physical analysis can be applied, with some success, to real wineglasses and other such vessels that can be found in any household. The analysis is, to be sure, rather unreasonably heavy in relation to the importance (or lack of it) of the specific topic, but it does exemplify the power of the energy method for the analysis of relatively complex vibrating systems.

One interesting feature is the way in which seemingly identical glasses (such as the set mentioned in Sec. VI) have distinctive and widely different frequencies of vibration. The natural frequency could well be used as an identifying label or "signature" for a glass or other vessel in cases where the original was rare and valuable and one wanted a simple noninvasive test to distinguish it from copies.

ACKNOWLEDGMENT

I wish to thank my son, Martin C. French, for helping to stimulate this investigation and for his partnership in all the observations.

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An electrostatic example to illustrate dimensional regularization and renormalization group technique

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An easy and intuitive introduction to the regularization and renormalization techniques in elementary particle physics is given. It is based on the use of a simple electrostatic problem.

I. INTRODUCTION

High-energy physics needs by its very nature an elaborate relativistic quantum field theory. In quantum electrodynamics the successes of this theory are remarkable. However, in all its perturbative aspects it has heavy divergence problems. We would like to show schematically what these problems are and how to solve them. We illustrate this last point precisely with an electrostatic example which shows the same type of divergence. The method used is dimensional regularization.[1] Since 1973, it is most used in applications. This example allows us to show the simplicity and the elegance of dimensional regularization. Moreover, it illustrates the crucial role played by dimensional analysis especially for any problem in which there is no fixed length (or energy) scale. We show that renormalization implies the introduction of a scale and therefore breaks the so-called scale invariance. We study in Sec. V the behavior of renormalized quantities such as an electrostatic potential and a dimensionless quantity of high-energy physics called R. By doing this, we illustrate a few aspects of the renormalization group techniques, which are commonly used in elementary particle physics.

II. DIVERGENCES IN QUANTUM FIELD THEORY

Every quantum field theory is basically of perturbative character. Without going into more details, we can say that it is most useful whenever we deal with a physical quantity which can be expressed as a truncated series. Every cross section and every decay width are of that type. The development parameter of this series is the coupling constant. It measures the intensity of the interaction under study. In quantum electrodynamics, this constant is[2]

$$\alpha = e^2/4\pi = 1/137$$

(the fine structure constant).

To be specific, let us consider the scattering between two particles interacting electromagnetically such as Compton scattering ($e^-\gamma \to e^-\gamma$), electron scattering ($e^-e^- \to e^-e^-$) or the annihilation process $e^-e^+ \to 2\gamma$. The cross section is proportional to the square of the scattering amplitude. It is this last quantity that quantum electrodynamics determines as a development in powers of α. Each term of this development can be calculated directly, but it is more convenient to write it as a sum of algebraic quantities represented by the so-called Feynman diagrams. These diagrams have a much more intuitive interpretation than the corresponding algebraic quantities. In the case of e^-e^- scattering the first nontrivial order (Born approximation) is represented by the very simple diagram of Fig. 1. With this diagram we can calculated its cross section. (Analogously for $e^-\gamma \to e^-\gamma$ and $e^-e^+ \to 2\gamma$.) The agreement with experiment is very good.

However, experimental data have sometimes such an accuracy that we must calculate the next-order corrections. These correspond to more complicated diagrams [with

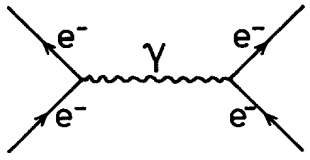

Fig. 1. Two electrons interact by exchanging a virtual (nonphysical, nonobservable) photon.

more vertices]. In the case of e^-e^- scattering, the second-order correction is given by the diagram of Fig. 2. We notice the occurence in this diagram of a loop corresponding to the electron–positron pair. What is happening in the corresponding algebraic expression? Let us consider the exchanged photon momentum. It is shared between the electron and the positron of the loop in an arbitrary way. One of the two momenta (e^- or e^+) is completely arbitrary. We must therefore integrate over it in the algebraic expression. The problem is that this integral is generally infinite. The theory looks therefore unacceptable to this order and we must modify it. The same is true for all higher-order terms.

First, we collect all the infinite amplitudes to each order and afterwards we modify them to get a finite quantity. The process by which we transform an infinite quantity into a finite one (called renormalized) has two steps. The first one is the regularization. We cannot deal directly with infinities and we would like to see more clearly the occurence of the divergence. We imagine therefore a new and more general expression called regularized. It is finite, but depends on a parameter such that we recover our infinite quantity if this parameter tends towards a particular value that we shall call starting value. (We shall see that in the dimensional regularization, this parameter is the space dimension.) The regularized expression is not unique because there are several regularization methods. The second step is the renormalization. It modifies the regularized expression by subtracting terms which diverge when the parameter tends towards its starting value. There is some arbitrariness in this subtraction. We fix it putting some constraint on the new expression (called renormalization condition). The finite limit of our new expression when the parameter tends towards its starting value is called the renormalized expression.

It is this two-step procedure that we are going to illustrate with an electrostatic example. We shall use dimensional regularization because it has many advantages and is mostly used nowadays. It is clear that this procedure must be used in all next-order calculations in quantum field theory (such as the anomalous magnetic moment of the electron). It applies not only to quantum electrodynamics but also to quantum chromodynamics and many other field theories as well.

To end this section, we must add that for a theory to be renormalizable, it is not sufficient to renormalize all infinite amplitudes order by order. The modifications of these amplitudes must be of a particular kind. The total number of divergent amplitudes to all orders must be finite and the modifications of these amplitudes order by order must lead to a finite theory to all orders.

III. DIMENSIONAL REGULARIZATION OF A DIVERGENT ELECTROSTATIC EXAMPLE

Let us consider a uniformly charged straight line. We would like to calculate the electrostatic potential created everywhere in the space.

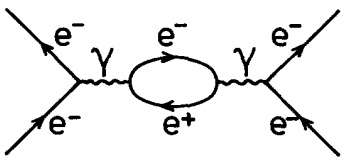

Fig. 2. Virtual exchanged photon turns temporarily into a virtual electron–positron pair (nonobservable).

First, we make a dimensional analysis of the problem. We know from physics, that the potential V must be proportional to the charge density λ and is a function of the radial variable r only (there is no other length scale[3–5]) so[2]

$$V(r) = (\lambda/2\pi)V'(r), \tag{1}$$

V' is dimensionless. Whatever the particular formula allowing the calculation of V may be, there are only three possibilities[6]:

(a) $V'(r) = c$ finite constant; (2)

(b) $V'(r) = \infty$;

(c) there exists an undetermined length scale Λ such that $V' = V'(r/\Lambda)$.

Let us now choose as a particular expression allowing a unique calculation of V the naïve classical formula:

$$V(\mathbf{y}) = \frac{1}{4\pi}\int \frac{\lambda(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|}\,dl; \tag{3}$$

it is such that $V \to 0$ if $|\mathbf{y}| \to \infty$. In our case, the appropriate form is

$$V(r) = \frac{\lambda}{4\pi}\int_{-\infty}^{+\infty} \frac{dx}{(x^2+r^2)^{1/2}}. \tag{4}$$

We choose this formula also because it does not involve any length scale. It respects therefore the symmetry of our example (no length scale). It formally satisfies scale invariance: if we change $r \to kr$

$$V(kr) = \frac{\lambda}{4\pi}\int_{-\infty}^{+\infty} \frac{dx}{(x^2+k^2r^2)^{1/2}} = \frac{\lambda}{4\pi}\int_{-\infty}^{+\infty} \frac{dy}{(y^2+r^2)^{1/2}} = V(r).$$

Formula (4) clearly illustrates the second case (b) above because it gives a constant infinite V. However, we know that it is possible to avoid the divergence of the integral because the electric field is finite. Our divergent formula (4) must therefore be renormalizable. Here, we know how to get a finite result. In general, whatever the renormalization method may be, the new formula must be finite and cannot correspond to zero electric field. Therefore, dimensional analysis (2) tells us that in any renormalized formula a length scale Λ must appear such that

$$V_{\text{ren}}(r) = V(r/\Lambda). \tag{5}$$

Λ is not given by the renormalization procedure, so that the result will no longer be unique.

We are going to apply the dimensional regularization method to this electrostatic example. Remember that the procedure involves two steps: dimensional generalization of the divergent integral (regularization), followed by the subtraction of the possible divergences (renormalization).

We write the integral in a l (integer)-dimensional space. l is our regularizing parameter. In order to keep the dimension of the potential fixed we must compensate the l-dependent dimension of the integral by an l-dependent length term:

$$V_l(r) = \frac{\lambda}{4\pi}\Lambda^{1-l}\int \frac{dx^l}{(x^2+r^2)^{1/2}},$$

where

$$x^2 = x_1^2 + \cdots + x_l^2. \tag{6}$$

Λ is a length scale which is unspecified. Our aim is to find a domain of l inside which the integral (6) is meaningful.

Dimensional regularization respects the spherical symmetry of the integrand, consequently we use spherical coordinates and we integrate over the angles. We have

$$dx^l = \rho^{l-1}\, d\rho\, d\Omega_l,$$

where ρ is the radial coordinate and $d\Omega_l$ the solid angle element[7]

$$\Omega_l = \int d\Omega_l = \frac{2\pi^{l/2}}{\Gamma(l/2)},$$

$$V_l(r) = \frac{\lambda}{4\pi} \Lambda^{1-l} \frac{2\pi^{l/2}}{\Gamma(l/2)} \int_0^\infty \frac{\rho^{l-1}\, d\rho}{(\rho^2 + r^2)^{1/2}}. \tag{7}$$

This last formula, thanks to the introduction of the Γ function, is meaningful even if l is noninteger. Given[8]

$$\int_0^\infty \frac{\rho^{l-1}\, d\rho}{(\rho^2 + r^2)^{1/2}} = \frac{1}{2} \frac{\Gamma(l/2)\Gamma[(1-l)/2]}{\sqrt{\pi}} r^{l-1}, \tag{8}$$

we get

$$V_l(r) = (\lambda/4\pi)\Lambda^{l-1} \pi^{(l-1)/2} \Gamma[(1-l)/2]\, r^{l-1}. \tag{9}$$

$V_l(r)$ is now finite except if l is an odd integer. Formula (9) is its regularized expression. If, in formula (9), the limit $l \to 1$ had been finite, the expression (4) would have been renormalized without the second step. Here, however, the second step is necessary because, in the limit $l \to 1$, the length scale Λ disappears. However, we know the finite expression necessary involves Λ.

The regularization method is not unique. Another one is to cut off the the integral range to get

$$V_L(r) = \frac{\lambda}{4\pi} \int_{-L}^{L} \frac{dx}{(x^2 + r^2)^{1/2}}. \tag{10}$$

L is the regularization parameter. But this procedure breaks one symmetry of the problem: translation invariance along the charged axis. Dimensional regularization respects all the symmetries of the problem. This is very important in quantum field theory and explains its interest.

IV. POTENTIAL RENORMALIZATION

In order to see very clearly the divergence we choose $l = 1 + \varepsilon$ in the regularized formula (9) and we develop it for $\varepsilon \ll 1$. We have[7]

$$\begin{aligned} &(r/\Lambda)^{l-1} \simeq 1 + \varepsilon \ln(r/\Lambda) + O(\varepsilon^2), \\ &\pi^{(l-1)/2} \simeq 1 + (\varepsilon/2)\ln\pi + O(\varepsilon^2), \\ &\Gamma\left(\frac{1-l}{2}\right) = \Gamma\left(-\frac{\varepsilon}{2}\right) \\ &\qquad = -\frac{2}{\varepsilon}\Gamma\left(1 - \frac{\varepsilon}{2}\right) \\ &\qquad = -\frac{2}{\varepsilon}\left(1 + \frac{\varepsilon}{2}\gamma + O(\varepsilon^2)\right). \end{aligned} \tag{11}$$

Therefore,

$$V_\varepsilon(r) = -\frac{\lambda}{2\pi}\left[\frac{1}{\varepsilon} + \frac{1}{2}\ln\pi + \frac{\gamma}{2} + \ln\left(\frac{r}{\Lambda}\right) + O(\varepsilon)\right]. \tag{12}$$

Physics tells us that we can always add a constant to any electrostatic potential. Here, we can take advantage of that to renormalize Eq. (12) by subtracting a constant such that the limit $\varepsilon \to 0$ is finite. The fact that the divergent term is a constant is essential for the potential to be renormalizable.[9]

We get

$$V_R^\varepsilon(r) = -(\lambda/2\pi)[\ln(r/\Lambda') + O(\varepsilon)], \tag{13}$$

where $\Lambda' = c\Lambda$, with c an arbitrary finite constant. The limiting value ($\varepsilon \to 0$) is

$$V_R(r) = -(\lambda/2\pi)\ln(r/\Lambda'). \tag{14}$$

It is our renormalized potential, showing the length scale Λ' unspecified by the procedure. Of course, it is the same formula that we should have calculated classically by means of the Gauss theorem without imposing $V \to 0$ when $r \to \infty$.

Whenever we add a constant to a potential we know that we change the boundary condition. In our case, we had [see Eq. (9)]

$$\begin{aligned} &V = 0 \quad \text{if } r \to \infty \quad (\text{if } \varepsilon < 0), \\ &V = 0 \quad \text{if } r = 0 \quad (\text{if } \varepsilon > 0), \end{aligned}$$

and now we have

$$V = 0 \quad \text{if } r = \Lambda', \tag{15}$$

which is an intermediate condition involving a length scale Λ'. In fact, it is the renormalization condition that gives a unique renormalized potential. We can see that $V_R(r)$ is the limit for $\varepsilon \to 0$ of

$$V_\varepsilon^R(r) = \frac{\lambda}{4\pi}\left[\pi^{\varepsilon/2}\Gamma\left(-\frac{\varepsilon}{2}\right)\left(\frac{r}{\Lambda'}\right)^\varepsilon - \pi^{\varepsilon/2}\Gamma\left(-\frac{\varepsilon}{2}\right)\right], \tag{16}$$

which satisfies $V_\varepsilon^R(\Lambda') = 0$. To this condition independent of ε corresponds clearly a subtraction constant which changes with ε and which is in fact divergent if $\varepsilon \to 0$:

$$b_\varepsilon = -\frac{\lambda}{4\pi}\pi^{\varepsilon/2}\Gamma\left(-\frac{\varepsilon}{2}\right)\left[1 + \left(\frac{1}{\Lambda'}\right)^\varepsilon - \left(\frac{1}{\Lambda}\right)^\varepsilon\right]. \tag{17}$$

(In a way, the singularity has been removed inside the additive constant.)

In conclusion, we see clearly that, starting with the infinite expression (4) that we cannot modify directly, the first step consists in writing it in a l-dimensional space, the second step involves a renormalization condition in the neighborhood of $l = 1$ such that (if possible) the limit $\varepsilon \to 0$ is finite. In quantum field theory, the expression (4) is replaced by a Feynman diagram with at least one loop. The procedure to regularize it is nevertheless entirely similar.

V. ILLUSTRATION OF RENORMALIZATION GROUP TECHNIQUE[10]

To be specific, let us consider the ratio R between the cross section of $e^+e^- \to$ anything and the cross section of $e^+e^- \to \mu^+\mu^-$. We assume that it no longer depends on the lepton masses when Q^2 is sufficiently big [$Q^2 > 1\ \text{GeV}^2$]:

$$R = \sum_x \sigma(e^+e^- \to x^+x^-)/\sigma(e^+e^- \to \mu^+\mu^-) = R(Q^2); \tag{18}$$

it has no dimension and depends only on Q^2 (here $Q/2$ is the electron center of mass momentum). The same dimensional analysis can be done for R as for $V'(r)$ [cf. Eq. (2)].[11] Either

(i) $R(Q^2) = c$ finite constant;

(ii) $R(Q^2) = \infty$;

(iii) there exists an energy scale Λ, unspecified such that

$$R = R(Q^2/\Lambda^2). \tag{19}$$

From quantum field theory, we know that R is expressed in the form of a series of Feynman diagrams. We know each term after the first order of perturbation is infinite. We can renormalize R as we did for V. Furthermore, we have reasons to believe that R cannot be a constant,[12–15] so the renormalized expression must be [cf. Eq. (19)]:

$$R_{\text{ren}} = R(Q^2/\Lambda^2) \text{ analogous to } V'_{\text{ren}}(r) = -\ln(r/\Lambda), \tag{20}$$

where Λ^2 is not specified by the theory. (We drop the subscript ren in the following.) We have

$$\frac{dR}{d\Lambda^2} = R'\left(\frac{Q^2}{\Lambda^2}\right)Q^2 \neq 0, \quad \frac{dV'}{d\Lambda} = \frac{1}{\Lambda} \neq 0. \tag{21}$$

Therefore, the absolute value of R [V'] depends on Λ^2 [Λ] and cannot be predicted by the theory. On the other hand, we know that R is a physical quantity and consequently a measure of R at a known Q^2 gives us Λ^2 (experimental quantity). [In electrostatics, $V(r)$ may be considered as the energy needed to move a test charge between a fixed point to the point r. The starting point fixes Λ.]

We can be more specific about the properties of the renormalized formula. Using relation (21), it can be assumed that the function R may be inverted (at least piecewise), we thus have

$$R = R\left(\frac{Q^2}{\Lambda^2}\right) \to \frac{\Lambda^2}{Q^2} = f(R);$$

$$V' = -\ln\frac{r}{\Lambda} \to \frac{\Lambda}{r} = \exp V', \tag{22}$$

where R is the measured value at the energy Q^2 and V' is the value at the distance r. This implies

$$\Lambda^2 = Q^2 f[R(Q^2)] = \mu^2 f[R(\mu^2)],$$
$$\Lambda' = r\exp[V'(r)] = \mu\exp[V'(\mu)], \tag{23}$$

where μ^2 is any energy and $R(\mu^2)$ is the value of R at this energy. [Similarly, μ is any distance and $V'(\mu)$ the value of V' at this distance.]

Consequences

(A) We have

$$R = R\left(\frac{Q^2}{\mu^2 f[R(\mu^2)]}\right) = F\left(\frac{Q^2}{\mu^2}, R(\mu^2)\right). \tag{24}$$

Thus our renormalized formula provides $R(Q^2)$ once we have measured R at any known energy μ^2. It cannot provide the absolute value of R but only its variation from a fixed point. We remark that the value of $R(Q^2)$ does not depend on the chosen energy μ becaue using Eqs. (24) and (23):

$$\frac{dR}{d\mu^2} = 0. \tag{25}$$

A renormalized formula contains less information than a convergent one; somehow, it considers R like a potential.

(B) We have

$$\Lambda^2 = \mu^2 f[R(\mu^2)]. \tag{26}$$

It is the relation between two different energy scales: μ^2 is arbitrary, Λ^2 is fixed, not by the theory but from experiment.[16]

(C) R satisfies a differential equation which determines completely its variation. This is the only thing that can be predicted. In fact, starting from Eq. (23), we get

$$\ln\frac{Q^2}{\mu^2} = \ln\left(\frac{f[R(\mu^2)]}{f[R(Q^2)]}\right), \tag{27}$$

and, by derivation, we obtain[17]

$$Q^2\frac{dR}{dQ^2} = -f(R)\frac{df}{dR} = \beta(R). \tag{28}$$

It is a basic differential equation of renormalization group.[19] This differential equation (if we know β) contains as much information as the renormalized formula. It can, in fact, be solved to get Eq. (24),

$$R(Q^2) = K^{-1}\{K[R(\mu^2)] + \ln(Q^2/\mu^2)\}, \tag{29}$$

where K is the primitive of $1/\beta$.

The same consequences are true for the electrostatic potential if we replace the energy μ by the length μ.

(i) The equation corresponding to Eq. (24) is

$$V'(r) = V'(\mu) - \ln(r/\mu). \tag{24'}$$

It does not depend on the distance μ where the potential is given

$$\frac{dV'}{d\mu} = 0. \tag{25'}$$

(ii) The equation analogous to Eq. (26) is

$$\Lambda = \mu\exp[V'(\mu)]. \tag{26'}$$

(iii) V' also satisfies a differential equation which determines its variations. Starting from

$$\ln(r/\mu) = V'(\mu) - V'(r), \tag{27'}$$

we obtain by derivation

$$2\pi r E(r) = \lambda, \quad \text{where } E(r) = -\frac{dV}{dr}. \tag{28'}$$

It corresponds to Eq. (28). This is a particular expression of the well-known Gauss theorem. We can of course start from Eq. (28') to get Eq. (24'). It is the classical way to calculate V'.

VI. CONCLUSION

We hope we have shown that some field-theoretic methods and concepts seemingly difficult to understand can be illustrated in a very simple way in the context of a purely classical and even elementary example. The dimensional regularization procedure gets an intuitive and almost physical interpretation. The renormalization group equations (25) and (28) express the arbitrariness of the chosen point on which the physical quantity is given.

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[1] G. 't Hooft and M. Veltman, Nucl. Phys. B **44**, 189 (1972).

[2] We use the natural unit system $\hbar = c = 1$. In the international system: $\alpha = e^2/4\pi\varepsilon_0\hbar c$.

[3] In a quantum field theory coupling several massless fields, there is also no mass scale if the coupling constants are dimensionless. Such a theory is renormalizable.[4,5]

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[6] In the cases (a) and (b) there is a scale invariance (for the scale transfor-

mation $r \to kr$). In case (c) the scale invariance is broken. A massless renormalizable theory leads first to scale invariance (for the scale transformation $Q^2 \to kQ^2$); but, just like the third case, this invariance is broken by renormalization.

[7] $\Gamma(x)$ is the Euler function. It is an analytic function of x except for $x = 0, -1, -2, \ldots$. If $\varepsilon \ll 1$: $\Gamma(1+\varepsilon) = 1 - \varepsilon\gamma + O(\varepsilon^2)$ where γ is the Euler constant $\gamma = 0.5772$.

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[16] In field theory, μ can be identified to "the subtraction constant" needed to renormalize each order of perturbation. Its appearance is the so-called dimensional transmutation phenomena.

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[18] M. Gell-Mann and F. E. Low, Phys. Rev. **95**, 1300 (1954).

[19] In quantum field theory, the renormalization scale μ necessarily enters in the calculated expression of R. [It plays the role of Λ' in the expression (14) of V_R.] Equation (25) is therefore fundamental. In fact, it implies that the explicit μ dependence in R must be compensated and leads to Eq. (24). We can then deduce Eq. (28).

Photon mass experiment

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A Coulomb null experiment is described that enables physics students to obtain rigorous upper bounds on photon mass. The experimenter searches for subnanovolt signals that would escape a closed shell were photon mass to be positive. The approach can be adapted for several college levels. At the simplest level, a "miniature" low-cost experiment allows a student to verify the exponent " -2" in Coulomb's law to eight or more decimal places. An advanced student given a full-size apparatus (at greater cost) can obtain mass bounds very close to the established laboratory limit.

I. INTRODUCTION

The idea that the photon mass is zero, as assumed in the classical theory of electromagnetism, must always be subject to experimental scrutiny. A particle is not massless until proven so. Indeed, the neutrino now presents laboratory[1] and possibly astronomical[2] evidence of having positive mass. For the photon there are no positive-mass claims at the present time. The history of the photon mass problem is fascinating, however, and is a beautiful introduction to the concept of a null experiment. Previous methods of obtaining mass bounds for the photon include: measurement of the speed of light versus wavelength,[3] measurement of pulsar light dispersion,[4] magnetic methods,[5,6] and laboratory verifications of Coulomb's law. This last method is the most relevant to the present treatment. Williams *et al.* obtained (1971) the mass bound[7]

$$m < 2 \times 10^{-47}\ \text{g},$$

while the present author and collaborators have recently improved this to[8]

$$m < 8 \times 10^{-48}\ \text{g},$$

by bounding the voltage of a radio-frequency signal that penetrates *into* a closed conducting shell. The present treatment describes a similar experiment having simpler, "inside-out" geometry suitable for student work. Generally speaking, the more advanced student can obtain a tighter mass bound, usually by way of relatively longer signal processing time. The experiment is rich in pedagogical physics and is good for demonstrating precisely what in Maxwell theory is dependent on Coulomb's law.

The typical bound obtained by methods in keeping with introductory physics classes is

$$m < 10^{-42}\ \text{g},$$

while for third- and fourth-year laboratory students, the bound can be lowered to

$$m < 7 \times 10^{-47}\ \text{g},$$

and serious students with research skill can approach the bound of Williams *et al.* stated above.

II. THEORY

When the zero-mass restriction on the photon is lifted, many interesting changes occur in the standard electrodynamical theory. Excellent treatments of the theory exist,[9] some major modifications are lised here: