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REMARKS ON A KNOWN EXAMPLE OF A MONOTONE CONTINUOUS FUNCTION

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In this note we are concerned with a well known example of a continuous monotone function. We have collected together a few properties of this function which is very well fitted for illustration of many important points of the theory of functions of a real variable. Some of these properties have been mentioned several times in the literature, some others, however, simple as they are, appear not to have been stated explicitly.

To simplify our formulas we shall consistently use the binary and ternary scales of notation. Thus ${}_3.101$ will mean $1/3 + 1/27 = 10/27$, while ${}_2.101 = 1/2 + 1/8 = 5/8$.

To define our function¹ we first construct a perfect set of points nowhere dense on the interval $(0, 1)$: Subdivide $(0, 1)$ into three equal parts and remove the interior of the middle part (1-st stage of the process); subdivide each of the remaining two parts into three equal parts and remove the interiors of the middle parts of each of them (2-nd stage) and repeat this process indefinitely (the p -th repetition will be called the p -th stage of the process).

It is seen at once that the number of intervals removed at the p -th stage is 2^{p-1} . We denote them (ordered from left to right) by

$$(1) \quad \delta_{pk} \quad (k = 1, 2, \dots, 2^{p-1}).$$

If we denote the length of the interval δ_{pk} by the same letter, then

$$(2) \quad \delta_{pk} = 3^{-p}.$$

With this notation we have

$$\delta_{11} = ({}_3.1, {}_3.2), \quad \delta_{21} = ({}_3.01, {}_3.02), \quad \delta_{22} = ({}_3.21, {}_3.22), \dots$$

The total number of the intervals δ_{pk} removed during the p first stages will be $1 + 2 + \dots + 2^{p-1} = 2^p - 1$.

Let E be the set of points of $(0, 1)$ which will not be removed. Then the complementary set $D = C(E)$ coincides with $\sum \delta_{pk}$ (where only the interior point of the intervals δ_{pk} are taken into account). The set E consists of all the end-points of the intervals δ_{pk} and of their limiting points. It is readily seen that E is identical with the set of points which are represented by infinite fractions

$$(3) \quad {}_3.a_1 a_2 a_3 \dots a_n \dots,$$

where only the digits 0 and 2 are admitted. Furthermore, the end-points of δ_{pk} are represented by the fractions (3) where all the digits after a certain place are all zeros or all two's, while the limiting points of the end-points will have infinitely many zeros and two's, except for the two extreme points

¹ Hobson, *The theory of functions of a real variable*, vol. 1, 3rd edition, 1927, pp. 123, 368. This is referred to as H in the sequel.

$$0 = {}_3.000 \dots \quad \text{and} \quad 1 = {}_3.222 \dots$$

For instance the end-points of the interval δ_{34} are

$${}_3.221 = {}_3.220222 \dots \quad \text{and} \quad {}_3.222 = {}_3.222000 \dots$$

Simultaneously with the intervals δ_{pk} we shall consider the intervals

$$(4) \quad \eta_{pk} \quad (k = 1, 2, \dots, 2^p)$$

which remain at the p -th stage. We assume the η_{pk} to be closed (while the δ_{pk} are open). The set E is always covered (in the large sense) by the intervals η_{pk} . All η_{pk} (for fixed p) are of the same length 3^{-p} and the sum of their lengths is $(2/3)^p$. Since this $\rightarrow 0$ as $p \rightarrow \infty$, the set E is of measure 0 (and even of Jordan content 0).¹

Since all the numbers of the type (3) can be approximated as closely as we please by numbers of the same type, the set E contains all its limiting points, and also, each point of E is a limiting point, which shows that E is perfect. On the other hand each subinterval of $(0, 1)$, no matter how small, contains parts which are free from points of E , whence E is nowhere dense on $(0, 1)$.

We proceed now to the definition of our function $\omega(x)$. We agree once for all to use the letter a to indicate the digits 0 or 2 and to designate by b the number $a/2$, so that b assumes only the values 0 and 1. If $x = {}_3.a_1a_2a_3 \dots a_n \dots$ is a point of the set E , we define

$$(5) \quad \omega(x) = {}_2.b_1b_2b_3 \dots b_n \dots$$

According to this definition $\omega(x)$ has equal values

$$(6) \quad \begin{aligned} \omega_{pk} &= \omega({}_3.a_1a_2 \dots a_m 0222 \dots) = {}_2.b_1b_2 \dots b_m 0111 \dots \\ &= {}_2.b_1b_2 \dots b_m 1000 \dots = \omega({}_3.a_1a_2 \dots a_m 2000 \dots) = (2k - 1)/2^p \end{aligned}$$

at the end-points of each interval δ_{pk} and we take this value as the value of $\omega(x)$ at all the points of the corresponding δ_{pk} , with the result that the intervals δ_{pk} are intervals of constancy of $\omega(x)$. Now the function $\omega(x)$ is defined at all the points of $(0, 1)$ and we may proceed to the enumeration of the properties of $\omega(x)$.

i. $\omega(x)$ is monotone (non-decreasing) on $(0, 1)$ and increases from 0 to 1 as x increases from 0 to 1. The intervals δ_{pk} are intervals of constancy of $\omega(x)$.

Proof: In proving the inequality

$$\omega(x'') \geq \omega(x') \quad \text{if} \quad x'' > x',$$

we may restrict ourself to the points of E , since $\omega(x)$ is constant on each δ_{pk} . Let

$$x' = {}_3.a'_1 a'_2 \dots, \quad x'' = {}_3.a''_1 a''_2 \dots$$

¹ H. p. 171. It follows then that $\text{meas. } D = 1$, which can be proved also by an immediate computation: $\text{meas. } D = \sum \delta_{pk} = 1/3 + 2/9 + \dots + 2^{p-1}/3^p + \dots = 1/3 \cdot 1/(1 - 2/3) = 1$.

If $x'' > x'$, there will be a value of the subscript n for which

$$a_1' = a_1'', \dots, a_{n-1}' = a_{n-1}'' \text{ but } a_n'' > a_n',$$

whence

$$\omega(x'') = {}_2.b_1'' b_2'' \dots \geq {}_2.b_1' b_2' \dots = \omega(x').$$

ii. $\omega(x)$ is continuous on $(0, 1)$.

Proof: We have to prove that

$$\omega(x') \rightarrow \omega(x) \text{ as } x' \rightarrow x;$$

and again we may consider only the case where x is a point of E and x' assumes only the values belonging to E . It will suffice to give the proof only in the case where $x' > x$.

Let

$$x = {}_3.a_1 a_2 \dots, x' = {}_3.a_1' a_2' \dots.$$

If now $x' > x$ but $x' \rightarrow x$, then there will be a value of the subscript n (where $n \rightarrow \infty$ as $x' \rightarrow x$) such that

$$a_1' = a_1, \dots, a_{n-1}' = a_{n-1} \text{ but } a_n' > a_n;$$

whence

$$\omega(x') = {}_2.b_1 b_2 \dots b_{n-1} b_n' \dots \rightarrow {}_2.b_1 b_2 \dots b_{n-1} b_n \dots = \omega(x).$$

iii. The function $\omega(x)$ is not absolutely continuous. Its λ -variation¹ on $(0, 1)$ is constant and equals 1.

Proof: To prove the last part of the statement it suffices to put $(\alpha_k, \beta_k) = \eta_{pk}$. The corresponding sum $\sum |\omega(\beta_k) - \omega(\alpha_k)| = \sum \{\omega(\beta_k) - \omega(\alpha_k)\} = 1$, while $\sum (\beta_k - \alpha_k) = \sum \eta_{pk}$ can be made as small as we please by taking p sufficiently large. The first part follows from the last one and the definition of the absolute continuity.²

iv. The function $\psi(x) = (x + \omega(x))/2$ gives a continuous one-to-one correspondence between the segments $(0, 1)$ on the X -axis and on the Y -axis, such that a set E of measure zero is transformed into a set of measure $> 0 (= \frac{1}{2})$.³

¹ Carathéodory, *Vorlesungen über reelle Funktionen*, 2nd edition, 1927, p. 511. By the λ -variation of a function $f(x)$ on $(0, 1)$ is meant the upper limit of the sum

$$\sum_{k=1}^m |f(\beta_k) - f(\alpha_k)|$$

of absolute values of increments of $f(x)$ over any finite set of non-overlapping sub-intervals (α_k, β_k) , $k = 1, 2, \dots, m$, whose total length

$$\sum_{k=1}^m (\beta_k - \alpha_k) \leq \lambda.$$

² H. p. 291.

³ See Carathéodory, loc. cit., p. 356.

Proof: Only the last statement of this property needs a justification. The transformation $y = \psi(x)$ makes to correspond to each interval δ_{pk} on the X -axis an interval of length $\delta_{pk}/2$ on the Y -axis, hence the set D of measure 1 is transformed into a set D_y of measure $\frac{1}{2}$. Then it is obvious that the set $E = C(D)$ (of measure 0) is transformed into a set $E_y = C(D_y)$ of measure $\frac{1}{2}$.

Remark: Since every set of measure >0 contains non-measurable subsets¹ the same function $\psi(x)$ gives an example of a continuous one-to-one transformation in which a measurable set (even a set of measure zero) is transformed into a non-measurable set.

v. *The derivative $\omega'(x)$ of the function $\omega(x)$ is zero almost everywhere on $(0, 1)$ (that is at all the points except for a set of points of measure zero).*

Proof: This is obvious since $\omega'(x) = 0$ at all the points of the set D , that is almost everywhere.²

Remark: Despite the fact that $\omega(x)$ has an integrable derivative almost everywhere, still

$$\int_0^x \omega'(x)dx = 0 \neq \omega(x) - \omega(0) = \omega(x).$$

vi. *The area under the curve $y = \omega(x)$ (that is the area limited by the curve, the X -axis and the ordinates $x = 0, x = 1$) is $\frac{1}{2}$.*

Proof: Since the set E is of measure zero the area in question is

$$\begin{aligned} A &= \int_0^1 \omega(x)dx = \int_E \omega(x)dx + \int_D \omega(x)dx = \int_E \omega(x)dx = \sum_p, k \omega_{pk} \delta_{pk} \\ &= \sum_{p=1}^{\infty} 3^{-p} \sum_{k=1}^{2^{p-1}} (2k - 1) 2^{-p} = \sum_{p=1}^{\infty} 6^{-p} \sum_{k=1}^{2^{p-1}} (2k - 1) = \sum_{p=1}^{\infty} 6^{-p} 2^{2p-2} \\ &= \frac{1}{4} \sum_{p=1}^{\infty} \left(\frac{2}{3}\right)^p = \frac{1}{6} \cdot 1 / \left(1 - \frac{2}{3}\right) = \frac{1}{2}. \end{aligned}$$

This also follows from the skew symmetry of our curve with respect to the line $x = \frac{1}{2}$.

vii. *The length of the arc of the curve $y = \omega(x)$ between the points $(0, 0)$ and $(1, 1)$ is 2.*

Proof: Since $\omega(x)$ is monotone, hence of bounded variation, our curve has a finite length³ which is defined in the usual manner, as the limit of the perimeter

¹ Carathéodory, loc. cit., p. 268.

² It is readily proved by considerations of a general nature (H., pp. 601-602) that the set of points at which $\omega'(x)$ is $+\infty$, is not denumerable. It is not difficult to exhibit a continuum of such points (which necessarily are distinct from the end-points of the intervals δ_{pk} , where the left (right-)hand derivative is $+\infty$ while the right (left-)hand derivative is 0, according as the point in question is a left-hand or a right-hand end-point of δ_{pk}). But the question of a complete determination of all the points of E at which $\omega'(x) = +\infty$ requires more delicate considerations and undoubtedly is related to the arithmetic properties of fractions representing such points.

³ H., pp. 338-339.

of an inscribed polygon. To prove property vii we shall show that the perimeter of any inscribed polygon (which does not cross itself) can not exceed 2, and, on the other hand, there exists an inscribed polygon whose perimeter is as near to 2 as we please. The first statement follows immediately from the fact that the perimeter of any inscribed polygon without double points can not exceed the sum of all the horizontal and vertical projections of its sides, which equals 2, provided the polygon starts from (0, 0) and ends at (1, 1). To prove the second statement take for the inscribed polygon the broken line whose vertices are at (0, 0), (1, 1) and at the end-points of the intervals δ_{pk} (n fixed, $p=1, 2, \dots, n; k=1, 2, \dots, 2^{p-1}$). The sum of the horizontal sides of this polygon is

$$\sum_{p,k} \delta_{pk} = \sum_{p=1}^n 2^{p-1} \cdot 3^{-p} = 1 - \left(\frac{2}{3}\right)^n$$

while all the inclined sides are equal, their common length being

$$(2^{-2n} + 3^{-2n})^{1/2} = 2^{-n} [1 + \left(\frac{2}{3}\right)^{2n}]^{1/2}$$

and the total number 2^n (the number of intervals η_{nk}). Hence the length l_n of our polygon is

$$1 - \left(\frac{2}{3}\right)^n + [1 + \left(\frac{2}{3}\right)^{2n}]^{1/2} \rightarrow 2 \text{ as } n \rightarrow \infty.$$

Remark: It is interesting to observe that the length of our curve can not be computed by the familiar formula

$$\int_0^1 [1 + \omega'(x)^2]^{1/2} dx = 1 \neq 2.$$

The failure of this formula is due to the fact that $\omega(x)$ is not absolutely continuous.

viii. *The function $\omega(x)$ satisfies a Lipschitz condition of order $\alpha = \log 2/\log 3$.* In other words, if x and $x+h$ are in (0, 1),

$$|\omega(x+h) - \omega(x)| \leq \Lambda |h|^\alpha; \quad \Lambda \leq \max \Lambda \leq 2; \quad \alpha = \log 2/\log 3.$$

Proof: If we set $\eta_{pk} = (x, x+h)$, then $\omega(x+h) - \omega(x) = 2^{-p}$ while $h = 3^{-p}$, and we see at once that the order α and the upper limit of the coefficient Λ can not be less than $\log 2/\log 3$ and 1, respectively. Let now x and $x+h$ be any pair of numbers in (0, 1); there will be no loss of generality in assuming $h > 0$. We also may restrict the discussion to the case where both points $x, x+h$ belong to the set E , since, if x or $x+h$ is an interior point of an interval δ_{pk} , we can replace x by the right-hand end-point and $x+h$ by the left-hand end-point of the corresponding δ_{pk} , respectively. This will not change the difference $\omega(x+h) - \omega(x)$ but will reduce h ; hence, if property viii is proved in the case where x and $x+h$ belong to E , it will hold true in the general case.

Let now

$$x = {}_3.a_1 a_2 \cdots a_m \cdots; \quad x+h = {}_3.a'_1 a'_2 \cdots a'_m \cdots,$$

and let n be determined by the condition

$$a'_1 = a_1, \dots, a'_{n-1} = a_{n-1}, \quad a'_n > a_n \quad \text{whence} \quad a'_n = 2, \quad a_n = 0.$$

We then have

$$\omega(x + h) - \omega(x) = {}_2.b_1b_2 \cdots b_{n-1}1 \cdots - {}_2.b_1b_2 \cdots b_{n-1}0 \cdots \leq 2^{-n+1},$$

while

$$h = {}_3.a_1a_2 \cdots a_{n-1}2 \cdots - {}_3.a_1a_2 \cdots a_{n-1}0 \cdots \geq 3^{-n}$$

and

$$[\omega(x + h) - \omega(x)]h^{-\alpha} \leq 2^{-n+1} \cdot 3^{\alpha n} = 2 \quad \text{if} \quad \alpha = \log 2 / \log 3.$$

Remark: The set of points x at which $\omega(x+h) - \omega(x) > h^\alpha$ for h sufficiently small is obviously a subset of E , hence it is of measure zero. This ought to be expected since otherwise the derivative $\omega'(x)$ would be $+\infty$ at a set of points of measure¹ > 0 .

The function $\omega(x)$ so far has been defined on the interval $(0, 1)$. It can be extended outside this interval by setting

$$\omega(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x \geq 1. \end{cases}$$

This will be assumed throughout the remainder of the present note. The difference $\omega(x+h) - \omega(x) = \phi_h(x)$ is of course ≥ 0 , and, as a function of x , is of bounded variation, since it equals a difference of two monotone functions.² Let $T(h)$ be the total variation of the function $\phi_h(x)$ and let

$$\Omega(z) = \max T(h) \quad \text{for} \quad 0 \leq h \leq z.$$

It is plain that $\Omega(z)$ does not increase when z decreases and the natural question arises as to what is $\lim_{z \rightarrow 0} \Omega(z) = \Omega_0$? If $\phi_h(x)$ were absolutely continuous, then, since $\omega'(x) = 0$ almost everywhere, we would have³

$$T(h) = \int_{-\infty}^{\infty} |\phi'_h(x)| dx = \int_{-\infty}^{\infty} |\omega'(x+h) - \omega'(x)| dx = 0,$$

so that

$$\Omega(z) = \Omega_0 = 0.$$

¹ H., p. 400.

² H., p. 329.

³ H., p. 605. In the general case, where $\omega(x)$ is any absolutely continuous function, by a fundamental property of Lebesgue integrals (H., p. 636),

$$T(h) = \int_{-\infty}^{\infty} |\omega'(x+h) - \omega'(x)| dx \rightarrow 0 \quad \text{as} \quad h \rightarrow 0$$

which would yield the same result, $\Omega_0 = 0$.

The situation is entirely different, however, in our case (due to the non-absolute continuity of $\omega'(x)$), which is shown by the property¹

ix. *The function $\Omega(z)$ as defined above is constant and equals 2.*

Proof: From the definition of the total variation of a function² it follows that $T(h)$, being a total variation of a difference $\omega(x+h) - \omega(x)$, can not exceed the sum of the total variations of the constituents $\omega(x+h)$, $\omega(x)$. Since $\omega(x)$ and $\omega(x+h)$ are monotone and increase from 0 to 1, they have the same total variation 1, whence

$$T(h) \leq 2.$$

Take now $h = 3^{-n}$. Since

$$(-\infty, \infty) = \delta_0 + \sum_{p=1}^n \sum_{k=1}^{2^{p-1}} \delta_{pk} + \sum_{i=1}^{2^n} \eta_{ni} + \delta_1; \quad \delta_0 = (-\infty, 0); \quad \delta_1 = (1, \infty),$$

we have, by the additive property of the total variation,³

$$T(h) = T(\delta_0) + \sum_{p,k} T(\delta_{pk}) + \sum_i T(\eta_{ni}) + T(\delta_1),$$

where each term of the right-hand member is the total variation of $\phi_h(x)$ over the corresponding interval. It is important to observe that our h equals the common length of the intervals η_{ni} and does not exceed the length of any of the intervals δ_{pk} ($p = 1, 2, \dots, n$). Hence, when x ranges over an interval η_{ni} , $(x+h)$ ranges over a part of the interval δ_{pk} that is adjacent to η_{ni} . Under these circumstances, $\omega(x+h)$ remains constant while $\omega(x)$ increases by 2^{-n} . Hence

$$T(\eta_{ni}) = 2^{-n}.$$

To compute $T(\delta_{pk})$, let $x_1 < x_2$ be the end-points of δ_{pk} . Subdivide δ_{pk} in two parts,⁴ $\delta' = (x_1, x_2 - h)$, $\delta'' = (x_2 - h, x_2)$ and denote by T' , T'' the total variations of $\phi_h(x)$ in the intervals δ' , δ'' respectively. When x ranges over δ' the functions $\omega(x)$ and $\omega(x+h)$ remain constant, so that $T' = 0$. When x ranges over δ'' , $(x+h)$ ranges over the interval η_{ni} that is adjacent to δ'' . Then $\omega(x)$ remains constant but $\omega(x+h)$ increases by 2^{-n} , whence $T'' = 2^{-n}$ and

$$T(\delta_{pk}) = T' + T'' = 2^{-n}.$$

As to the terms $T(\delta_0)$ and $T(\delta_1)$, we find in exactly the same fashion that

$$T(\delta_0) = 2^{-n}, \quad T(\delta_1) = 0.$$

¹ In the notation of the theory of Stieltjes integrals we can state this property as follows:

$$\Omega(z) = \max_{0 \leq h \leq z} \int_{-\infty}^{\infty} |d\omega(x+h) - d\omega(x)| = 2.$$

In a recent important note [*Eine Kennzeichnung der totalstetigen Funktionen*, Crelle's Journal, vol. 160 (1929), pp. 26-32]. A. Plessner proved that the condition $T(h) \rightarrow 0$ as $h \rightarrow 0$ is necessary and sufficient for the absolute continuity of the function $\omega(x)$.

² H., p. 325.

³ H., p. 330.

⁴ The first part δ' exists only if $\delta_{pk} > h$, that is if $p < n$.

On combining all these facts and observing that the number of the intervals η_{ni} is 2^n while that of the intervals δ_{pk} is $2^n - 1$, it follows at once that

$$T(h) = 2^{-n}(1 + 2^n + 2^n - 1) = 2$$

whence

$$\Omega(z) = \Omega_0 = 2.$$

The last and perhaps the most interesting property of our function $\omega(x)$ is in connection with its "Fourier-Stieltjes coefficients." If $f(x)$ is any function given on $(0, 1)$ we may call its Fourier-Stieltjes coefficients the integrals¹

$$(7) \quad f'_n = \int_0^1 e^{2\pi inx} df(x) \quad (n = 0, \pm 1, \pm 2, \dots).$$

If $f(x)$ is absolutely continuous, then integrals (7) reduce to the classical Fourier coefficients of $f'(x)$:

$$f'_n = \int_0^1 e^{2\pi inx} f'(x) dx ;$$

and, by the fundamental Riemann-Lebesgue theorem,² $f'_n \rightarrow 0$ as $|n| \rightarrow \infty$.

In a more general case where $f(x)$ is only of bounded variation, we still have right to integrate by parts:

$$(8) \quad f'_n = \int_0^1 e^{2\pi inx} df(x) = [f(1) - f(0)] - 2\pi in \int_0^1 e^{2\pi inx} f(x) dx,$$

so that

$$(9) \quad f_n = \frac{[f(1) - f(0)]}{2\pi in} - \frac{f'_n}{2\pi in}$$

is the Fourier coefficient of $f(x)$.

There is an essential difference between the two cases just mentioned, which is shown by the property

x. ³The Fourier-Stieltjes coefficient ω'_n of the function $\omega(x)$ does not tend to 0 as $|n| \rightarrow \infty$.

¹ We refer as to the definition and fundamental properties of Stieltjes integrals to H. Here we deal exclusively with the Riemann-Stieltjes integrals. A Riemann-Stieltjes integral of a function $g(x)$ with respect to the function $f(x)$ is defined as the limit (in case it exists) of the sum:

$$\int_0^1 g(x) df(x) = \lim \sum_{i=1}^m g(\xi_i) [f(x_i) - f(x_{i-1})]; \quad x_0 = 0, \quad x_m = 1,$$

where (x_{i-1}, x_i) , $i = 1, 2, \dots, m$ is any subdivision of the interval $(0, 1)$ such that the maximum length of the intervals $(x_{i-1}, x_i) \rightarrow 0$ as $m \rightarrow \infty$ and ξ_i is an arbitrary point of the interval (x_{i-1}, x_i) , the end-points inclusive. The existence of this limit is assured if $g(x)$ is continuous and $f(x)$ is of bounded variation.

² H., vol. 2 (2nd edition), 1926, p. 514.

³ This is a special case of an example of Carleman, *Sur les équations intégrales singulières à noyau réel et symétrique*, Uppsala Universitets Årsskrift, 1923, No. 3, pp. 223-226.

Proof: By definition we have

$$\omega'_n = \int_0^1 e^{2\pi i n x} d\omega(x) = \lim_{m \rightarrow \infty} \sum_{s=1}^m e^{2\pi i n \xi_s} [\omega(x_s) - \omega(x_{s-1})] = \lim_{m \rightarrow \infty} \Sigma_m.$$

In computing this limit we can take any special type of subdivisions of $(0, 1)$; for instance, we may subdivide $(0, 1)$ into $2^p = m$ equal parts. Then the set $\{(x_{s-1}, x_s)\}$ will consist partly of the intervals η_{pk} and partly of the intervals $\delta_{jk} (j=1, 2, \dots, p)$ and their subdivisions. Since $\omega(x)$ is constant on each δ_{jk} this second part will give no contribution to the sum Σ_m . As to the points ξ_s we shall make them to coincide with the left-hand end-points of the corresponding intervals η_{pk} . They will be designated (in increasing order) by $\alpha_k, k=1, 2, \dots, 2^p$.

Since $\omega(x)$ increases by 2^{-p} when x ranges over an interval η_{pk} ,

$$\Sigma_m = \Sigma_{2^p} = 2^{-p} \sum_{k=1}^{2^p} e^{2\pi i n \alpha_k}.$$

It is readily seen that the set of points $\{\alpha_k\}$ consists of all the *finite* fractions of the form

$$.3.a_1 a_2 \dots a_p \quad (a_j = 0 \text{ or } 2).$$

The summation over all such values of a_j will be designated simply by $\sum_{(a)}$.

Hence

$$\begin{aligned} \Sigma_{2^p} &= 2^{-p} \sum_{(a)} \exp. \left[2\pi i n \sum_{j=1}^p a_j 3^{-j} \right] = 2^{-p} \sum_{(a)} \prod_{j=1}^p \exp. [2\pi i n a_j 3^{-j}] \\ &= 2^{-p} \prod_{j=1}^p \{1 + \exp. (4\pi i n 3^{-j})\} = \exp. \left(\sum_{j=1}^p 2\pi i n 3^{-j} \right) \prod_{j=1}^p \cos (2\pi n 3^{-j}). \end{aligned}$$

This yields the final result

$$\omega'_n = \lim_{p \rightarrow \infty} \Sigma_{2^p} = e^{\pi i n} \prod_{j=1}^{\infty} \cos (2\pi n 3^{-j})$$

since

$$2 \sum_{j=1}^{\infty} 3^{-j} = 1.$$

In the preceding computation n was an arbitrary number (not necessarily an integer). Now we set $n = 3^q$ where q is a positive integer. Then

$$\omega'_n = \omega_{3^q}' = - \prod_{j=1}^{\infty} \cos (2\pi 3^{q-j}) = - \prod_{\nu=1}^{\infty} \cos (2\pi/3^\nu).$$

The infinite product of the left-hand member converges absolutely and contains no zero factor;¹ therefore it is different from 0. On the other hand it does not

¹ A necessary and sufficient condition for the absolute convergence of the infinite product $\prod_\nu (1+u_\nu)$ is given by the absolute convergence of the series $\sum_\nu u_\nu$. This condition is satisfied in the present case since $|u_\nu| = |1 - \cos (2\pi/3^\nu)| = 2\sin^2(\pi/3^\nu) < 2\pi^2 3^{-2\nu}$.

depend on q . If now we make $q \rightarrow \infty$ the corresponding Fourier-Stieltjes coefficient $\omega_{3q'}$ of $\omega(x)$ will not tend to 0.

Remark: The function

$$\chi(x) = \omega(x) - x$$

gives an example of a periodic continuous function (of bounded variation) such that, if χ_n is the Fourier coefficient of $\chi(x)$, the product $n\chi_n$ does not tend to any limit as $|n| \rightarrow \infty$.

This follows immediately from (9) and property x.

The interest of this example lies in the fact that if $f(x)$ is continuous and periodic and if nf_n tends to a limit as $|n| \rightarrow \infty$ then this limit¹ is necessarily 0.

AN ALGEBRAIC METHOD OF DIFFERENTIATION

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It is the purpose of this paper to present a method of obtaining the formulas of the differential calculus by purely algebraic means, without the use of limiting processes. The method is rather obvious, and is essentially equivalent to those used by the mathematicians of the eighteenth century, before the logical rigor which we associate with the name of Weierstrass came into favor.² The method here presented is rigorous, however, being based on the theory of analytic functions of a hypercomplex variable.

Consider the hypercomplex number system (or linear algebra), analogous to the ordinary complex number system, whose basal units are 1 and j , where $j^2 = 0$. Because of its many geometric applications, the function theory of this algebra has been much studied. It has been shown by Scheffers³ that the most general analytic function of one variable in this algebra has the form

$$(1) \quad f(x + yj) = \phi(x) + [\phi'(x)y + \psi(x)]j,$$

where $\phi(x)$ and $\psi(x)$ are real functions of a real variable. (The terms real and imaginary will be used to distinguish x and yj , to keep the analogy with the theory of functions of a complex variable. It would be possible to allow x and y to be complex, and in this case the terms *scalar* and *nilpotent* would be less confusing.) If now f is a function which is real for real values of the argument, we have, setting $y = 0$, that $\psi(x) = 0$ and $\phi(x) = f(x)$, which gives us

¹ Neder, *Über die Fourierkoeffizienten der Funktionen von beschränkter Schwankung*, *Mathematische Zeitschrift*, vol. 6 (1920), pp. 270–273; Steinhaus, *Bemerkung zu der Arbeit des Herrn Neder . . .*, *ibidem*, vol. 8 (1920), pp. 320–322; Alexits, *Zwei Sätze über Fourierkoeffizienten*, *ibidem*, vol. 27 (1927), pp. 65–67. Another example of a continuous periodic function $f(x)$ for which $\lim (nf_n)$ does not exist was given by F. Riesz, *ibidem* vol. 2 (1918), pp. 312–315. Riesz's example, however, is of entirely different nature.

² See the interesting paper of Professor James Pierpont, *Mathematical Rigor, Past and Present* in the *Bulletin of the American Mathematical Society*, vol. 34 (1928), p. 23.

³ *Mathematische Annalen* vol. 60 (1905), p. 529.