

Asymptotic analysis of the moments of the Cantor distribution[☆]

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Abstract

We use known recursive expressions for the moments of the Cantor distribution to derive asymptotic expansions for these moments. This is done by a combination of a method based on Mellin transform and the saddle point method.

Keywords: Cantor distribution; Mellin transform; Saddle point method

The Cantor distribution with parameter $\vartheta, 0 < \vartheta < 1$ can be described by a random series

$$\frac{\bar{\vartheta}}{\vartheta} \sum_{i \geq 1} X_i \vartheta^i,$$

where the X_i are independent with the distribution

$$\mathbb{P}\{X_i = 0\} = \mathbb{P}\{X_i = 1\} = \frac{1}{2},$$

and $\bar{\vartheta} = 1 - \vartheta$. The essential result of Lad and Taylor (1992) is the following recursion for the moments $\mathbb{E}(X^N)$:

$$\mathbb{E}(X^N) = \frac{1}{2(1 - \vartheta^N)} \sum_{i=0}^{N-1} \binom{N}{i} \vartheta^i \bar{\vartheta}^{N-i} \mathbb{E}(X^i), \quad N \geq 1, \quad \mathbb{E}(X^0) = 1. \quad (1)$$

The aim of the present note is to solve the recursion (1). We abbreviate $a_N = \mathbb{E}(X^N)$ and rewrite the recursion as

$$2a_N - \vartheta^N a_N = \sum_{i=0}^N \binom{N}{i} \vartheta^i \bar{\vartheta}^{N-i} a_i, \quad N \geq 1, \quad a_0 = 1. \quad (2)$$

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Now we introduce the exponential generating function

$$A(z) = \sum_{N \geq 0} a_N \frac{z^N}{N!}.$$

Recursion (2) translates into

$$2A(z) - 2 - A(\vartheta z) + 1 = e^{\bar{\vartheta}z} A(\vartheta z) - 1,$$

or

$$A(z) = \frac{1 + e^{\bar{\vartheta}z}}{2} A(\vartheta z). \quad (3)$$

We can solve this equation by iteration, and we obtain (note that $A(0) = 1$)

$$A(z) = \prod_{k \geq 0} \frac{1 + e^{\bar{\vartheta}^k z}}{2}. \quad (4)$$

For $\vartheta = \frac{1}{2}$, the product collapses to

$$A(z) = \frac{e^z - 1}{z},$$

so that $a_n = 1/n + 1$, which of course can be seen directly, since the Cantor distribution is then just the uniform distribution over the interval $[0, 1]$. Since a_N is the coefficient of $z^N/N!$, we have solved the recursion (1). More useful, however, is the asymptotic equivalent for a_N , which we are going to derive in the sequel. For that, we need also the Poisson generating function

$$B(z) = e^{-z} A(z) = \sum_{N \geq 0} b_N \frac{z^N}{N!}.$$

Eq. (3) translates then into

$$B(z) = \frac{1 + e^{-\bar{\vartheta}z}}{2} B(\vartheta z), \quad (5)$$

yielding

$$B(z) = \prod_{k \geq 0} \frac{1 + e^{-\bar{\vartheta}^k z}}{2}.$$

We note here that the result obtained above could be deduced easily by interpreting the Cantor distribution as the distribution function of a 2-additive function given by

$$f\left(\sum_{\ell=0}^L \varepsilon_\ell 2^\ell\right) = \bar{\vartheta} \sum_{\ell=0}^L \varepsilon_\ell \vartheta^\ell.$$

The notion of q -additive functions was introduced by Delange (1972), where a necessary and sufficient condition for the existence of a distribution function is given. Formula (4) is an immediate consequence of

Delange’s formula for the Fourier transform of this distribution function. Furthermore notice that for $\vartheta < \frac{1}{2}$ the moment generating function $A(z)$ can be written as the integral

$$A(z) = \int_{\mathcal{C}} e^{zt} d\mathcal{H}(t),$$

where \mathcal{C} denotes the Cantor set given by the closure of the image $f(\mathbb{N})$ and \mathcal{H} denotes the normalized Hausdorff measure of dimension $\log_{\frac{1}{\vartheta}} 2$. The function $h(x) = \int_{[0,x] \cap \mathcal{C}} d\mathcal{H}(t)$ is a singular function, i.e. it is monotonically increasing, thus differentiable almost everywhere and the derivative vanishes almost everywhere despite of the fact that $h(0) = 0$ and $h(1) = 1$.

Let us now start the asymptotic analysis of a_N . For this purpose we compute the Mellin transform (cf. Doetsch, 1958; Flajolet et al., 1995) of the logarithm of $B(z)$

$$(\log B)^*(s) = \int_0^\infty \log B(x) x^{s-1} dx = \frac{\vartheta^{-s}}{1 - \vartheta^{-s}} \int_0^\infty \log \left(\frac{1 + e^{-x}}{2} \right) x^{s-1} dx \tag{6}$$

for $-1 < \Re s < 0$. The remaining integral is easily computed as

$$\int_0^\infty \log \left(\frac{1 + e^{-x}}{2} \right) x^{s-1} dx = \Gamma(s) \zeta(s+1) (1 - 2^{-s}),$$

again for $-1 < \Re s < 0$. Thus by Mellin’s inversion formula we have

$$\log B(z) = \frac{1}{2\pi i} \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} \Gamma(s) \zeta(s+1) (1 - 2^{-s}) \frac{\vartheta^{-s}}{1 - \vartheta^{-s}} z^{-s} ds.$$

This formula holds for every z with $|\arg z| < \pi/2$. By shifting the line of integration to the right and taking the residues at $s = 0$ (double pole!) and $s = \chi_k := 2k\pi i / \log 1/\vartheta$ for $k \in \mathbb{Z} \setminus \{0\}$ (simple poles) into account we obtain

$$\begin{aligned} \log B(z) &= -\log_{\vartheta} 2 \cdot \log \vartheta z - \frac{\log 2}{2} - \frac{\log 2 \cdot \log_{\vartheta} 2}{2} \\ &+ \frac{1}{\log \vartheta} \sum_{k \in \mathbb{Z} \setminus \{0\}} \Gamma(\chi_k) \zeta(1 + \chi_k) (1 - 2^{-\chi_k}) e^{-\chi_k \log \vartheta} e^{2k\pi i \log_{\vartheta} z} \\ &+ \frac{1}{2\pi i} \int_{M-i\infty}^{M+i\infty} \Gamma(s) \zeta(s+1) (1 - 2^{-s}) \frac{\vartheta^{-s}}{1 - \vartheta^{-s}} z^{-s} ds, \end{aligned} \tag{7}$$

where $\Theta = 1/\vartheta$ and M is any positive real number. Since the remaining integral is convergent by the well-known asymptotic behaviour of the Γ -function, the remainder term is a $O(z^{-M})$ for any $M > 0$. From this we derive

$$B(z) = F(\log_{\vartheta} z) z^{-\log_{\vartheta} 2} (1 + O(z^{-M}))$$

with an infinitely differentiable 1-periodic function $F(x)$.

The “de-Poissonization” technique (cf. Grabner, 1993; Rais et al., 1993) suggests the approximation $a_N \sim B(N)$. Applying it to every term of the Fourier expansion of $F(x)$ separately (the Fourier series is uniformly convergent), we obtain

$$a_N = F(\log_{\vartheta} N) N^{-\log_{\vartheta} 2} \cdot \left(1 + O\left(\frac{1}{N}\right) \right). \tag{8}$$

In order to derive an explicit expression for the Fourier coefficients of $F(x)$ we take the Mellin transform of (5)

$$B^*(s) = \frac{1}{2} \Theta^s B^*(s) + \frac{1}{2} \int_0^\infty B(\vartheta x) e^{-\vartheta x} x^{s-1} dx.$$

By the above computations we know that this transform exists for $0 < \Re s < \log_\Theta 2$. The last equation yields

$$B^*(s) = \frac{1}{2 - \Theta^s} \int_0^\infty B(\vartheta x) e^{-\vartheta x} x^{s-1} dx,$$

where the integral converges for every s with $\Re s > 0$. By Mellin’s inversion formula the Fourier coefficients of $F(x)$ are equal to the negative residues of $B^*(s)$ at $s = \log_\Theta 2 + \chi_k, k \in \mathbb{Z}$. These are given by

$$\hat{F}(k) = -\text{Res } B^*(s) \Big|_{s=\log_\Theta 2 + \chi_k} = \frac{1}{2 \log \Theta} \int_0^\infty B(\vartheta x) e^{-\vartheta x} x^{\log_\Theta 2 - 1 + \chi_k} dx.$$

These integrals can be easily computed numerically, because they are rapidly convergent for $x \rightarrow \infty$ and the behaviour for $x \rightarrow 0$ is regular.

The value

$$\hat{F}(0) = \frac{1}{2 \log \Theta} \int_0^\infty B(\vartheta x) e^{-\vartheta x} x^{\log_\Theta 2 - 1} dx$$

is of special interest, as it is the *mean value* around which the periodic function $F(x)$ fluctuates. Since the amplitudes of these fluctuations are usually quite small, we can write

$$a_N \simeq \hat{F}(0) \cdot N^{-\log_\Theta 2},$$

which is suggestive, but not quite correct, as it ignores the fluctuations.

Remark. Notice that $\hat{F}(0)$ can be viewed as the following limit

$$\hat{F}(0) = \lim_{N \rightarrow \infty} \frac{1}{\log N} \sum_{n=1}^N a_n n^{\log_\Theta 2 - 1}.$$

This is due to the fact that $\sum_{n < N} n^{-1+it} = O(1)$ for $t \neq 0$.

To illustrate the results, we give a table of $\hat{F}(0)$, for several values of ϑ , and compare them with the values of the recursion, for $N = 50, N = 100$ and $N = 200$ (see Table 1).

Furthermore we present a plot of $a_N N^{\log_{10} 2}$ versus $\log_{10} N$ for $\vartheta = 0.1$ to illustrate the fluctuating behaviour of this function. (see Fig. 1).

For the reader’s convenience we summarize our findings.

Theorem. *The moments $\mathbb{E}(X^N)$ of the Cantor distribution with parameter ϑ are given as the coefficient of $\frac{z^N}{N!}$ in*

$$\prod_{k \geq 0} \frac{1 + e^{(1-\vartheta)\vartheta^k z}}{2}.$$

Asymptotically, we have

$$\mathbb{E}(X^N) = F(\log_{1/\vartheta} N) N^{-\log_{1/\vartheta} 2} \cdot \left(1 + O\left(\frac{1}{N}\right) \right),$$

where $F(x)$ is a periodic function of period 1 and known Fourier coefficients. The mean of $F(x)$ is given by

$$-\frac{1}{2 \log \vartheta} \int_0^\infty \prod_{k \geq 1} \frac{1 + e^{-(1-\vartheta)\vartheta^k x}}{2} e^{-(1-\vartheta)x} x^{\log_{1/\vartheta} 2 - 1} dx.$$

We note here that in the case $\vartheta = 2^{-1/m}$ for a positive integer m the function F is constant, because all Fourier coefficients in (7) vanish (except the one for $k = 0$). This corresponds to the fact that $A(z)$ can be

Table 1

	$\hat{F}(0)$	$a_{50} \times 50^{\log_{\vartheta} 2}$	$a_{100} \times 100^{\log_{\vartheta} 2}$	$a_{200} \times 200^{\log_{\vartheta} 2}$
$\vartheta = 0.7$	3.746	3.5373	3.6374	3.6892
$\vartheta = 0.6$	1.532	1.4842	1.5077	1.5197
$\vartheta = 0.4$	0.800	0.7902	0.7953	0.7979
$\vartheta = 0.333$	0.734	0.7260	0.7297	0.7320
$\vartheta = 0.3$	0.711	0.7041	0.7084	0.7094
$\vartheta = 0.2$	0.670	0.6682	0.6656	0.6726
$\vartheta = 0.1$	0.657	0.6534	0.6688	0.6513

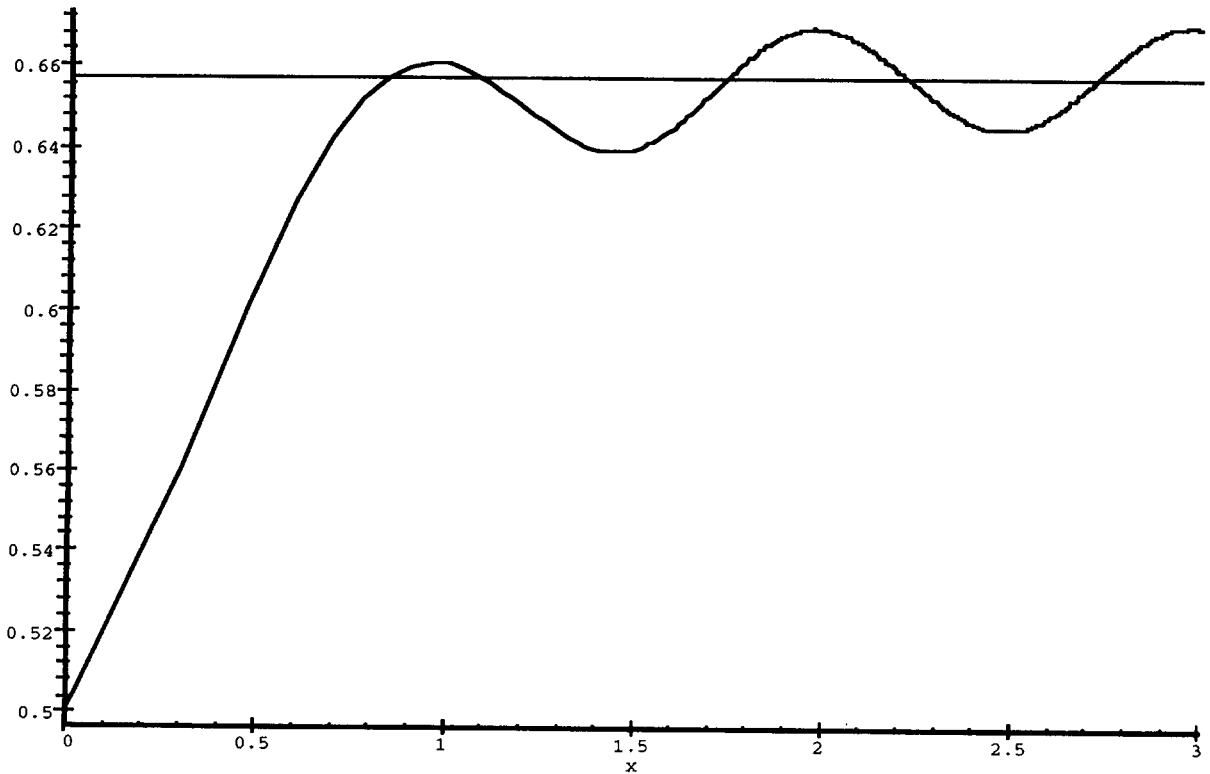


Fig. 1.

given explicitly by

$$A(z) = \bar{\vartheta}^{-m} 2^{-m+1/2} z^{-m} \prod_{r=0}^{m-1} \left(e^{2\bar{\vartheta}^r z} - 1 \right) \sim \bar{\vartheta}^{-m} 2^{-m+1/2} z^{-m} e^z,$$

which yields

$$a_N \sim \bar{\vartheta}^{-m} 2^{-m+1/2} N^{-m}$$

for this case.

Furthermore, we note that in the context of the order statistics of the Cantor distribution a recursion occurred in Hosking (1994) which was solved in Knopfmacher and Prodinger (1994). In this case, a somewhat more direct approach could be used, since explicit formulæ for the quantities of interest were available, whereas here we have only the generating function $A(z)$, from which we have to extract the necessary information.

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