(x). \quad (32)$$

We remark that this proof does not require that x belong to the spectrum of H . The second part of the lemma follows easily from the last equation (7) for the coefficients R_j :

$$\prod_{j=1}^m R_j = \prod_{j=1}^{2m} R_j. \quad \square$$

Lemma 6. *The solutions $\psi(x)$ of the eigenproblem (29) have the property:*

$$\psi_n(-x) = (-1)^n \psi_n(x).$$

Proof. Since $\psi_0 = 1$, $\psi_1 = R_1^{-1/2}x$, the property is verified for $n = 0, 1$. By induction, supposing the lemma is verified for $n = 0, \dots, p$, and using (29), one obtains

$$\sqrt{R_{p+1}}\psi_{p+1}(-x) = x\psi_p(-x) - \sqrt{R_p}\psi_{p-1}(-x) = (-1)^p \psi_{p+1}(x). \quad \square$$

Thanks to this technical lemma, we can prove the following lemma.

Lemma 7. *The spectrum is symmetric: $x \in \sigma(H)$ implies $-x \in \sigma(H)$.*

Proof. Suppose that $\psi(x)$ is an eigenfunction, that is, it satisfies (29) and it is polynomially bounded. Then, $\psi(x)$ obtained in Lemma 6 is also polynomially bounded, and hence $-x$ belongs to the spectrum. \square

In the next section, we will derive crucial properties of the spectral measure.

4. The spectral measure associated to H

We are now able to prove an important result regarding the measure $d\mu$ with respect to which the polynomials $\psi_n(x)$ are orthogonal.

Lemma 8. *The components $\psi_n(x)$ of the solutions of the spectral equation (27) are normalized orthogonal polynomials associated with the balanced measure on the Julia set $d\mu$, defined by*

$$\int d\mu(x) f(x) = \int d\mu(x) f(x^2 - \lambda), \quad (33)$$

for any continuous function f .

Proof. To prove this lemma, we must show the spectral formula

$$I_{n,m} = \int \psi_n(x) \psi_m(x) d\mu(x) = \delta_{n,m}. \quad (34)$$

Firstly, we consider the case $I_{n,0}$. It is immediate that n must be even, because of parity. Using (33) with $f(x) = \psi_n(x)$, one obtains

$$I_{n,0} = I_{n/2,0}.$$

If $\frac{1}{2}n$ is odd, the integral is null. Otherwise, the procedure is repeated, showing that

$$I_{n,0} = \delta_{n,0}.$$

One can now proceed by induction, supposing that (34) is verified for all n , and for $m = 0, \dots, p$, and proving it for $m = p + 1$.

If $p + 1$ is even, n odd, parity assures the result. If n is even, it follows very easily that

$$I_{n,p+1} = I_{n/2,(p+1)/2},$$

allowing use of the induction hypothesis.

If $p + 1$ is odd, n even, parity assures the result. If n is odd, one must twice make use of the eigenvalue equation, to reduce the degree of the orthogonal polynomials, and apply the induction hypothesis. We skip this rather tedious calculation. \square

We will now derive other important properties of the spectrum.

Lemma 9. *The spectrum has nonlinear scaling properties: $x \in \sigma(H)$ implies $x^2 - \lambda \in \sigma(H)$, $\pm\sqrt{x + \lambda} \in \sigma(H)$.*

Proof. The first assertion is immediate to prove, since, if $\psi(x)$ is a polynomially bounded solution of the spectral problem, then $D\psi(x)$ is also polynomially bounded, and verifies $HD\psi(x) = (x^2 - \lambda)D\psi(x)$.

The second assertion is also easy to prove. Let $H\psi(y) = y\psi(y)$, and take $y = \pm\sqrt{x+\lambda}$. Applying the previous result, we know that $HD\psi(y) = (y^2 - \lambda)D\psi(y) = xD\psi(y)$, that is, $D\psi(y) = \psi(x)$. If $x \in \sigma(H)$, then $\psi(x)$ is polynomially bounded, and hence the subsequence $\psi_{2n}(y) = \psi_n(x)$ is also polynomially bounded. Because of the eigenvalue equation, we can prove that the odd subsequence $\psi_{2n+1}(y)$ is also polynomially bounded.

This last result can be also used to construct the eigenfunction $\psi(\pm\sqrt{x+\lambda})$ if $\psi(x)$, $x \in \sigma(H)$, is known:

$$\begin{aligned}\psi_{2n}(\pm\sqrt{x+\lambda}) &= \psi_n(x), \\ \psi_{2n+1}(\pm\sqrt{x+\lambda}) &= (\pm\sqrt{x+\lambda})^{-1} [\sqrt{R_{2p+2}}\psi_{p+1}(x) + \sqrt{R_{2p+1}}\psi_p(x)].\end{aligned}\quad (35)$$

Lemma 10. *The spectrum is bounded: $\sigma(H) \subset [-a(\lambda), a(\lambda)]$.*

Proof. Because of symmetry, it is enough to consider the positive part of the spectrum. Let $x \in \sigma(H)$. Firstly, we notice that $|x| \leq \lambda$; otherwise $\sqrt{x+\lambda}$ would be a complex number, which is prevented because the spectrum must be real. This first bound ensures that there surely exists a quantity $a(\lambda) \leq \lambda$ which bounds the spectrum. Lemma 9 imposes the inequality $a^2 - \lambda \leq a$, whence

$$a \leq \frac{1}{2}(1 + \sqrt{1 + 4\lambda}) = a(\lambda), \quad (36)$$

where we have defined the bound $a(\lambda)$. To see that this bound is optimal, we take any $x \in \sigma(H)$, and consider the sequences x_n of values in the spectrum defined by

$$x_{n+1} = \sqrt{x_n + \lambda}, \quad x_0 = x.$$

This sequence is bounded, monotonically nondecreasing, $x_{n+1} \geq x_n$, and hence admits a limit which is easily computed as the fixed point of the recursion relation, and is $a(\lambda)$: $x_n \rightarrow a(\lambda)$. \square

Lemma 11. *The spectrum is the Julia set of the polynomial $T(x) = x^2 - \lambda$.*

Proof. This lemma follows from the previous Lemmas 9 and 10. We notice that this Julia set can be obtained as the Iterated Functions System associated to the two nonlinear maps $w_{\pm}(x) = \pm\sqrt{x+\lambda}$, which allow a rapid construction of the associated invariant measure. The Iterated Functions System construction [4] also allows to prove that this set is perfect. \square

In particular, we can label the spectrum by the infinite sequence $\{\sigma_i\}$, $i = 0, 1, \dots$, $\sigma_i = \pm 1$. In this way, any point $x \in \sigma(H)$, labelled $x(\underline{\sigma})$, is the limit of the sequence

$$x_n = \sigma_0\sqrt{\lambda + \sigma_1\sqrt{\dots\sigma_n\sqrt{\lambda + \eta}\dots}} = (w_{\sigma_0} \circ w_{\sigma_1} \circ \dots \circ w_{\sigma_n})(\eta), \quad (37)$$

which converges to the same limit $x(\underline{\sigma})$ for any $\eta \in [-\lambda, \lambda]$.

The action of the polynomial T on the spectrum corresponds to the shift operator S on the sequence $\underline{\sigma}$: $(S\sigma)_n = \sigma_{n+1}$. This follows immediately from the definition (37):

$$T(x(\underline{\sigma})) = x(S\underline{\sigma}), \quad (38)$$

moreover,

$$w_{\pm}(x(\underline{\sigma})) = x(\pm 1, \underline{\sigma}), \quad (39)$$

where $(\pm 1, \underline{\sigma})$ is a shorthand notation for the sequence σ' defined by $\sigma'_0 = \pm 1$, $\sigma'_{n+1} = \sigma_n$.

It is now important to introduce the spectral measure $d\tilde{\mu}$, relative to the mean value of the resolvent in the state e_0 , defined by

$$(e_0, (1 - zH)^{-1}e_0) = \int (1 - zx)^{-1} d\tilde{\mu}(x). \quad (40)$$

It turns out that this measure is identical to the previous measure $d\mu$ (with respect to which ψ_n are orthogonal).

Lemma 12. *The spectral measure $d\tilde{\mu}$ is identical to the balanced measure of the Julia set of the polynomial $T(x) = x^2 - \lambda$.*

The measure $d\tilde{\mu}$ is invariant under T and w_{\pm} :

$$\int d\tilde{\mu}(x) f(x) = \int d\tilde{\mu}(x) f(x^2 - \lambda) = \frac{1}{2} \int d\tilde{\mu}(x) [f(w_+(x)) + f(w_-(x))],$$

for any continuous function f .

Proof. Since $H^m \psi(x) = x^m \psi(x)$, we can write

$$(e_0, H^m \psi) = x^m (e_0, \psi) = x^m \psi_0(x) = x^m. \quad (41)$$

Let us consider the moments μ_m of the invariant measure of the Julia set $d\mu$:

$$\mu_m = \int d\mu(x) x^m.$$

The left-hand side of (41) can be differently computed:

$$(e_0, H^m \psi) = \sum_{k=0}^{\infty} (e_0, H^m e_k) (e_k, \psi) = \sum_{k=0}^m (e_0, H^m e_k) \psi_k(x), \quad (42)$$

where the summation over k has been restricted to $0, \dots, m$ because H^m is a banded matrix with band size $2m + 1$. We now integrate (41) and (42) with respect to $d\mu$, remembering that $\psi_n(x)$ are orthonormalized polynomials with respect to this measure (Lemma 8). Because of this, only one term at the right-hand side of (42) survives, and we obtain

$$\mu_m = (e_0, H^m e_0). \quad (43)$$

The moments μ_m can be seen as the expectation values of the operator H^m over the state e_0 . It is now immediate to show that the moment generating function $G(z)$, defined via

$$G(z) = \sum_{k=0}^{\infty} \mu_k z^k, \quad (44)$$

can be rewritten, thanks to (43) as

$$G(z) = \sum_{k=0}^{\infty} (e_0, H^k e_0) z^k = (e_0, (1 - zH)^{-1} e_0). \quad (45)$$

This proves that the Stieltjes transforms of $d\mu$ and $d\tilde{\mu}$ are identical. It then follows that $d\tilde{\mu}$ has the same balanced properties of $d\mu$, which is the thesis. \square

A corollary of this lemma is the functional equation for the resolvent function:

$$G(z) = \frac{1}{1 - \lambda z^2} G\left(\frac{z^2}{1 - \lambda z^2}\right). \quad (46)$$

Inserting the moment representation (44) into the functional equation (46), we derive the equality

$$(1 - \lambda z^2) \sum \mu_k z^k = \sum \mu_l z^{2l} (1 - \lambda z^2)^{-l}. \quad (47)$$

Making use of the binomial series expansion for $(1 - x)^{-l}$ and identifying terms with equal powers of z , we obtain a recursion relation in the moments:

$$\mu_k - \lambda \mu_{k-2} = \sum_{2(j+l)=k} \mu_l \lambda^j \binom{l+j-1}{j}, \quad (48)$$

where the sum is extended to all positive or null integers j, l whose sum is k . The above relation is initialized with $\mu_0 = 1, \mu_1 = 0$, and produces recursively all moments of the measure $d\mu$. In particular, all odd moments turn out to be null, as they should.

It must be observed that the moments μ are polynomial in λ with integer coefficients. When λ is also an integer, we are therefore considering a diophantine moment problem [7].

To end this section, we detail the calculations for the case $\lambda = 2$. In this case, the sequence of coefficients R_n is $R_0 = 0, R_1 = 2, R_m = 1, m \geq 2$, as is easily shown by induction. Consequently, the spectral equation (29) becomes $P_1(x) = x, P_2(x) = xP_1(x) - 2$ and

$$xP_n(x) = P_{n+1}(x) + P_{n-1}(x), \quad n \geq 2. \quad (49)$$

This set of equations is easily recognized to be associated to the Chebyshev polynomials $T_n(x)$:

$$P_n(x) = 2T_n\left(\frac{1}{2}x\right), \quad n \geq 1. \quad (50)$$

The invariant measure of the Julia set generated by the polynomial map $T(z) = z^2 - 2$ is the absolutely continuous measure with density $\sigma(x)$ given by

$$\sigma(x) = \frac{1}{\pi} \frac{1}{\sqrt{4 - x^2}}. \quad (51)$$

5. Complexity of the eigenfunctions

The important spectral properties derived in the previous section allow us to discuss the amount of “randomness” of the eigenfunctions. We remark that, for the eigenfunctions to be random, it is not necessary that the Hamiltonian matrix elements be random variables.

To do this, for any $x \in \sigma(H)$ we consider the sequence of real numbers $\psi_n(x) = (\psi(x), e_n)$, the components of the related eigenfunction on the canonical basis. Following the approach of algorithmic complexity [11], we will translate such a sequence into a binary digits string, by fixing an arbitrary precision $\delta = 2^{-p}$ for each of the ψ_n components. The symbolic sequence obtained in this fashion will be simply denoted by $[\psi_n]_\delta$. We will also consider all finite truncations $n = 0, \dots, N$ of such digit strings. The complexity of such truncations is then defined as the length (in binary symbols) of the shortest program, which, coded on a universal computer, is capable of outputting such a sequence.

The complexity of a sequence is by logical consequence [11] uncomputable, but we can nevertheless put bounds on its value, producing particular codings of given length. We will call *random* (or *chaotic*, or *complex*) all sequences whose complexity is of the order of the truncation length N [1].

In the study of iterative systems (like the one we are now considering), estimating the complexity amounts to determining the sensitivity of the sequence ψ_n to the initial condition, and to the eigenvalue x . In fact, a simple code of the truncated sequence $[\psi_n(x)]_\delta$, $n = 0, \dots, N$, consists of the recursion relation and initial conditions (27), which can be translated into a finite, constant-length code, and of the specification of N and x .

While N is easily coded by $\log_2 N$ bits, one needs to determine how many bits of x are needed to code $[\psi_n]_\delta$ exactly, for n as large as N . As the simplest example, let us consider the sequence $\phi_n(x) = 2\phi_{n-1}(x) \bmod 1$, $\phi_0(x) = x$. This is a random sequence, because taking the truncation at precision $\delta = 2^{-1}$, we see that $[\phi_n]_\delta$ is the n th binary digit of x . Hence, to code the sequence up to $n = N$ we need N bits of information.

The motion on the Julia set of the polynomial $T(x) = x^2 - \lambda$, for $\lambda > 2$, gives another example of random sequence. In fact, because of Lemmas 5 and 11, this set is labelled by the binary string $\underline{\sigma}$, and $x \rightarrow T(x)$ is equivalent to the shift on such a sequence. This is crucial to prove the following lemma.

Lemma 13. *The subsequences*

$$\eta_n = \psi_{2^m n}(x) = \psi_m(T^{(n)}(x)),$$

$x \in \sigma(H)$, m integer, are random.

Proof. To compute η_0, \dots, η_N requires $\sigma_0(x), \dots, \sigma_N(x)$. \square

On the other hand, this does not imply that the full sequence $\psi_n(x)$, that is, the eigenfunction, is random. In fact, this does not prevent the fact that the values $[\psi_j]$, $2^n \leq j \leq 2^{n+1}$, can be derived from the digit string corresponding to $T^n(x)$, that is, $S^n \underline{\sigma}(x)$.

It is particularly easy to show this fact in the limiting case $\lambda = 2$. Here, $\psi_n(x)$ is proportional to $\arccos(n \cos(\frac{1}{2}x))$. The constant of proportionality, via (28), is trivially computed, and does not depend on n . Now, an algorithm to compute the Chebyshev polynomial is easily coded with a finite

binary length program. The computation of $\psi_n(x)$ is therefore related to the computation of the argument $\frac{1}{2}nx \bmod 2\pi$. This leads to another well-known dynamical system, the ergodic rotation on the circle: $\theta_{n+1} = \theta_n + \omega$. Any truncation of this sequence has a complexity which scales as $\log_2 N$, and hence it is nonchaotic.

6. Time evolution generated by H

The Schrödinger–Jacobi Hamiltonian H generates a time evolution in the space l^2 via the Schrödinger equation

$$-i\partial_t\phi(t) = H\phi(t), \quad (52)$$

with $\phi(0) = \phi_0 \in l^2$ as initial condition. Since H is time-independent, the solution of (52) is

$$\phi(t) = e^{iHt} \phi_0. \quad (53)$$

To evaluate this expression, we project on the states $\psi(x)$, use the closure relation (34), and get

$$\phi(t) = \int e^{ixt} (\psi(x), \phi_0) \psi(x) d\mu(x). \quad (54)$$

We then project on the basis state e_n , as follows:

$$\phi_n(t) = (e_n, \phi(t)) = \int e^{ixt} (\psi(x), \phi_0) \psi_n(x) d\mu(x). \quad (55)$$

To obtain an expression in terms only of components on the canonical basis, we develop the scalar product $(\psi(x), \phi_0)$, and we get

$$\phi_n(t) = \sum_{k=0}^{\infty} \phi_k \int e^{ixt} \psi_k(x) \psi_n(x) d\mu(x), \quad (56)$$

which can be rewritten as

$$\phi_n(t) = \sum_{k=0}^{\infty} \phi_k C_{n,k}(t), \quad (57)$$

where we have defined the time-dependent coefficient $C_{n,k}(t)$:

$$C_{n,k}(t) = \int e^{ixt} \psi_k(x) \psi_n(x) d\mu(x). \quad (58)$$

Knowledge of these coefficients enables us to solve the full time-dependent problem. The coefficient $C_{0,0}$ gives the autocorrelation function of the time evolution of e_0 . For reasons of simplicity, we only develop the analysis of this case. The analysis extends without significant modifications to the general case.

When $n = 0$, $k = 0$, we recognize in (58) the Fourier transform of the measure $d\mu$:

$$\hat{\mu}(t) = \int e^{ixt} d\mu(x). \quad (59)$$

The study of the long-time behaviour of the motion generated by H is therefore equivalent to the fine structure analysis of the measure $d\mu$. Physically, this is equivalent to the structure of the X-ray analysis of a fractal measure.

The balanced properties of $d\mu$ enable us to prove the following lemma.

Lemma 14. *The Fourier transform of the spectral measure $d\mu$, $\hat{\mu}(t)$, satisfies the integral equation*

$$\hat{\mu}(t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-i[x^2/4t + \lambda t - \pi/4]} \hat{\mu}(x) dx. \quad (60)$$

Proof. We insert in (60) the explicit definition of $\hat{\mu}(x)$, and we change the order of integration:

$$\hat{\mu}(t) = \frac{1}{2\sqrt{\pi t}} \int d\mu(y) \int_{-\infty}^{\infty} e^{-i[x^2/4t + \lambda t - \pi/4] + ixy} dx.$$

Then, we explicitly calculate the Gaussian integral; this leads to

$$\hat{\mu}(t) = \int d\mu(y) e^{-i[\lambda t - y^2 t]}.$$

We finally use the balanced property (33), and we obtain the thesis

$$\int d\mu(y) e^{-i[\lambda t - y^2 t]} = \int d\mu(y) e^{-iyt} = \hat{\mu}(t). \quad \square$$

This lemma is the corner-stone of the analytical treatment of this problem. We notice here only some of its main consequences. Separating real and imaginary part in (60), and making use of the symmetry of $d\mu$, we obtain

$$\hat{\mu}(t) = \frac{1}{\sqrt{\pi t}} \int_0^{\infty} \cos\left(\frac{x^2}{4t} + \lambda t - \frac{1}{4}\pi\right) \hat{\mu}(x) dx \quad (61)$$

and

$$0 = \frac{1}{\sqrt{\pi t}} \int_0^{\infty} \sin\left(\frac{x^2}{4t} + \lambda t - \frac{1}{4}\pi\right) \hat{\mu}(x) dx. \quad (62)$$

We can give a different form to $\hat{\mu}(t)$, by defining the integrals

$$I_1(t) = \frac{1}{\sqrt{\pi t}} \int_0^{\infty} \cos\left(\frac{x^2}{4t}\right) \hat{\mu}(x) dx, \quad (63)$$

$$I_2(t) = \frac{1}{\sqrt{\pi t}} \int_0^{\infty} \sin\left(\frac{x^2}{4t}\right) \hat{\mu}(x) dx. \quad (64)$$

This leads to

$$\begin{aligned}\hat{\mu}(t) &= \cos(\lambda t - \tfrac{1}{4}\pi)I_1 - \sin(\lambda t - \tfrac{1}{4}\pi)I_2, \\ 0 &= \sin(\lambda t - \tfrac{1}{4}\pi)I_1 + \cos(\lambda t - \tfrac{1}{4}\pi)I_2,\end{aligned}\tag{65}$$

and finally to

$$\hat{\mu}(t) = \frac{1}{\cos(\lambda t - \tfrac{1}{4}\pi)} \int_0^\infty \cos\left(\frac{x^2}{4t}\right) \hat{\mu}(x) dx = -\frac{1}{\sin(\lambda t - \tfrac{1}{4}\pi)} \int_0^\infty \sin\left(\frac{x^2}{4t}\right) \hat{\mu}(x) dx.\tag{66}$$

In the special case $\lambda = 2$, the Fourier transform of $d\mu$ is the entire function $\hat{\mu}(t) = J_0(2t)$. Inserting this function into (66) leads to a well-known identity [5, p.110, eq. 27]. Equation (66) may lead to an iterative solution scheme. The full details of this study are presently under investigation.

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