

ANALOGS OF RENORMALIZATION GROUP TRANSFORMATIONS IN RANDOM PROCESSES*

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We review the properties of a real-space renormalization group transformation of the free energy, including the existence of oscillatory terms multiplying the non-analytic part of the free energy. We then construct stochastic processes which incorporate into probability distributions the features of the free energy scaling equation. (The essential information is obtainable from the scaling equation and a direct solution for a probability is not necessary.) These random processes are shown to be generated directly from Cantor sets. In a spatial representation, the ensuing random process exhibits a transition between Gaussian and fractal behavior. In the fractal regime, the trajectories will, in an average sense, form self-similar clusters. In a temporal representation, the random process exhibits a transition between an asymptotically constant renewal rate and fractal behavior. The fractal regime represents a frozen state with only transient effects allowed and is related to charge transport in glasses.

1. Introduction

We begin by considering a real-space renormalization group transformation for the free energy F^1),

$$F(K) = l^{-d}F(K') + G(K), \quad (1)$$

where K is an interaction parameter in the original system, $K' = K'(K)$ is the interaction parameter in the transformed system, l is the decimation length and d the dimension (so that the transformed system is l^d times larger than the original system), and G is a regular function of K . In terms of a scaling field u , eq. (1) becomes

$$F(u) = l^{-d}F(\lambda u) + G(u), \quad (2)$$

with λ a real, relevant eigenvalue greater than unity. The assumption that the scaling field u is a regular function of K and that G is a regular function of u

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has been used. Iterating eq. (2) n times and taking the limit $n \rightarrow \infty$ gives

$$F(u) = \lim_{n \rightarrow \infty} \left[l^{-nd} F(\lambda^n u) + \sum_{j=0}^{n-1} l^{-jd} G(\lambda^j u) \right]. \quad (3)$$

It is usually assumed as a boundary condition that the first term on the r.h.s. vanishes, so that

$$F(u) = \sum_{j=0}^{\infty} l^{-jd} G(\lambda^j u). \quad (4)$$

Thus all singular behavior will be contained in the sum involving G .

A solution for the singular part of $F(u)$ is obtained by writing

$$F_{\text{sing}}(u) = A(u)|u|^{d/y}, \quad (5)$$

which when substituted into eq. (1) leads to the relations

$$y = \ln \lambda / \ln l \quad (6)$$

and

$$A(u) = A(\lambda u) = \sum_n A_n \exp(2\pi i n \ln u / \ln \lambda), \quad (7)$$

where $A(u)$ need not be constant and may in general be periodic in $\ln u$ with period $\ln \lambda$. The relevant eigenvalue λ is greater than unity, so y will be positive. The existence of the oscillatory solutions was first noted by Jona-Lasinio²), Nauenberg³), and Niemeijer and van Leeuwen¹). In critical phenomena the eigenvalue λ is chosen to depend on l , so that the exponent y will not depend on l . The oscillatory terms are discarded because their period depends through $\lambda(l)$ on the decimation length l , while the free energy should be independent of l . Nevertheless, the oscillations are present in the general mathematical solution.

We shall now derive and examine the behavior of random processes which obey scaling equations like eq. (2). We consider a spatial representation for the random process in section 2, and a temporal representation in section 3. In section 4 we show how the self-similarity is an embodiment of a Cantor set.

2. Spatial scaling

For our random process we consider a Markovian random walk on an infinite, one-dimensional periodic lattice with unit bond length. The probability $P_{n+1}(l)$ of being at site l after $n + 1$ steps is given by

$$P_{n+1}(l) = \sum_{l'} P_n(l') p(l - l'), \quad (8)$$

where $p(l)$ is the probability of a vector displacement l at any step. A discrete

Fourier transform of eq. (8) leads to

$$\tilde{P}_n(k) = \sum_l \exp(ikl)P_n(l) = [\tilde{p}(k)]^n \tilde{P}_0(k), \tag{9}$$

where

$$\tilde{p}(k) = \sum_l \exp(ikl)p(l) \tag{10}$$

is called the structure function. Thus, except for the initial position, $\tilde{p}(k)$ contains all the information about the random walk.

We shall incorporate scaling into $\tilde{p}(k)$ by choosing⁴⁾ for the normalized single step probability distribution

$$p(l) = \frac{a-1}{2a} \sum_{n=0}^{\infty} a^{-n} [\delta_{l,b^n} + \delta_{l,-b^n}], \tag{11}$$

with $a, b > 1$ and b an integer. The corresponding structure function is

$$\tilde{p}(k) = \frac{a-1}{a} \sum_{n=0}^{\infty} a^{-n} \cos(b^n k) \tag{12}$$

and satisfies the following scaling relation, similar to eq. (2):

$$\tilde{p}(k) = a^{-m} \tilde{p}(b^m k) + \frac{a-1}{a} \sum_{j=0}^{m-1} a^{-j} \cos(b^j k), \tag{13}$$

where m is an arbitrary positive integer. The reader should contrast eq. (12) with a nearest-neighbor hopping random walk, for which $\tilde{p}(k) = \cos(k)$ and $P_n(l)$ is Gaussian for large n . Eq. (11) allows for jumps of all orders of magnitude, with each succeeding order of magnitude displacement occurring with an order of magnitude less probability. The walker makes about “ a ” jumps of unit length in a region, forming a cluster of sites visited, before making a jump of length “ b ” to begin a new cluster. Then about “ a ” clusters, each with about “ a ” sites and separated from each other by a distance “ b ”, are formed before a jump of length b^2 occurs, etc. At least for a modest number of steps, clustering will occur. We now consider the conditions under which the set of sites visited will remain in the form of self-similar clusters as the number of steps taken tends to infinity, and how this is related to eq. (13), the scaling equation for $\tilde{p}(k)$.

First, the usual Gaussian behavior (and so no clustering) is obtained after many steps when the second moment \bar{l}^2 of $p(l)$ is finite, where

$$\bar{l}^2 \equiv \sum l^2 p(l) = -\frac{\partial^2}{\partial k^2} \tilde{p}(k=0). \tag{14}$$

If \bar{l}^2 is finite, then near $k = 0$ the structure function $\bar{p}(k)$ will have the expansion

$$\bar{p}(k) = 1 - \frac{1}{2}\bar{l}^2 k^2 + O(k^2). \tag{15}$$

(Our random walk is symmetric, so the first moment is zero.) Second, the probability of reaching any site in a walk of infinite duration must be less than unity, and this is known⁵⁾ to be equivalent to the probability R of eventual return to the origin being less than unity, where

$$R = 1 - u^{-1} \tag{16}$$

and in one dimension⁶⁾

$$u = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{dk}{1 - \bar{p}(k)}. \tag{17}$$

Thus if self-similar clustering of the sites visited is to occur it is necessary that

$$(i) \bar{l}^2 = \infty; \quad \text{and} \quad (ii) u < \infty.$$

The second of these conditions requires, in view of eq. (17), that $k/[1 - \bar{p}(k)] \rightarrow 0$ as $k \rightarrow 0$. The two conditions depend on the behavior of $\bar{p}(k)$ and its first derivative near $k = 0$.

We return to eq. (13) for $\bar{p}(k)$ and examine its similarity to the renormalization group scaling equation (2). First, we note that $b > 1$ is necessary, which corresponds to $\lambda > 1$ (i.e. λ a relevant eigenvalue) in eq. (2). When clustering occurs, the number of sub-clusters within a distance b^m of the origin is, on the average, about a^m . This corresponds to l^d in eq. (2), the number of blocks in the old system which fit into a single block of the new system. In the limit $m \rightarrow \infty$ in eq. (13) we have an explicit example of the behavior postulated in eq. (4) that the first term on the r.h.s. of eq. (13) goes to zero, while all the singular behavior of $\bar{p}(k)$ now resides in the infinite sum over analytic functions. We expect interesting singular behavior from eq. (12) because it is simply related to the lacunary Taylor series⁷⁾

$$S(z) = \sum_{n=0}^{\infty} a^{-n} z^{b^n} \tag{18}$$

which cannot be analytically continued beyond its circle of convergence $|z| = 1$ because it has a dense set of singular points at $z = \exp(2\pi i h/b^m)$ for all integer h and m . The singular part of $\bar{p}(k)$ satisfies the equation

$$\bar{p}_{\text{sing}}(k) = a^{-m} \bar{p}_{\text{sing}}(b^m k), \tag{19}$$

with the solution

$$\tilde{p}_{\text{sing}}(k) = |k|^H Q(k), \tag{20}$$

where

$$H = \ln a / \ln b \tag{21}$$

and

$$Q(k) = Q(bk), \tag{22}$$

i.e. $Q(k)$ is periodic in $\ln |k|$ with period $\ln b$, in analogy with eqs. (5)–(7). In view of the correspondence between eqs. (12) and (18) it is clear that $Q(k)$ cannot be identically constant, and a complete determination of $Q(k)$ is given in appendix A. If $b^2 > a$, then $0 < H < 2$ and

$$\tilde{p}(k) = 1 + |k|^H Q(k) + \mathcal{O}(k^2) \quad \text{as } k \rightarrow 0, \tag{23}$$

which is consistent with $\bar{l}^2 = \infty$ since

$$\bar{l}^2 = \frac{a-1}{a} \sum_{n=0}^{\infty} (b^2/a)^n = -\frac{\partial^2}{\partial k^2} \tilde{p}(k=0). \tag{24}$$

In fact, $\tilde{p}(k)$ is Weierstrass' example of a function which is continuous, but nowhere differentiable when $b/a \geq 1$, and eq. (23) is just a demonstration of this highly non-analytic behavior at $k = 0$. The fact that \bar{l}^2 is infinite shows that $p(l)$ decays very slowly.

We note that H corresponds to the fractal (Hausdorff–Besicovitch) dimension⁸⁾ of the clusters, since there are “ a ” sub-clusters per cluster, each scaled down by the factor “ b ”. For our one-dimensional walk, in the case $0 < H < 1$, the conditions $\bar{l}^2 = \infty$ and $u < \infty$ are met and a clustering of the sites visited endures for a walk of infinite duration. In two dimensions, an appropriate analog of the walk⁹⁾ yields clusters when $0 < H < 2$. If \bar{l}^2 is finite then $P_n(l)$ will be a Gaussian, with variance $n\bar{l}^2$.

The scaling equation for $\tilde{p}(k)$ gives rise to a fractional exponent H , but the existence of a fractional exponent does not imply that scaling will hold. For example, if b and β are unequal integers, then the structure function

$$\tilde{p}(k) = \frac{1}{2} \sum_{n=0}^{\infty} \left[\frac{a-1}{a^{n+1}} \cos(b^n k) + \frac{\alpha-1}{\alpha^{n+1}} \cos(\beta^n k) \right] \tag{25}$$

will not satisfy a scaling equation of the form (12). The random walk defined by eq. (25) corresponds to the mixing of two distinct hierarchies of self-similar clusters, and the resultant trajectory will not be self-similar. However, if

$$\ln a / \ln b = \ln \alpha / \ln \beta = H, \tag{26}$$

$\bar{p}(k)$ still has an expansion with a fractional exponent as $k \rightarrow 0$,

$$\bar{p}(k) - 1 \sim \frac{1}{2}|k|^H [Q_b(k) + Q_\beta(k)], \tag{27}$$

where $Q_b(k)$, $Q_\beta(k)$ are periodic in $\ln |k|$ with periods $\ln b$ and $\ln \beta$ respectively, but the fractional exponent H does not give a complete qualitative description of the trajectory of the random walk [unlike the walk generated by eq. (12)].

3. Temporal scaling

Consider a random process consisting of a sequence of events, with $\psi(t)$ the waiting time density between events, i.e. $\psi(t) dt$ is the probability that an event occurs in the time interval $(t, t + dt)$, given that the most recent previous event occurred at time $t = 0$. The probability density for the time of occurrence of the n th event is

$$\psi_n(t) = \int_0^t \psi_{n-1}(t - \tau)\psi(\tau) d\tau, \tag{28}$$

so that in Laplace transform space, with

$$\psi^*(\epsilon) = \int_0^\infty e^{-\epsilon t} \psi(t) dt, \quad \psi_n^*(\epsilon) = \int_0^\infty e^{-\epsilon t} \psi_n(t) dt, \tag{29}$$

we have, for a process started at $t = 0$,

$$\psi_n^*(\epsilon) = [\psi^*(\epsilon)]^n. \tag{30}$$

These equations are the temporal analogs of eqs. (8)–(10) and constitute, in the terminology of applied probability theory, a renewal process. We assume here that the process is non-defective, i.e. that there is probability zero that the process terminates at any stage. The standard example of such a process has the exponential density

$$\psi(t) = b e^{-bt}, \tag{31}$$

giving a Poisson distribution for the number of events occurring up to time t .

We now incorporate scaling into the waiting time density $\psi(t)$ by writing

$$\psi(t) = \frac{1-a}{a} \sum_{n=1}^\infty a^n b^n \exp(-b^n t), \tag{32}$$

with $a, b < 1$. The evolution of the process is governed by a set of competing

rates, the rate b^n occurring with a probability proportional to a^n , $n = 1, 2, \dots$. The scaling inherent in eq. (32) is most easily exhibited in Laplace transform space:

$$\psi^*(\epsilon) = \frac{1-a}{a} \sum_{n=1}^{\infty} \frac{(ab)^n}{b^n + \epsilon} \tag{33}$$

and

$$\psi^*(\epsilon) = a^m \psi^*(\epsilon/b^m) + \frac{1-a}{a} \sum_{j=1}^m \frac{a^j}{1 + \epsilon/b^j}, \tag{34}$$

where m is an arbitrary positive integer. This is an analog of eq. (2). For $m = \infty$ the first term on the r.h.s. vanishes and any singular behavior must reside in the second term, as is assumed in eq. (4). If the first moment \bar{t} of $\psi(t)$ is finite, then for small ϵ ,

$$\psi^*(\epsilon) = 1 - \epsilon \bar{t} + \mathcal{O}(\epsilon). \tag{35}$$

To maintain the self-similar clustering between event times one needs $\bar{t} = \infty$. The decay of $\psi(t)$ must be slow enough that there is no finite time scale.

The singular behavior of $\psi^*(\epsilon)$ satisfies the equation

$$\psi_{\text{sing}}^*(\epsilon) = a^m \psi_{\text{sing}}^*(\epsilon/b^m), \tag{36}$$

so that

$$\psi_{\text{sing}}^*(\epsilon) = \epsilon^H K(\epsilon), \tag{37}$$

with

$$H = \ln a / \ln b \tag{38}$$

and $K(\epsilon)$ a periodic function of $\ln \epsilon$ with period $\ln(b^{-1})$. If $a > b$, then $0 < H < 1$ and

$$\psi^*(\epsilon) = 1 + \epsilon^H K(\epsilon) + \mathcal{O}(\epsilon), \tag{39}$$

which is consistent with $\bar{t} = \infty$, since

$$\bar{t} = \frac{1-a}{a} \sum_{n=1}^{\infty} (a/b)^n = -\frac{\partial}{\partial \epsilon} \psi^*(\epsilon = 0). \tag{40}$$

The complete behavior of $\psi^*(\epsilon)$ is given in appendix B. On the average, a^{-1} events occur, separated by a time b^{-1} , before an event separated by a time b^{-2} ; a^{-1} clusters of this type occur before a waiting time of b^{-3} , etc., with the proper weighting of long waiting times to make $\bar{t} = \infty$. Thus H can be viewed, in an average sense, as a fractal (Hausdorff-Besicovitch) dimension.

Since moments of order less than H are convergent and those of order

greater than H are divergent, the waiting time density defined by eq. (32) is effectively $\mathcal{O}(t^{-1-H})$ at large times. Long-tailed distributions have been very successful in describing charge transport in xerographic films¹⁰⁻¹³). Under the influence of an electric field, the transient current from photo-generated charges is measured and found to have a universal behavior, as well as other interesting properties. The current $I(t)$ is universal in the sense that plots of $\ln[I(t)/I(t_0)]$ vs. $\ln(t/t_0)$ [where t_0 is a transit time across the sample] are identical for different xerographic materials, independent of the sample size and the electric field strength. This can be shown to imply that the mobility of the charges must depend on these parameters, which is not the case in standard transport theories. The current is modelled in terms of charges hopping between spatially disordered sites. The transition rate for a jump will depend on the random distribution of neighboring sites, and a wide distribution of hopping rates is to be expected. The hopping time density is found to be, in the experimental time regime,

$$\psi(t) \sim \text{constant} \cdot t^{-1-H}, \quad 0 < H < 1, \quad (41)$$

so \bar{t} is infinite. Under these circumstances, the distance travelled by a charge in time t , $\langle l(t) \rangle$, and the fluctuation in this distance, $\sigma(t)$, have the same long-time power-law behavior:

$$\langle l(t) \rangle \propto \sigma(t) \propto t^H. \quad (42)$$

Thus the amount of fluctuation per unit distance travelled is independent of time. This is the origin of the universal behavior of the current traces. Such universal behavior would not occur if the motion were governed by statistics with \bar{t} and \bar{t}^2 finite, since then $\sigma(t)/\langle l(t) \rangle \propto t^{-1/2}$ for a random walk with a bias.

4. Cantor sets

Cantor¹⁴) originally was interested in examining whether or not trigonometric series could represent functions which had divergent or discontinuous behavior at a set of points. He soon became more interested in the way he could choose sets of points than in the trigonometric series, and developed his theory of transfinite numbers. In his search for an understanding of the continuum he introduced in 1883 his famous ternary set $\{Z\}$, the set of all points in the interval $[0, 1]$ which can be written in base 3 without the use of the number 1, i.e.

$$Z = \frac{c_1}{3^1} + \frac{c_2}{3^2} + \cdots + \frac{c_n}{3^n} + \cdots, \quad (43)$$

where $c_n = 0, 2$. This set can be put into a one-to-one correspondence with the real numbers, is closed, and every point is a limit point, but the set has measure zero and there is no interval in which the points of $\{Z\}$ are dense. The Cantor set can be viewed graphically in terms of the Cantor bar⁸) as shown in fig. 1. The remaining points appear in a self-similar pattern. The Cantor set may be defined over the positive real numbers instead of the finite interval $[0, 1]$ if after each iteration we expand our interval by a factor of 3, so that the left-most bar is always of unit length. After an infinite number of iterations, this ternary Cantor bar has a fractal dimension of $\ln 2/\ln 3$, since each bar is broken into 2 pieces and each piece is scaled down by a factor of 3. The self-similarity of a Cantor bar has been directly incorporated into a functional form in our probability distributions $p(l)$ in eq. (11) and $\psi(t)$ in eq. (32), except that now the cut-outs are made such that the final set has a fractal dimension $H = \ln a/\ln b$.

The relationship with the Cantor set is most straightforward when a [for $p(l)$] or a^{-1} [for $\psi(t)$] is integral (corresponding to a or a^{-1} subclusters per cluster respectively), but for general values of a the relationship is also evident if we choose any integer $N > 1$ such that

$$H = \ln a/\ln b = \ln N/\ln(b^{\ln N/\ln a}). \tag{44}$$

Our fractal dimension H in sections 2 and 3 is then derived from a Cantor bar for which each line is cut into N parts, with each part scaled down by a factor $b^{\ln N/\ln a}$. The precise Cantor set clustering is used to determine the way jump distances or jump rates enter a probability distribution, and the clustering in space or time which results is only in an average sense that occurring in the Cantor set.

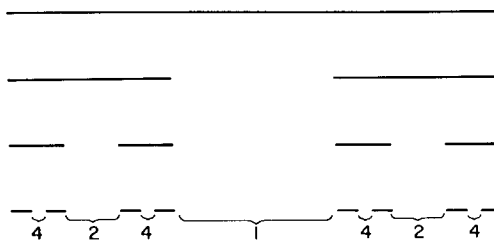


Fig. 1. The Cantor bar. The middle third of each line segment is cut out in each iteration. The first three iterations are shown here. The longest interval between points of the set occurs 2^0 times. The next longest interval is a factor of 3^1 smaller and occurs 2^1 times. The shortest interval is 3^2 times smaller than the longest interval and occurs 2^2 times as often. The points remaining after an infinite number of iterations form a self-similar pattern with fractal dimension $\ln 2/\ln 3$.

5. Conclusions

We have shown how the form and properties of a real-space renormalization group scaling equation for the free energy can be incorporated into random processes. However the existence of a scaling equation for a random process is not in itself sufficient to ensure a transition away from the standard spatial or temporal behavior. For spatial scaling it is necessary in addition that the second moment of the jump distribution be infinite, and for temporal scaling it is necessary that the first moment of the waiting time between events be infinite. Under these conditions, for a random walk with spatial scaling, the sites that are visited form (on the average) a fractal set of self-similar clusters, provided that the probability of reaching any given site is less than unity. The fractal dimension of the clusters is the exponent of $|k|$ appearing in the dominant term of the small k expansion of $1 - \bar{p}(k)$, where $\bar{p}(k)$ is the structure function. Similarly, when temporal scaling occurs the set of times between events shows (on the average) self-similar clustering, the pattern possessing an average fractal dimension which appears as the exponent of ϵ in the small ϵ expansion of $1 - \psi^*(\epsilon)$, where $\psi^*(\epsilon)$ is the Laplace transform of the waiting time distribution. We have given examples in which clustering arises from long-tailed distributions which have infinite moments. Furthermore, we have shown how the self-similar scaling has its mathematical basis in the distribution of points in a Cantor set. We have chosen and weighted parameters in distribution functions according to the length and weight of intervals in Cantor sets.

Finally, we have shown in the spatial scaling case, strong enough disorder in the jump distribution can yield an ordering of the number of sites visited in space. A strong enough randomness in the waiting-time distribution leads to a freezing of motion (i.e. the mean time between events is infinite), except for transient effects. In a rapid quench from a disordered to a partially ordered phase, the system can first acquire some degree of order and then reach a frozen (metastable) state. However, if the quench is sufficiently rapid, the disordered phase may be frozen in^{15,16}). Perhaps these types of processes can be fruitfully viewed as a competition between the onset of fractal spatial behavior and the onset of fractal temporal behavior.

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Appendix A

The structure function

An expansion of the cosine in eq. (12) in powers of k^2 fails when $b^2 > a$ because all the coefficients of powers of k^2 will be infinite. We know that the correct expansion must have non-analytic behavior consistent with eqs. (20)–(22) and we give only a sketch of its derivation here, since a rigorous discussion appears elsewhere⁹). We replace $\cos(b^n k)$ in eq. (12) by its inverse Mellin transform, so that

$$2\pi i \tilde{p}(k) = \frac{a-1}{a} \sum_{n=0}^{\infty} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s) \cos(\pi s/2)}{a^n b^{sn} |k|^s} ds, \tag{A.1}$$

for $0 < c = \text{Re } s < 1$. Interchanging the sum and integral we obtain

$$2\pi i \tilde{p}(k) = \frac{a-1}{a} \int_{c-i\infty}^{c+i\infty} \frac{|k|^{-s} \Gamma(s) \cos(\pi s/2)}{1 - a^{-1} b^{-s}} ds. \tag{A.2}$$

The integrand has poles at $s = 0, -2, -4, \dots$ from the $\Gamma(s) \cos(\pi s/2)$ factor, and poles from the denominator at $s = -\ln a / \ln b \pm 2\pi i n / \ln b$ ($n = 0, 1, 2, \dots$). Translating the contour to $\text{Re } s = -\infty$ and taking account of the residues at the poles crossed, we arrive at

$$\tilde{p}(k) = 1 + |k|^{\ln a / \ln b} Q(k) + \frac{a-1}{a} \sum_{n=1}^{\infty} \frac{(-1)^n k^{2n}}{(2n)! [1 - a^{-1} b^{2n}]}, \tag{A.3}$$

where

$$Q(k) = \frac{a-1}{a \ln b} \sum_{n=-\infty}^{\infty} \Gamma(\kappa) \cos(\pi \kappa/2) \exp(-2\pi i n \ln |k| / \ln b) \tag{A.4}$$

and $\kappa = -\ln a / \ln b + 2\pi i n / \ln b$. When $\ln a / \ln b \leq \frac{1}{2}$, the series (A.4) has to be summed using a convergence factor⁹). The borderline case $\ln a / \ln b = 2$ is easily analysed by similar means.

Appendix B

The waiting time density

When $b < a$, a straightforward expansion of the Laplace transform $\psi^*(\epsilon)$ of $\psi(t)$ in integer powers of ϵ fails, the coefficients of each power of ϵ being infinite. However if we replace $(b^n + \epsilon)^{-1}$ in eq. (33) by its inverse Mellin

transform, so that

$$2\pi i \psi^*(\epsilon) = \frac{1-a}{a} \sum_{n=1}^{\infty} (ab)^n \int_{c-i\gamma}^{c+i\gamma} \frac{\pi}{\sin(\pi s)} \epsilon^{-s} b^{n(s-1)} ds \quad (\text{B.1})$$

($0 < c = \text{Re } s < 1$), an interchange of sum and integral yields

$$2\pi i \psi^*(\epsilon) = \frac{1-a}{a} \int_{c-i\gamma}^{c+i\gamma} \frac{\epsilon^{-s} \pi a b^s}{\sin(\pi s) [1 - ab^s]} ds. \quad (\text{B.2})$$

The integrand has simple poles from $\sin(\pi s)$ at $s = 0, \pm 1, \pm 2, \dots$ and from the other factor in the denominator when $s = -\ln a / \ln b \pm 2\pi i n / \ln b$ ($n = 0, 1, 2, \dots$). Translating the contour to $\text{Re } s = -\infty$ and taking account of the poles crossed we find for $\epsilon < 1$ that

$$\psi^*(\epsilon) = 1 + \epsilon^{\ln a / \ln b} K(\epsilon) + \frac{1-a}{a} \sum_{n=1}^{\infty} \frac{(-1)^n a \epsilon^n}{b^n - a}, \quad (\text{B.3})$$

where

$$K(\epsilon) = -\frac{1-a}{a \ln b} \sum_{n=-\infty}^{\infty} \frac{\pi a b^x}{\sin(\pi x)} \exp(-2\pi i n \ln \epsilon / \ln b) \quad (\text{B.4})$$

and $x = -\ln a / \ln b + 2\pi i n / \ln b$.

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