

Tuning Anomalous Photon Transport with Deterministic Aperiodic Structures

Luca Dal Negro^{1,2*,+} and Sandeep Inampudi^{1,3,+}

¹Boston University, Department of Electrical and Computer Engineering and Photonics Center 8 Saint Mary's Street, Boston Massachusetts 02215, United States of America

²Boston University, Division of Materials Science and Engineering, 15 Saint Mary's Street, Brookline Massachusetts 02446, United States of America

³Northeastern University, Electrical and Computer Engineering Department, 360 Huntington Ave, Boston Massachusetts 02215, United States of America

+these authors contributed equally to this work

*corresponding.dalnego@bu.edu

ABSTRACT

The propagation of optical pulses through deterministic aperiodic structures is numerically studied in the time domain using the rigorous transfer matrix method. We demonstrate tunable anomalous photon transport properties, including the elusive logarithmic Sinai sub-diffusion regime in photonic systems for the first time. Our results are in excellent agreement with the scaling theory of wave transport in aperiodic media with fractal spectra, and can be modeled within the analytical framework of fractional diffusion-wave equations. In particular, we discuss the connection between tunable optical pulse transport in deterministic aperiodic media and fractional transport equations, and we show that non-Markovian fractional kinetic models can provide analytical insights into the complex photon transport in optical media with tunable aperiodic order. Engineering anomalous photon transport phenomena in non-homogeneous systems and metamaterials with aperiodic spatial correlations can provide a novel strategy to enhance light-matter interactions and tailor transport phenomena beyond the standard diffusion regime.

Please note: Abbreviations should be introduced at the first mention in the main text – no abbreviations lists. Suggested structure of main text (not enforced) is provided below.

Introduction

In 1855, Adolf Fick proposed his laws governing mass transport through diffusive media.¹ In particular, Fick's second law predicted how the concentration $\phi(x, t)$ of a diffusing substance changes with space and time. Fick showed that in one spatial dimension $\phi(x, t)$ obeys the well-known diffusion equation $\partial_t \phi(x, t) = D \partial_{xx} \phi(x, t)$, where D is the diffusion coefficient. Almost 60 years later, in his theoretical study of the Brownian motion Albert Einstein unveiled the microscopic origin of diffusion by introducing a memoryless (i.e., Markovian) random walk model.² It is now well-known that the microscopic dynamics of Markovian random walks obeys a stochastic differential equation which, in the continuum limit of vanishingly small steps, reduces to the Fick's diffusion equation.³ Conversely, the fundamental solution (i.e., the Green's function) for the Cauchy problem of the linear diffusion equation can be interpreted as a Gaussian Probability Density Function (PDF) in space, which evolves in time. All moments of this PDF are finite and its variance, or Mean Square Displacement (MSD), is proportional to the first power of time, i.e., $\langle x^2(t) \rangle \propto t$, which is the hallmark of standard diffusion. However, after Richardson's 1926 pioneering work on diffusion in turbulent media,⁴ many natural phenomena were discovered that exhibit *anomalous diffusion* characterized by a nonlinear scaling of the MSD according to the power law:⁵ $\langle x^2(t) \rangle \propto t^\beta$, with β a real number in the interval $[0, 2]$. In particular, anomalous sub-diffusion occurs when $\beta < 1$ and anomalous super-diffusion when $\beta > 1$. The extremal cases $\beta = 1$ and $\beta = 2$ correspond to standard diffusion and ballistic wave transport, respectively.

At the microscopic level, anomalous diffusion processes are described by generalized Continuous Time Random Walks (CTRWs) that capture the non-Markovian correlation effects between the different steps of a walker in non-homogeneous random media.⁶ Contrary to standard (i.e., uncorrelated) random walks, CTRW models allow for the possibility of incorporating separate statistical distributions for the waiting times and step sizes of the random walker, including long-tailed non-Gaussian distributions with algebraic decays that yield divergent moments, as in the case of Lévy flights.⁷

At the continuum level, CTRWs converge to *fractional transport equations* with space and time derivatives of fractional order.⁸ These are integro-differential operators with power-law kernels that account for the space correlations and the time

memory effects that are important when wave excitations are multiply scattered in strongly non-homogeneous environments.^{11,12} Remarkable examples of anomalous transport have been recently provided in various scientific domains such as turbulent plasmas, viscoelasticity, percolation and transport through fractals and porous media, amorphous solids, biological systems, and even internet traffic.¹³ In optics, super-diffusive photon transport has recently been demonstrated in artificial media called Lévy glasses.¹⁴ However, anomalous sub-diffusion has not been reported in engineered optical structures so far. This ability could lead to the development of novel active media that systematically exploit non-Markovian correlation effects as additional degrees of freedom to tailor photon transport and to enhance light-matter coupling over broader frequency spectra.

In this paper, by numerically investigating optical pulse propagation through dielectric structures, we demonstrate anomalous transport largely tunable from sub- to super-diffusion in complex media with long-range aperiodic order. In particular, we focus on one-dimensional (1D) dielectric multilayer structures with aperiodic refractive index modulations. Such structures can be rigorously described by well-established transfer matrix and trace map techniques, and have been extensively investigated to explore the rich physics of aperiodic scattering potentials in electronic and optical systems.^{15,16} Following this approach, we additionally demonstrate that 1D dielectric systems with aperiodic pseudo-randomness support tunable logarithmic photon transport with a temporal scaling of the MSD described by the law: $\langle x^2 \rangle \sim \log^\nu(t)$. Such an intriguing regime, which was theoretically investigated by Sinai in the context of random walks in non-homogeneous random media,¹⁷ is referred to as *ultra-slow Sinai sub-diffusion*. It is interesting to notice that at the continuum level, Sinai logarithmic sub-diffusion can also be modeled by fractional diffusion equations but only when generalized to distributed-order.^{18–20}

Anomalous transport and fractional wave propagation

The nature of transport in complex photonic structures is investigated by considering the time scaling of the MSD and the temporal autocorrelation function (ACF) of optical wave packets that propagate across photonic multilayers with periodic, quasi-periodic and pseudo-random positional order. The structures considered in this work are simply generated by binary inflation rules that act on two constituent layers (say A and B) with refractive indices n_1 and n_2 , respectively. With no loss of generality the optical thickness of each layer is set to be $\lambda_0/4$, and we chose $\lambda_0 = 1550\text{nm}$. The spatio-temporal electric field distribution of propagating pulses and the optical transmission spectra of the investigated structures are calculated using the rigorous transfer matrix and method over a wide frequency range centered at $\omega_0 = 2\pi c/\lambda_0$. Additional details on this well-established technique can be found in our methods section.

Anomalous transport of electron wave packets through 1D quasi-periodic potentials with fractal energy spectra has been intensively investigated in recent years using the transfer matrix approach,²¹ and the connection between the spreading of wave packets and the fractality of energy spectra and eigenmodes has been established.^{23,24} In particular, both scaling analysis and numerical simulations suggest that the exponent β for quantum wave packets diffusing through a 1D quasi-periodic sequence of scattering potentials can be expressed as $\beta = 2D_2^\mu/D_2^\psi$, where D_2^μ and D_2^ψ are fractal dimensions of the spectrum and of the wavefunctions, respectively.²⁴ This simple relation implies already that the transport in a system with fractal energy spectrum can be largely controlled by the fractality of its eigenmodes.

Due to the isomorphism between the Schrödinger and Helmholtz equations in 1D scattering potentials, we expect classical wave scattering to share fundamental similarities with its electronic counterpart. Using full-vector electromagnetic modeling, we demonstrate an entirely analogous transport behavior in 1D photonic systems with deterministic aperiodic order and show that, for each investigated structure, photon transport can be largely controlled by the refractive index contrast.

In analogy with the electronic case, we define the MSD of optical wave packets as:^{24,25}

$$\langle x^2(t) \rangle = \frac{1}{L} \int_0^L |E(x,t)|^2 |x - x_0|^2 dx \quad (1)$$

where L represents the total length of the sample, and $E(x,t)$ is the normalized electric field inside the sample as a function of space and time (methods section). Moreover, we study the temporal ACF of optical wave packets defined as:²³

$$C(t) = \frac{1}{t} \int_0^t |\langle E(x,t') | E(x,0) \rangle|^2 dt' \quad (2)$$

where \langle, \rangle indicates the inner product $\langle E(x,t) | E(x,0) \rangle = \sum_i E(x_i,t) E(x_i,0)$.

The temporal scaling of the above quantities fully reveals the anomalous nature of diffusive transport in arbitrary 1D systems. In particular, it is well-known for the electron transport across quasi-periodic structures with fractal spectra that the asymptotic scaling of $C(t)$ decays algebraically as $\sim t^{-\delta}$, where the exponent δ is equal to the fractal dimension of the spectrum.^{23,24}

While many fascinating results have been established in recent years by resorting to extensive numerical simulations of electronic fractal systems, a comprehensive analytical framework that effectively captures their asymptotic transport properties

has only recently been developed based on generalized kinetic equations.²⁸ This powerful approach exploits the connection between anomalous transport and fractional calculus,^{11,12} which provides an analytical underpinning for transport phenomena in the presence of long-range spatio-temporal correlations.²⁸

In the context of our paper, we show that the large tunability of the anomalous photon transport in deterministic aperiodic systems is effectively described by the asymptotic solutions of the *fractional diffusion-wave equation* (FDWE): $D_*^\beta \phi(x, t) = D \partial_{xx} \phi(x, t)$ with $0 < \beta \leq 2$. In this equation D stands for a generalized diffusion coefficient while D_*^β is the Caputo-type fractional time-derivative of (real) order β , which is defined as:¹⁰

$$D_*^\beta \phi(x, t) = \begin{cases} \frac{1}{\Gamma(1-\beta)} \int_0^t \left[\frac{\partial}{\partial \tau} \phi(x, \tau) \right] \frac{d\tau}{(t-\tau)^\beta} & \text{if } 0 < \beta < 1 \\ \frac{1}{\Gamma(2-\beta)} \int_0^t \left[\frac{\partial^2}{\partial \tau^2} \phi(x, \tau) \right] \frac{d\tau}{(t-\tau)^{\beta-1}} & \text{if } 1 < \beta < 2 \end{cases} \quad (3)$$

and Γ is the Euler's gamma function. When β is an integer ($\beta = 1, 2$) the right-hand-side in the above definition reduces to the corresponding partial derivative of integer order and we recover the standard diffusion equation for $\beta = 1$ and the wave equation for $\beta = 2$. When $1 < \beta < 2$ the fractional equation is expected to interpolate continuously between a diffusion and wave propagation. The large tunability of wave transport associated to the fractional order β manifests the microscopic non-Markovian nature of the photon transport in complex aperiodic systems. The FDWE kinetics provides an effective model to account for the complex photon correlations and memory effects established by the phases of multiply scattered waves in strongly inhomogeneous aperiodic media. The connection between field propagation inside multilayered structures with fractal spectra and fractional transport is discussed in our methods section in terms of the behavior of the scattering map.⁴⁵ In particular, it is possible to show based on simple scaling arguments that the time-dynamics of optical pulses for the systems discussed in this paper is determined by a non-local fractal operator in time-domain.^{28,29} Therefore, the resulting pulse dynamics obeys a fractional operator that depends, yet in a complex and non-analytical fashion, on the aperiodic refractive index modulation.

The fundamental solution of the FDWE can be obtained in closed-form using well-established Fourier-Laplace integral inversion methods (i.e., Mellin-Barnes integrals) and it can be expressed analytically in terms of the transcendental Wright functions or using the Fox H-functions.^{9,10} The reduced Green's function (see methods section), can be interpreted as a symmetric spatial PDF evolving in time with a stretched exponential relaxation, which provides the following expression for the MSD:⁹

$$\sigma^2 = \frac{2t^\beta}{\Gamma(\beta + 1)} \quad (4)$$

We notice that the transport processes described by the FDWE model are widely tunable since they can be modulated from anomalous sub-diffusion when $0 < \beta < 1$ to anomalous super-diffusion for $1 < \beta < 2$. However, this picture cannot properly describe Sinai logarithmic sub-diffusion. Fractional transport equations of distributed order are necessary for this purpose.¹⁸

The distributed-order time fractional diffusion equation can be obtained from the FDWE by integrating with respect to the order β the fractional time derivative as follows:

$$\int_0^1 p(\beta) D_*^\beta \phi(x, t) d\beta = D \partial_{xx} \phi(x, t) \quad (5)$$

where $p(\beta)$ is a non-negative dimensionless weight function subject to the normalization condition $\int_0^1 p(\beta) d\beta = 1$. When the weight is a delta function the distributed-order time fractional diffusion equation reduces to the FDWE as a special case.

Ultra-slow kinetic processes with tunable logarithmic MSD scaling $\langle x^2 \rangle \sim \log^\nu(t)$ are modelled by the asymptotic solution of the distributed-order diffusion equation with a power law weight function $p(\beta) = \nu \beta^{\nu-1}$ ($\nu > 0$) associated to a broad distribution of localization traps (i.e., localized modes). At the microscopic level, these kinetic equations originate from CTRWs processes with an extremely broad waiting time PDF, characteristic of strongly non-homogeneous environments. The case $\nu = 4$ yields the exact MSD scaling of the original Sinai model.¹⁸

1 Transport across localized photon states in periodic structures

In order to validate our method we first considered as a reference case the optical pulse transport through localized states in photonic coupled micro-cavities with structural defects embedded periodically along an otherwise regular photonic crystal structure. The defects consisted in additional layers of type A positioned along the periodic AB layer sequence. The investigated

system can be symbolically represented as $[(AB)_M A]_N (BA)_M$, where M and N indicates the number of repeated units inside the parenthesis. This photonic structure supports a number of resonant defect modes forming a comb within the fundamental Bragg gap. The number of localized states in the gap equals the number N of cavity defects introduced in the system, which are regularly spaced with respect to the central frequency ω_0 as shown by the calculated transmission spectrum shown in Figure 1(a) overlapped with the spectrum of the incident pulse. The transmission spectrum in Figure 1(a) contains a large number of regularly separated resonant states with very close spacings. However, even for large systems with several hundred defect layers, the resonant states overlap resulting in coherent photon tunneling (i.e., resonant tunneling) across the entire structure. This phenomenon is analogous to the well-known formation of transmission minibands for the electron transport in semiconductor superlattices. Therefore, despite the large number of resonant modes, their spectrum does not display a fractal structure and does not support fractal eigenstates. This is consistent with the general behavior of periodic 1D systems for which $D_2^H/D_2^Y = 1$. According to the scaling theory of electron transport in 1D, no anomalous diffusion is expected to take place under such condition and wave transport occurs ballistically. This qualitative picture is confirmed by our numerical simulations that are obtained by computing the spatial-temporal evolution of a Gaussian optical pulse transported across the structure (details in methods section). In Figure 1 (b) we show few representative snapshots of the pulse intensity that propagates inside the structure. The normally incident input pulse tunnels into the system and propagates ballistically while spreading in time until it

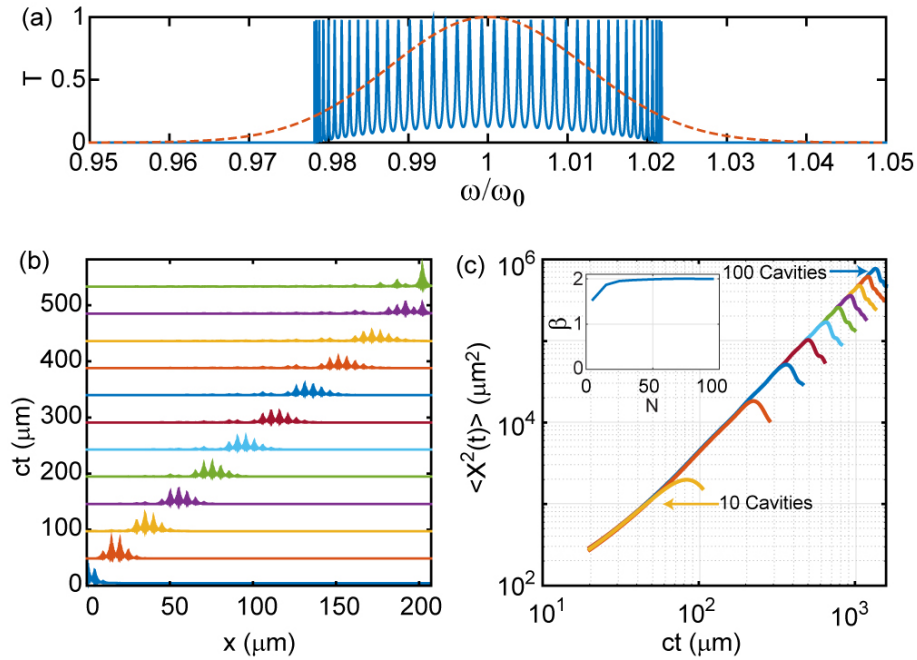


Figure 1. (a) Transmission spectrum of $N = 40$ cavities of $\lambda_0/2n_1$ thickness each $\lambda_0 = 1550nm$. Cavities are separated from each other by periodic Bragg reflectors made of 20 alternating layers of refractive indices $n_1 = 1.5$, $n_2 = 1.75$ and thickness $\lambda_0/4n_1$ and $\lambda_0/4n_2$, respectively. Dashed line represents the frequency spectrum of the incident optical pulse. (b) Intensity distribution ($|E|^2$) of an optical pulse propagating through the layers at different times. (c) Calculated MSD of the pulse in coupled cavity systems with varying cavities ranging from $N=10$ to $N=100$ (in steps of 10). (Inset) Calculated scaling exponent of the MSD as a function of N .

reaches the output face of the structure. At any spatial position in the sample, the envelope of the pulse remains Gaussian with a high-frequency spatial modulation that results from the reshaping due to the interference inside the structure.³⁰ The calculated MSD for pulses transported across coupled cavities with different numbers of defect states are displayed in Figure 1 (c). The MSD time traces feature a power-law scaling with a constant slope up to 100 coupled cavities, irrespective of the refractive index contrast. This behavior demonstrates coherent transport of the wave packet with an asymptotic scaling exponent $\beta = 2$, which clearly demonstrates ballistic character. The inset in Figure 1 (c) summarizes the results obtained for different numbers N of cavity states. Notice that the values of the transport exponent converge to the ballistic value $\beta = 2$ when the structures are sufficiently large to avoid spurious reflections from their end facets, which cause a drop in the MSD values. In all our simulations we exercised particular care to verify that the coupling of the pulse to the end facet of the structures is negligible, so that the values of the transport exponents (obtained by power-law fitting) do not depend neither on the launching conditions of the pulses nor on the length of the system. The ballistic nature of pulse transport in periodic coupled cavity systems is

further confirmed by the study of the ACF time decay, which was found to follow the power law $C(t) \sim t^{-\delta}$ with $\delta = 1$, irrespective of the index contrast.

2 Tunable anomalous photon transport in quasi-periodic structures

We will now address the optical pulse transport in quasi-periodic photonic media. In contrast to periodic structures, we show that 1D quasi-periodic photonic systems exhibit largely tunable anomalous transport as a function of the refractive index contrast between the constituent layers A and B .

As a representative case we consider a quasi-periodic Fibonacci photonic system where the scattering layers are arranged according to the well-known Fibonacci inflation rule:^{15,16} $A \rightarrow AB$ and $B \rightarrow A$ starting with A . The optical transmission spectrum of this structure is truncated at $N = 2048$ layers, and it is shown in Figure 2(a) overlapped with the spectrum of the input pulse. It is known that in the limit of $N \rightarrow \infty$ (where N is the number of layers) the highly fragmented spectrum of the Fibonacci multilayer converges to a self-similar Cantor set with zero Lebesgue measure and with fractal dimension $d_f = \ln 2 / \ln 3$, irrespective of the refractive index contrast.^{31,32} Moreover, differently from periodic systems, Fibonacci multilayers support distinctive fractal modes that are known as *critical states*, with fractal dimensions that strongly vary with frequency and with the strength of the scattering potential.³¹ This fact provides yet unexplored opportunities to control the transport exponent β in Fibonacci optical systems by varying their index contrast, which modulates the fractal dimensionality of the critical eigenmodes. We note that, despite the fact that exact fractality of the spectrum (at all frequencies) only occurs in

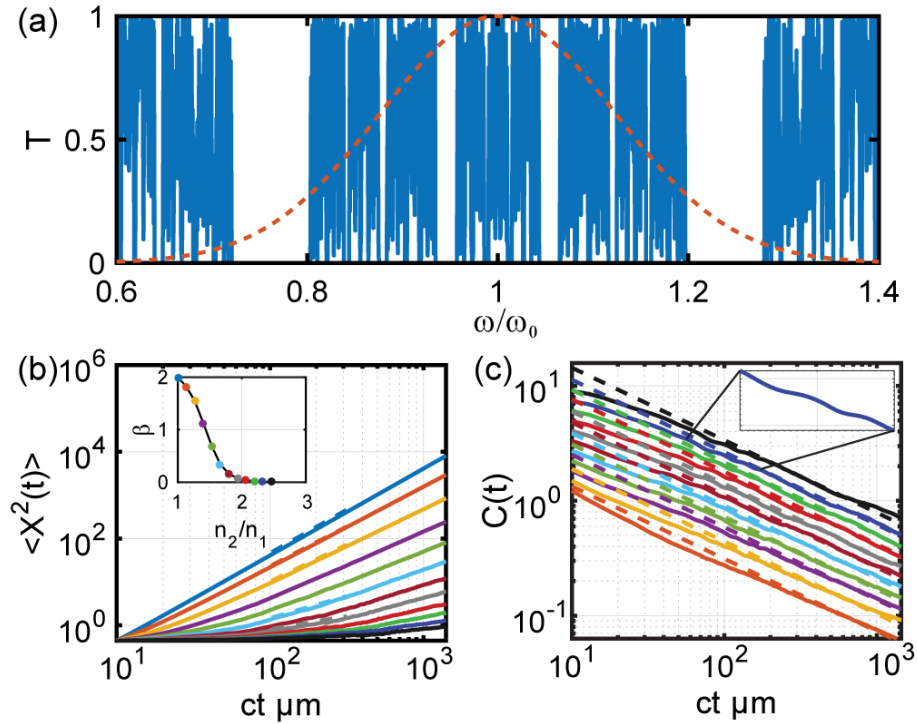


Figure 2. (a) Representative transmission spectrum of a Fibonacci multilayer with refractive indices $n_1 = 1.5$ and $n_2 = 1.9$ with layer thickness $\lambda_0/4n_1$ and $\lambda_0/4n_2$ respectively. Dashed line represents the frequency spectrum of the incident optical pulse with central wavelength at 1550nm. (b) Computed MSD values as a function of time and refractive index contrast between the constituent layers ($n_1 = 1.5$ in all cases), Inset shows the scaling exponent of the MSD obtained using numerical fitting of the data to power law. The quality of the fitting is demonstrated with representative dashed lines overlapping the data in main panel. (c) Computed ACF as function of time. Solid lines represent the numerical data and dashed lines are theoretical prediction (Lines are scaled vertically for better visualization). The inset shows a magnified portion of the initial-time ACF decay curve that support fractal log-periodic oscillations.

the limit of infinite layers, almost perfect self-similarity across broad spectral regions can be obtained even for relatively short Fibonacci structures with only few hundred layers. For our study we selected a layer number that gives rise to a constant fractal dimension across the portion of the frequency spectrum that overlaps with the envelope of the propagating pulse. Due to the fractality of the Fibonacci spectrum and the broad distribution of fractal dimensions of its critical eigenmodes,³³ we show that

it is possible to achieve anomalous transport with the exponent β controlled by the refractive index contrast. To demonstrate this property we perform numerical simulations of the MSD scaling, which are shown in Figure 2 (b) for different values of the refractive index contrast. We notice that the slopes of the MSD curves, plotted on a double logarithmic scale, strongly depend on the choice of the refractive index contrast. The strong dependence of the transport exponents β on the values of the refractive index contrast is summarized in the inset of Figure 2 (b). Our data clearly demonstrate that the transport properties of optical Fibonacci systems can be tailored from super-diffusion ($\beta > 1$) to sub-diffusion ($\beta < 1$) depending on the refractive index contrast, in close analogy with the tunable quantum dynamics of quasi-periodic electronic systems, which was qualitatively explained using renormalization group arguments.²¹ It should be emphasized that the observed tunability of photon transport is not associated with the excitation of a single critical mode but rather with the distribution of fractal eigenstates that overlap with the spectrum of the incident pulse.

The anomalous photon transport behavior demonstrated by the Fibonacci structure can be regarded as an instance of fractional photon transport, which is largely tunable as predicted by Equation (4). As we can appreciate in Figure 2, our numerical results on the MSD time scaling can be accurately reproduced by the asymptotic solutions of the analytical FDWE model, that are represented by the dashed lines. An effective FDWE approach has also been recently introduced to describe the multiple scattering of acoustic waves in one-dimensional multiscale media with long-range spatial correlations.³⁴ In this context, the FDWE associated to a lossless inhomogeneous (non-ergodic) random media describes an effective medium with power-law frequency dependent attenuation coefficient.³⁴ While the fractional wave transport approach is well-established in other domains such as viscoelasticity and seismic wave propagation,³⁵ it has not been systematically explored in optics. From a physical standpoint, the multiple scattering of optical waves in 1D media with long-range aperiodic order, such as Fibonacci quasi-crystals, establishes a fractal memory kernel in the field equation that governs the dynamics of the electric field. The asymptotic analysis of the MSD and ACF scaling of photonic systems with fractal spectra is detailed in our methods section. Note that alternative transport approaches based on CTRW with coupled waiting time and jump length PDFs have also been considered in the literature.^{36,37} Fractional transport equations with coupled space-time memory kernels are discussed further in references.^{38–40}

The fractal nature of the Fibonacci transport is directly revealed by the ACF decay, displayed in Figure 2(c) for systems with varying refractive index contrast. All the ACF curves exhibit an inverse power law scaling with constant slope. The decay exponent δ is found to be independent on the refractive index contrast, in agreement with the analytical scaling law $C(t) \sim t^{-\delta}$ previously introduced for electronic systems. We further validated our numerical results by computing the fractal dimension of the Fibonacci transmission spectrum using the accurate wavelet transform modulus maxima technique.^{41,42} In this case the exact frequency positions of the Fibonacci transmission resonances were determined using the efficient trace map method.⁴³ This analysis confirms that the calculated fractal dimension ($d_f = \ln 2 / \ln 3$) of the Fibonacci spectrum equals the exponent δ that we also derived from the power-law fitting of the ACF decay. The agreement between these two approaches to determine δ was excellent across a large range of refractive index contrast, which unambiguously demonstrates the anomalous nature of photon transport in 1D Fibonacci structures.

Finally we plot in the inset of Figure 2 (c) a magnified portion of a representative ACF decay that displays initial-time log-periodic oscillations in close analogy to the case of electronic systems.²¹ Such oscillations have been recently explained based on the analytic theory of spectral zeta functions on fractals.^{26,27} Physically, they can be interpreted as the manifestation of resonant scattering phenomena between neighboring lattice clusters that share similar local geometrical structures in fractal environments. Indeed, it is well-known that the number of different neighbourhoods for excitations (i.e., electrons, or photons) to tunnel along a 1D Fibonacci structure is restricted to only four situations with four different local eigenmodes. The existence of a hierarchy of resonance oscillations with different time scales in Fibonacci systems follows from the Conway's theorem for quasi-periodic structures. This important result states that if one selects a cluster of length L within a Fibonacci sequence, then it is possible to find an identical copy of the selected cluster within a distance of the same order L . Therefore, resonant oscillations develop at every length scale in a quasi-periodic Fibonacci structure. This intriguing phenomenon was also discussed in the context of the electron transport across Fibonacci chains using the renormalization group approach.²²

3 Tunable logarithmic photon transport in pseudo-random structures

We will now address Sinai logarithmic sub-diffusion of optical wave packets in Rudin-Saphiro (RS) photonic structures and in binary pseudo-random media. Currently, no engineered optical structures are known to display logarithmic Sinai sub-diffusion. In this paper we will show that deterministic aperiodic Rudin-Shapiro and pseudo-random binary dielectric multilayers provide such an intriguing opportunity. Differently from quasi-periodic systems, these structures are characterized by absolutely continuous spatial Fourier spectra and by singular (i.e., delta-like) optical transmission spectra, as in the case of disordered random media. However, such structures are deterministic and possess perfect long-range order with multi-fractal correlations.⁴⁴ RS multilayers are generated from a two-letter alphabet subject to the simple inflation rule: $AA \rightarrow AAAB$, $BB \rightarrow BBBA$, $AB \rightarrow AABA$, $BA \rightarrow BBAB$, starting from the initial seed AA .¹⁶ The transmission spectrum of the RS photonic

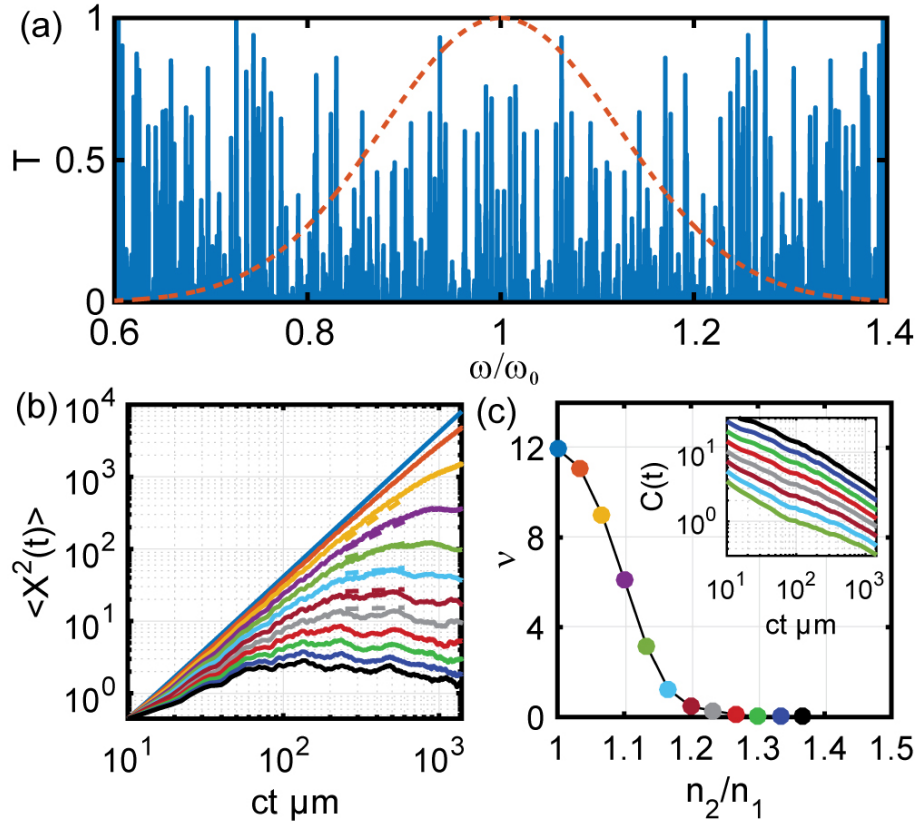


Figure 3. (a) Representative transmission spectrum of an RS multilayer with $n_1 = 1.5$ and $n_2 = 1.75$. The dashed line represents the frequency spectrum of the incident optical pulse at 1550nm. (b) Calculated MSD scaling curves for different values of the refractive index contrast between the constituent layers. (c) Calculated scaling exponent of the MSD curves shown in panel (b) using the logarithmic Sinai model. The quality of the fitting is demonstrated with representative dashed lines overlapping the data in panel (b). Inset in (c) is calculated ACF for few representative refractive index contrasts. The lines are scaled vertically for better visualization.

structure ($N=2048$ layers) shown in Figure 3 (a) is characterized by a singular distribution of narrow resonant peaks. Figure 3 (b) summarizes our results for the MSD of optical wave packets propagating through the systems. The data are plotted in a double logarithmic scale and demonstrate a clear logarithmic scaling behavior, which remarkably contrasts the case of Fibonacci quasi-periodic systems. The dashed lines are best fits obtained according to the general Sinai model $\langle x^2 \rangle \sim \log^\nu(t)$, which can be rigorously derived from the asymptotic solution of the distributed-order fractional diffusion equation.¹⁸ Notice that also in the case of the RS structure distinctive oscillations are present in the MSD curves, which reflect the correlated fractal nature of the aperiodic order. We additionally found that the Sinai transport exponent ν can largely be controlled by the refractive index contrast. In particular the original Sinai model is recovered for $\nu = 4$. According to the microscopic random walk theory of Sinai transport the parameter ν effectively describes the degree of spatial non-homogeneity of the transition rates or scattering potentials, which can be tailored over a large range for photon scattering. The values of ν as a function of the refractive index contrast are plotted in Figure 3 (c). In the inset we show the calculated ACF decay curves obtained for different values of refractive index contrast, vertically translated for better visualization. Our data demonstrate an inverse power-law scaling of the ACF decay of the RS structure, which is the hallmark of fractal spectra. However, differently from the more regular quasi-periodic Fibonacci case, whose spectrum is mono-fractal, the ACF decay of RS systems features a weaker oscillatory behavior that can be understood based on their non-homogeneous multi-fractal nature with broader singularity spectra.⁴⁴

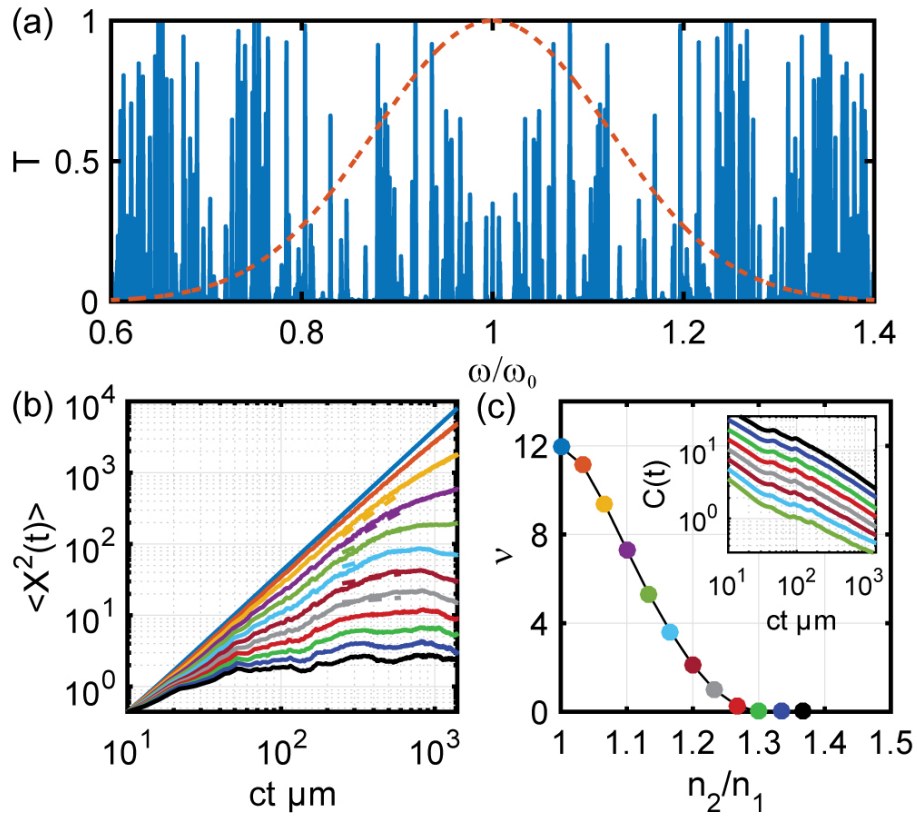


Figure 4. (a) Representative transmission spectrum of an pseudo-random multilayer with $n_1 = 1.5$ and $n_2 = 1.75$. The dashed line represents the frequency spectrum of the incident optical pulse at 1550nm. (b) Calculated MSD scaling curves for different values of the refractive index contrast between the constituent layers. (c) Calculated scaling exponent of the MSD curves shown in panel (b) using the logarithmic Sinai model. The quality of the fitting is demonstrated with representative dashed lines overlapping the data in panel (b). Inset in (c) is calculated ACF for few representative refractive index contrasts. The lines are scaled vertically for better visualization.

We also observed logarithmic photon sub-diffusion in binary pseudo-random structures generated using a random number generator, as shown in Figure 4. The transmission spectrum for one realization of this pseudo-random system is displayed in Figure 4 (a), which has a singular nature in the frequency range spanned by input Gaussian pulse. We found that the MSD grows logarithmically with time and that Sinai sub-diffusion can be controlled by again properly choosing the refractive index contrast. The numerically calculated MSD curves along with the fitting curves from the model $\langle x^2 \rangle \sim \log^\nu(t)$ are shown

in Figure 4(b), while the values of the scaling exponent ν are displayed as a function of refractive index contrast in Figure 4 (c). The corresponding ACF decay curves are summarized in the inset and translated vertically for better visualization. Initial-time oscillations develop in the ACF decay traces as well as in the MSD curves. Such features reveal the deterministic pseudo-random nature of the structure that establishes multi-scale spatial correlations, which will be absent in truly random, uncorrelated and homogeneous structures. Our extensive numerical simulation of pseudo-random structures suggests that Sinai logarithmic sub-diffusion of light could be a generic attribute of wave propagation in multi-fractal or multi-scale aperiodic media. It is interesting to realize that tunable sub-diffusion such as the logarithmic Sinai process is characterized by an effective diffusion constant that vanishes asymptotically, i.e., $D = \lim_{t \rightarrow \infty} \frac{\langle x^2(t) \rangle}{t}$. Therefore, the photon sub-diffusion regime achieves a localization mechanism that fundamentally differs from the traditional Anderson light localization picture that relies on interference and multiple scattering in truly random media. From our analysis it is apparent that complex aperiodic photonic structures with multi-fractal spatial correlations give rise to logarithmic Sinai sub-diffusion of light, even for very small values of the refractive index contrast.

An important aspect is related to the role of interference effects in Sinai sub-diffusion, since the term can be misleading when considering 1D photonic systems where interference effects cannot be neglected. In order to address this issue we show in Figure 5 the numerically and analytically estimated localization length and photon mean free path evaluated for a pseudo-random and a RS structure as a function of the refractive index contrast. These parameters are computed using analytical results for 1D random structures, as detailed in the methods section. The data in Figure 5 demonstrate excellent agreement between the analytical theory valid for 1D random structures^{47,48} and the numerical estimates obtained for pseudo-random media. In both cases, our results clearly indicate that the scattering mean free path and the localization length are significantly smaller than the total length L of the samples, at least for the investigated range of refractive index contrast. This implies that logarithmic photon transport in the investigated 1D pseudo-random systems originates from spatial correlations in wave interference, which already develops for small values of refractive index contrast (this is possible because there is no threshold for wave localization in sufficiently disordered large-scale 1D systems). Our discussion makes it clear that the FDWE modelling of anomalous diffusion of optical waves must be considered as an effective analytical description that accounts for the complex physics of multiple wave interference in 1D scattering systems with largely tunable aperiodic spatial correlations.

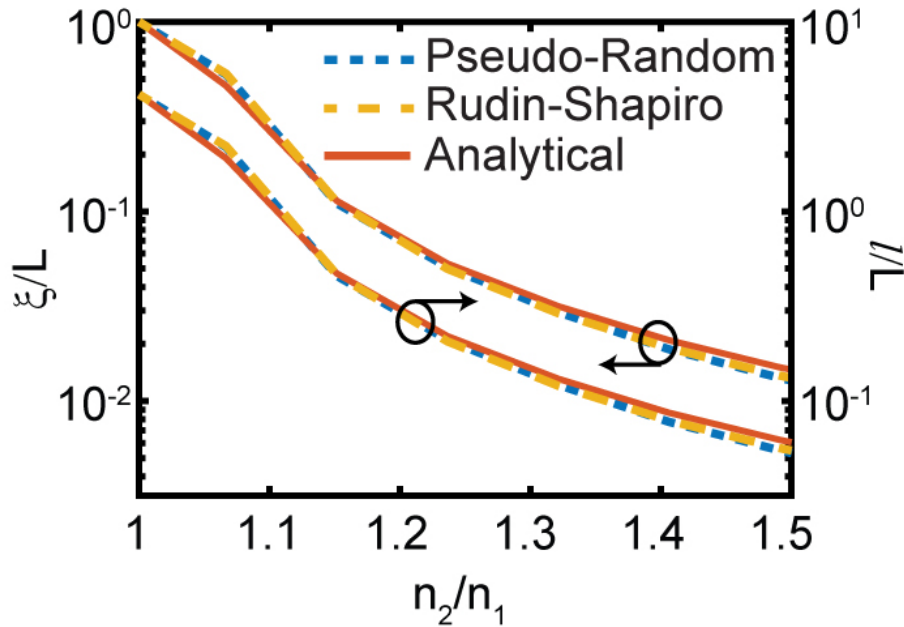


Figure 5. Localization length of the RS and pseudo-Random samples as a function of refractive index contrast. Dashed lines represent localization length normalized to the sample length. Solid lines represent the same calculated using the analytical formula derived for random systems.⁴⁸

4 Conclusions

In conclusion, we numerically investigated the propagation of optical wave packets through periodic, quasi-periodic and pseudo-random 1D photonic multilayer structures and we demonstrated tunable anomalous photon transport, including the logarithmic Siani sub-diffusion for the first time. In particular we numerically and analytically investigated optical pulse propagation through dielectric structures and demonstrated anomalous transport largely tunable from sub- to super-diffusion in complex media with long-range aperiodic order. Excellent agreement with the scaling theory of anomalous transport has been obtained. The connection with the asymptotic solutions of non-Markovian fractional transport equations of single and distributed order has been established and provides insights into the complex behavior of multiple light scattering in non-homogeneous aperiodic media with tunable structural correlations. Our results are first steps towards the control of the fractional photon dynamics in optical structures with aperiodic order. The engineering of tunable anomalous photon transport including logarithmic sub-diffusion in artificial optical materials provides novel degrees of freedom to tailor light-matter interaction and localization in photonic systems with designed transport properties.

5 Methods

The calculation of the electromagnetic field through the multilayer structure has been performed using scattering matrix method. An electromagnetic pulse in time-domain is obtained by Fourier synthesis of different frequency components. The central frequency of the pulse is considered to be at $\omega_0 = 2\pi c/\lambda_0$, where $\lambda_0 = 1550\text{nm}$. The incident field ($E_i(x=0)$) is assumed with a gaussian spectral distribution expressed as: $E_i(x=0, \omega) = E_0 \exp(-a^2(\omega - \omega_0)^2)$, with $E_0 = 1$ and $a = 33\text{fs}$. The propagation of waves occurs normally to the layers (i.e., along x axis) and the layers are in the yz -plane. The electric field is polarized along y axis. Under these conditions, the electric field at a given position x within the sample can be expressed as:

$$E(x, t) = \int_0^\infty E_i(0, \omega) e^{-i\omega t} \psi(x, \omega) d\omega \quad (6)$$

where $\psi(x, \omega)$ is the so-called *scattering map*,⁴⁵ which provides the internal field amplitude of the scattering resonances at each position in the sample and at every frequency.

Fractal scaling analysis: for our purposes it is essential to realize that, for large enough samples, the kernel function $\psi(x, \omega)$ of aperiodic structures with fractal or, more generally, multi-fractal spectra defines a self-affine two-dimensional function of space and frequency. A self-affine function describes non-homogeneous fractals with different scale-invariant symmetry in the x - and ω -directions. As we can observe in Figure (6) for a large sample length the $\psi(x, \omega)$ can be well approximated by the fractal transmission spectrum of the corresponding aperiodic structure. Additionally, for a fixed value of ω , the function $\psi(x, \omega)$ yields the spatial profile of the corresponding critical mode of the aperiodic structure. Since critical modes are fractal functions with tunable scaling exponents (i.e., fractal dimensions) that depend on the choice of the frequency ω , the function $\psi(x, \omega)$ is asymptotically self-affine with scale-invariance symmetry: $\psi(ax, b\omega) = a^\alpha b^\nu \psi(x, \omega)$ for α and ν positive real numbers. This functional relation is satisfied by the asymptotic power-law scaling function $\psi(x, \omega) \sim C x^{\alpha(\omega)} \omega^\nu$, where C is a constant and $\alpha(\omega)$ and ν are the fractal dimensions of the resonant modes at frequency ω and of the transmission spectrum, respectively.

For an arbitrary layered structure, $\psi(x, \omega)$ can be expressed in closed form as:

$$\psi(x, \omega) = \left[\prod_{m=1}^{j-1} t_m(\omega) \right] \left[e^{ikn_j(x-x_{j-1})} + r_j(\omega) e^{-ikn_j(x-x_j)} \right] = \Phi(\omega; x) \chi(x, \omega) \quad (7)$$

where $x_{j-1} < x < x_j$ are inside the j^{th} layer in the structure, k is the free space wavenumber and n_j represents the refractive index of j^{th} layer. The complex internal reflection and transmission functions $r_j(\omega)$ and $t_j(\omega)$ are obtained iteratively for each layer using the recursive relations:

$$r_j(\omega) = \frac{(n_{j+1} - n_j)\phi_j + 2n_j\phi_{j+1}r_{j+1}(\omega)t_{j+1}(\omega)}{n_{j+1} + n_j} \quad (8)$$

$$t_j(\omega) = \frac{2n_j\phi_{j-1}}{(n_{j-1} + n_j) + (n_{j-1} - n_j)\phi_j r_j(\omega)} \quad (9)$$

assuming $r_N = 0$ and $t_N = 1$, where N represents the total number of layers and $\phi_j = \exp(ikn_j(x_j - x_{j-1}))$ is the propagation phase inside each layer. Note that equations (8) and (9) represent a compact form of scattering matrix equations for waves

propagating at normal incidence to the layers.⁴⁶ The product that appears in the first square bracket in equation (7) provides the net transmission of forward propagating waves in the structure up to the j^{th} interface. The first exponential term in the second square bracket represents the propagation phase up to coordinate x inside the layer and the quantity $r_j(\omega)$ in the second term gives the net reflection of the waves into j^{th} layer that is determined by the other side of the sample (i.e., from layer $j+1$ to layer N). The exponential term is the propagation phase up to position x inside the layer for waves that are propagating in the backward direction. Representative examples of the scattering map $\psi(x, \omega)$ for the investigated aperiodic photonic structures are shown in Figure (7).

By inverse Fourier transforming Equation (7) we can express the scattering map in the space-time domain as a non-local convolution of the form:

$$\psi(x, t) = \tilde{\Phi}(t; x) \otimes \tilde{\chi}(x, t) \quad (10)$$

where $\tilde{\Phi}(t; x)$ and $\tilde{\chi}(x, t)$ are the inverse Fourier transforms of the functions $\Phi(\omega; x)$ and $\chi(x, \omega)$ defined in Equation (7).

Despite the non-analytic and complex character of the functions involved, equation (10) clearly reveals the long-range correlated (fractional) nature of the optical transport when considering aperiodic fractal structures supporting singular continuous spectra. This connection can be appreciated by realizing that fractal spectra display self-similarity, which can be regarded as a symmetry under a discrete scaling transformation in frequency space.⁴⁹ In particular, it has been shown that the counting function of a fractal spectrum centered around the central frequency ω_0 follows the power-law scaling relation: $\Gamma_{\omega_0}(\omega) = |\omega - \omega_0|^\nu G(\ln|\omega - \omega_0|/\gamma)$ where G is a log-periodic function, ν and γ are scaling constants that depend on the choice of the central frequency.⁴⁹

At this point we notice that both the $r_j(\omega)$ and the product of the $t_j(\omega)$ are self-similar functions exhibiting a power-law scaling behavior around the central frequency ω_0 of the spectrum, as discussed above and vividly illustrated by the data in Figure (6). Since *power law functions are invariant under both Fourier transformation and convolution operations*, and since $\psi(x, \omega) \sim Cx^{\alpha(\omega)}\omega^\beta$, it follows that also the scattering map expressed in the space and time-domain $\psi(x, t)$ is a self-affine function, asymptotically. This means that $\psi(x, t)$ exhibits asymptotic (i.e., large time and large sample size limit) power law behavior versus space and time. Representative examples of the scattering map $\psi(x, t)$ for the investigated aperiodic photonic structures are shown in Figure (8).

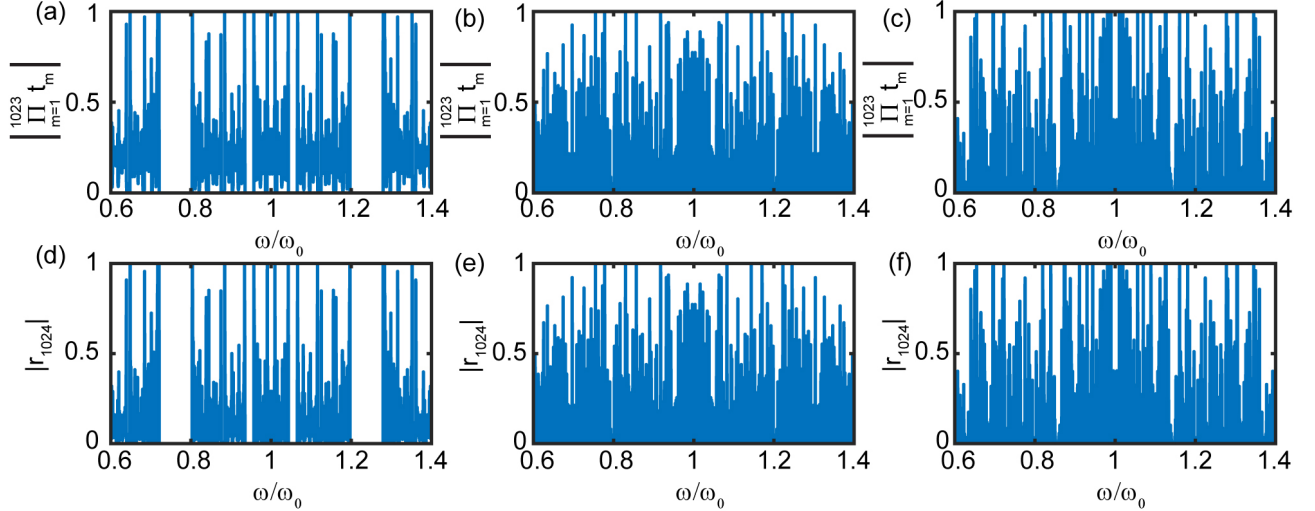


Figure 6. Net transmission (a,b,c) and reflection (d,e,f) spectrum defined using equations (9) and (8) respectively in the middle layer of the samples investigated in the main text. Each structure has 2048 layers and the panels display the net reflection and transmission values at the 1024^{th} layer. (a),(d) correspond to Fibonacci multilayers with $n_1 = 1.5$ and $n_2 = 1.9$. (b), (e) correspond to Rudin-Shapiro multilayers with $n_1 = 1.5$ and $n_2 = 1.75$ and (c), (f) correspond to the pseudo-random multilayers with $n_1 = 1.5$ and $n_2 = 1.75$

The above scaling analysis is also sufficient to deduce that both the ACF and the MSD of photonic structures with fractal spectra exhibit power-law scaling asymptotically. For this purpose we just need to remember that any causal power function of the form $\Phi_\mu(t) = t^{\mu-1}/\Gamma(\mu)$ transforms into $\Phi_{\mu+1}(t)$ when integrated between zero and t . Since the ACF definition involves the time integral of the scattering map, it follows that asymptotically the ACF also scales with a power law with exponent, as obtained in our numerical simulations. A similar argument explains the power-law scaling behavior of the MSD as well. In fact,

the time scaling behavior of the MSD results from the integration, at any time instant, of the scattering map $\psi(x, t)$ with respect to its spatial coordinate. Therefore, the long-time behavior of the MSD involves a spatial integral over the sample length of the scattering map in the form: $\psi(x, t) \sim C' x^{\alpha'(\omega)} t^{\beta'}$. However, differently from the ACF case the scaling behavior of the MSD is not a pure spectral property of the fractal dimension of the spectrum. In fact, the exponent $\alpha'(\omega)$ in the integrand shows that the MSD is additionally influenced by the fractal spectrum of the excited resonant modes of the structure. This important aspect is at the origin of the wide tunability of the transport mechanism that we discussed in the main text. However, no analytical expression captures this complex interplay since $\alpha'(\omega)$ depends additionally on the excitation conditions. However, our scaling analysis has fully justified the tunability of the anomalous transport that we have investigated numerically in deterministic aperiodic structures with fractal spectra.

Connection with fractional dynamics: we have proven in the previous section that both the space-frequency $\psi(x, \omega)$ and the space-time $\psi(x, t)$ scattering maps of fractal structures are self-affine fractal objects that can be asymptotically obtained by a convolution integral with power-law kernels. This feature is the hallmark of fractional calculus since fractional operators are defined by convolution integrals with non-local power-law kernel functions.¹² In particular, the fractional (non-local) character of the time-dynamics of optical transport across fractal structures can be exemplified by the scattering maps $\psi(x, t)$ plotted in Figure 7. The non-Markovian (i.e., correlated) nature of the pulse dynamics is clearly evidenced by the fractal distributions of internal pulse reflections that characterize aperiodic systems (Figure 7b-d) compared to the more coherent transport across periodic defect states (Figure 7a).

The anomalous nature of transport demonstrated in the aperiodic systems reflects the complex multiple scattering events that give rise to complex fractal patterns of internal reflections due to the fractal distribution of the dielectric scattering interfaces.

Finally, it is interesting to remark that anomalous transport in multi-interface structures can also be modeled as a diffusion process in comb-like structures with non-homogeneous distributions of teeth lengths. Diffusion in these complex media is described microscopically by a CTRW approach for a given statistical distribution of teeth lengths in the comb, while macroscopically it is modeled by fractional diffusion-wave equations. Considering combs with power-law distributions, which describe broad distributions of trapping states, it is also possible to describe these systems within the framework of anomalous sub-diffusion, including Sinai logarithmic transport.⁵⁰

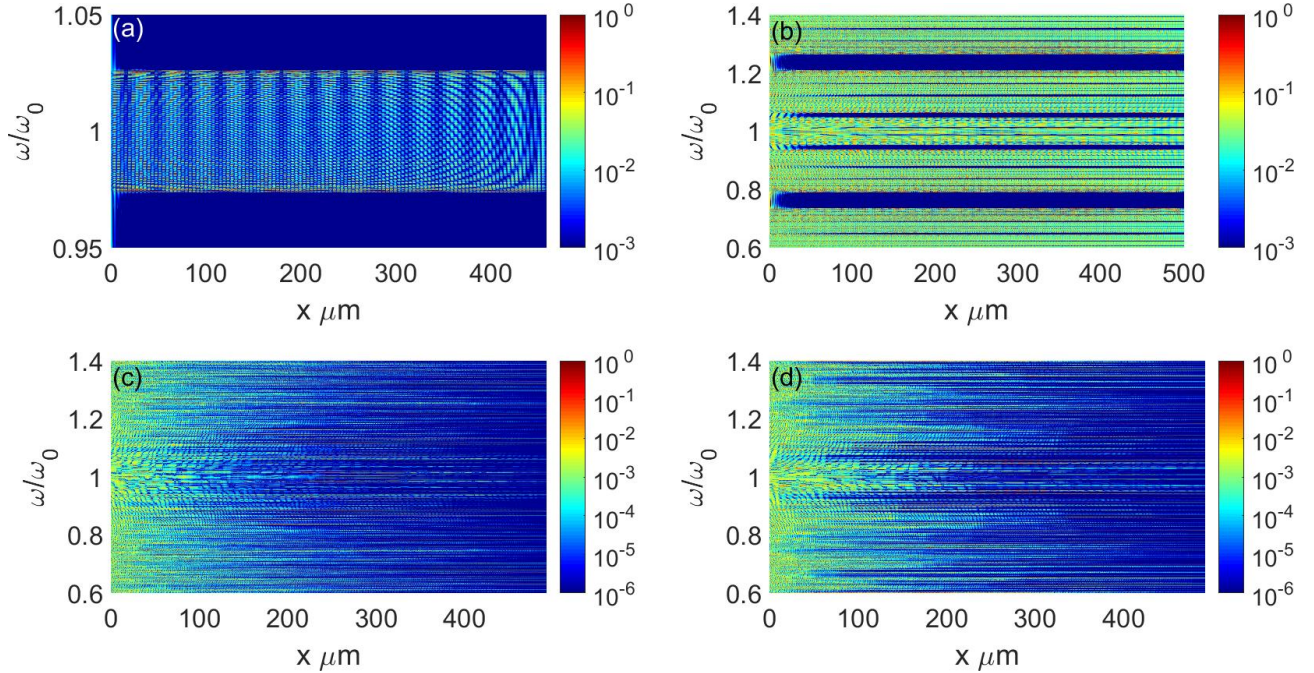


Figure 7. Scattering Map of field intensities in different sequence of layers. (a)Periodic Micro cavities, (b) Fibonacci, (c) Rudin-Shapiro and (d)Pseudo-Random. All the structures has 2048 layers with refractive index components $n_1 = 1.5$ and $n_2 = 1.75$. Color represents $|\psi(x, \omega)|^2$ calculated using equation (7).

This approach evidently captures many aspects of the multiple scattering in optical multi-layered systems with power-law (i.e., fractal) distributions of critically localized modes, each described by a different photon trapping time $\tau = Q/\omega$, where Q is the resonance quality factor of the mode at (angular) frequency ω . The fundamental connection between diffusion in disordered systems with fractal transition rates and anomalous transport of photons can provide a powerful approach to model

complex optical transport phenomena in non-homogeneous aperiodic media.

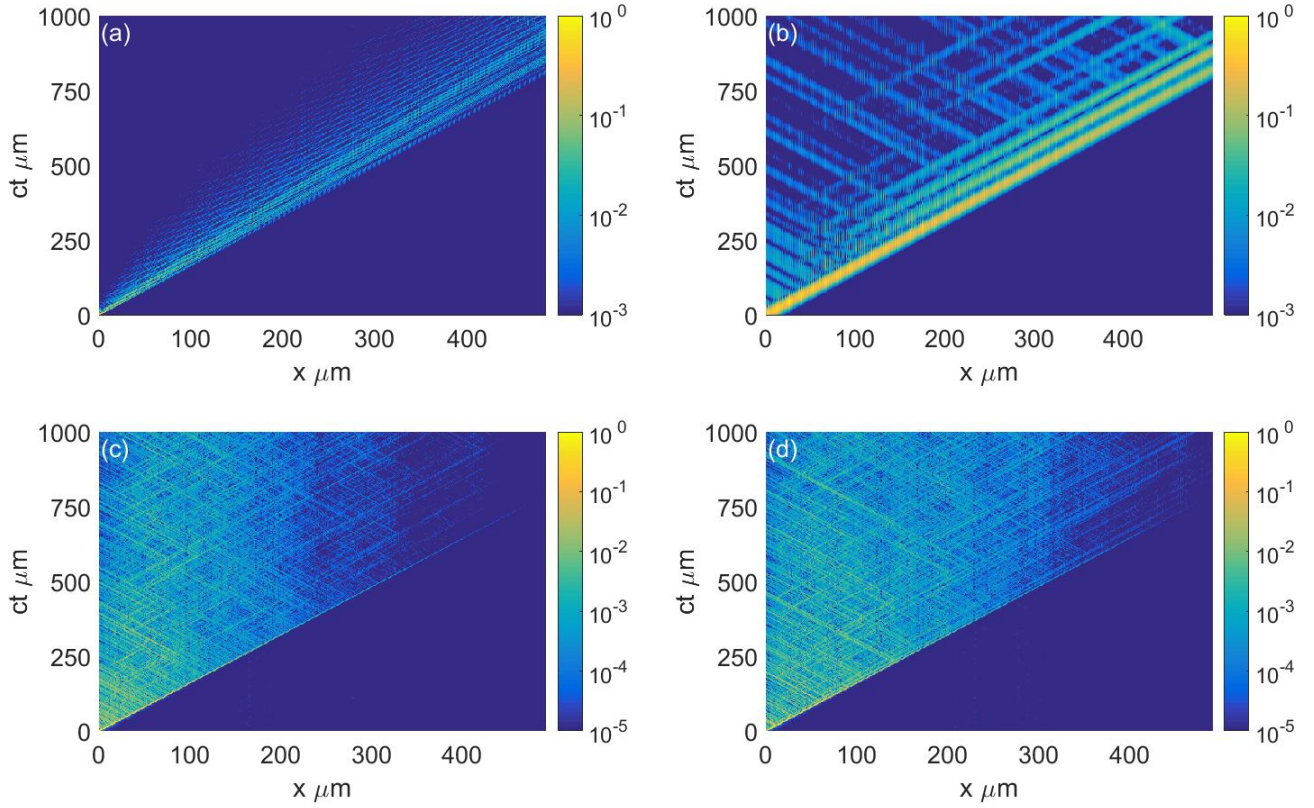


Figure 8. Time evolution of optical pulse in different sequence of layers. (a) Periodic Micro cavities, (b) Fibonacci, (c) Rudin-Shapiro and (d) Pseudo-Random. All the structures has 2048 layers with refractive index components $n_1 = 1.5$ and $n_2 = 1.75$. Color represents $|E(x,t)|^2$ calculated using equation (6).

Solutions of the FDWE: the fundamental solutions (i.e., Green's functions) corresponding to the Cauchy problems of the FWDE in Equation (3) are denoted by $G_\beta^{(j)}(x,t)$ ($j = 1, 2$) and can be analytically obtained.¹⁰ In particular, by introducing the similarity variable $x/t^{\beta/2}$ we can express the two Green's functions in terms of the one-variable *reduced Green's functions* $K_\beta^{(j)}(x)$ ($j = 1, 2$) as follows:

$$G_\beta^{(1)}(x,t) = t^{-\beta/2} K_\beta^{(1)}(x/t^{\beta/2}) \quad (11)$$

$$G_\beta^{(2)}(x,t) = t^{-\beta/2+1} K_\beta^{(2)}(x/t^{\beta/2}) \quad (12)$$

A convenient expression for the Green's functions as convergent power series has been recently derived:¹⁰

$$K_\beta^{(j)}(x) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-x)^n}{n! \Gamma[-\beta n/2 + (j - \beta/2)]} \quad j=1,2 \quad (13)$$

Both the Green functions above have been shown to be non-negative and normalized, so they can be interpreted as probability density functions.

The localization length of pseudo-random and RS structures is calculated using the ensemble-averaged transmission over the frequency spectrum and it is defined by:⁴⁷ $\xi/L = -1 / \langle \ln T \rangle$, where T is the transmission coefficient of the sample and L is the total length of the sample. To calculate the localization length in 1D random multilayers we averaged over the frequency spectrum of the optical pulse the analytical expression provided in:⁴⁸

$$\xi(\lambda) = 2(d_A + d_B) \left\{ \ln \left[(3n_A^2 + n_B^2) (n_A^2 + 3n_B^2) + \frac{3(n_A^2 - n_B^2)^2}{4 \cos(\pi \lambda_0 / \lambda) - 5} \right] - 2 \ln(4n_A n_B) \right\}^{-1} \quad (14)$$

The scattering mean free path l is estimated using the approximate expression valid for 1D random system:

$$\xi \simeq (1 + \pi)l \quad (15)$$

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Author contributions statement

LDN conceived the research activities, SI and LDN performed complementary simulations and analysed all the results. All authors reviewed the manuscript.

Additional information

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