

Participation ratio in the nonlinear σ -model representation of localization

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Wegner's analysis of the singularities in participation ratios near the mobility edge is extended to the case of spin-flip scattering and of random complex hopping. The critical exponents for the participation ratio and inverse participation ratio in $2+\epsilon$ dimensions are respectively given by $\mu_2=(2\epsilon)^{-1/2}+O(\epsilon)$ and $\pi_2=\epsilon^{-1}-(2\epsilon)^{-1/2}+O(1)$. The calculation is based on a unitary nonlinear σ -model representation for this localization problem.

I. INTRODUCTION

Some time ago Wegner¹ analyzed the concept of participation ratio in the mobility edge problem. This concept was originally introduced as an alternative quantity (besides the conductance) which distinguishes between localized and extended electronic eigenstates.² Wegner's analysis was based on a field-theoretic description (nonlinear σ model) of the Anderson model of electronic disorder which was made possible due to the replica method.³⁻⁶ Making use of the field-theoretic results obtained by Brezin *et al.*,⁷⁻⁹ Wegner showed that the singularity in the (inverse) participation ratio near the metal-insulator transition is very unusual from the point of view of ordinary second-order phase transitions. This singularity derives in a very peculiar fashion from the zero-component field theory in that it is associated with the occurrence of critical operators of dimension larger than the spatial one. Wegner's analysis is both elegant and necessary: his results are very hard (if not impossible) to obtain via other means; furthermore, his analysis is necessary if this field-theoretical description is to give a complete and consistent account on the physics of the problem.

Surprisingly, Wegner's analysis has not been checked for another candidate for the mobility edge problem: namely, the zero-component $U(m+n)/U(m)\times U(n)$ nonlinear σ model, which shows asymptotic freedom in two dimensions.⁷ This field theory applies to problems in which time-reversal invariance is broken; in particular, in case spin-flip scattering is significantly present¹⁰⁻¹³ and in case of random complex hopping in tight-binding models.^{4,14,15} Furthermore, in the more recent analysis of the integral quantum Hall effect¹⁶⁻¹⁸ this field theory plays a central, nonperturbative role.

In this paper, I consider the critical behavior in the (inverse) participation ratio for the aforementioned localization problems. In Sec. II I briefly review Wegner's scaling equations and state the results of this work. In Sec. III, I give the details of the calculation of critical exponents, using the analysis of Brezin *et al.*⁷ as a starting point.

B. Scaling relations and results

As a general remark, information on the nature of the electron's eigenstates is contained in the two-particle Green's function^{1,2}

II. PARTICIPATION RATIO

A. Model

This paper concerns the following one-particle Hamiltonians:

$$H = \frac{1}{2m} p^2 + \vec{h}(r) \cdot \vec{\tau}, \quad (1)$$

where $\vec{h}(r)$ denotes a random field due to magnetic impurities and $\vec{\tau}$ denotes the three Pauli matrices. Furthermore,

$$H = f(r, \sigma, r', \sigma'), \quad (2)$$

which is defined on a lattice; f is a random Hermitian matrix with matrix indices r (lattice site) and $\sigma = \pm 1$ (spin degree of freedom). The Hamiltonians of Eqs. (1) and (2) have in common that they are random and Hermitian. In both cases, the effective Lagrangian for the description of the transport quantities is given by the σ model with unitary symmetry,^{4,10,14,15,12} as mentioned earlier. This is in contrast to the original Anderson model, which is defined in terms of a real-symmetric Hamiltonian. The appropriate effective field theory for the latter is given by a σ model with orthogonal symmetries, specifically $O(n,m)/O(n)\times O(m)$ in the limit $n,m \rightarrow 0$.^{5,6}

Quite commonly, the Hamiltonians given in Eqs. (1) and (2) are referred to as localization problems in which time-reversal symmetry is broken. Such a statement is true only in a local sense. At macroscopic length scales, however, all possible random configurations do occur (we assume short-ranged correlated randomness) and from this we can conclude that time-reversal symmetry has been restored. This is a subtle but essential difference with the localization problem in which a constant magnetic background field is present. In this case, time-reversal symmetry is truly broken, i.e., at microscopic as well as macroscopic length scales. The consequences for scaling have been discussed in Refs. 16-18.

$$K(E, \eta) = \left\langle \left[\text{Tr} \langle r | \left[\frac{1}{E-H+i\eta} - \frac{1}{E-H-i\eta} \right] | r \rangle \right] \right\rangle_{\text{av}}, \quad (3)$$

where $\langle \rangle_{\text{av}}$ indicates an ensemble average over disorder and E, H denote the Fermi energy, respectively, the Hamiltonian of the disordered electronic system. The trace is over the spin degrees of freedom and r denotes the spatial position. That is, the quantity

$$P(E) = \frac{1}{2\pi\rho(E)} [\eta K(E, \eta)]_{\eta \rightarrow 0}, \quad (4a)$$

with $\rho(E)$ representing the density of electronic levels, is the ensemble averaged version of the inverse participation ratio. For a single realization of the ensemble, this quantity is defined as

$$P_i = \sum_r [\text{Tr} |\Psi_i(r)|^2]^2, \quad (4b)$$

with Ψ_i representing the wave function at energy E_i . It is a measure for the (averaged) inverse of the volume that is taken by the state with energy E_i .

On the other hand,

$$p(E) = \frac{4\pi^2\rho(E)^2}{K(E, \eta \rightarrow 0)} \quad (5a)$$

has been introduced as the averaged version of the participation ratio

$$p_i = \frac{1}{N \sum_r [\text{Tr} |\Psi_i(r)|^2]^2}, \quad (5b)$$

which measures the volume fraction taken by the state E_i (N = number of sites). For an extended electronic level, $P(E)$ vanishes in the thermodynamic limit whereas $p(E)$ is presumably positive (and has a maximum value of one). For a localized electronic state, $P(E)$ is positive and $p(E)$ vanishes.

In the neighborhood of the mobility edge, E_c , one expects that scaling behavior¹

$$K(E, \eta) \sim |E - E_c|^{x\nu} f(\eta / |E - E_c|^{d\nu}) \quad (6)$$

if E_c is approached from the extended states regime. The participation ratio falls off to zero, such as

$$P(E) \sim |E - E_c|^{\mu_2}, \quad (7)$$

where the critical exponent μ_2 will be calculated in Sec. III. The result in $2 + \epsilon$ dimensions is

$$\mu_2 = -x\nu = \frac{1}{\sqrt{2\epsilon}} + O(\sqrt{\epsilon}), \quad (8)$$

which is to be compared with Anderson localization¹ (or potential scattering)

$$\mu_2 = +2 + O(\epsilon). \quad (9)$$

The unusual feature of this scaling result is that it derives from a critical operator with a negative critical exponent x .

Approaching the mobility edge from the side of local-

ized states, one expects a scaling behavior¹

$$[\eta K(\eta, E)]_{\eta \rightarrow 0} \sim |E - E_c|^{(x+d)\nu} \left[\frac{\eta}{|E - E_c|^{d\nu}} f(\eta / |E - E_c|^{d\nu}) \right]_{\eta \rightarrow 0} \quad (10)$$

such that in this case, this inverse participation ratio vanishes according to

$$P(E) \sim |E - E_c|^{\pi_2}, \quad (11)$$

$$\pi_2 = (x+d)\nu = \epsilon^{-1} - (2\epsilon)^{-1/2} + O(1), \quad (12)$$

for the spin-flip scattering or complex hopping case. For potential scattering, one has in contrast¹

$$\pi_2 = \epsilon^{-1} - 1 + O(\epsilon). \quad (13)$$

III. NONLINEAR σ -MODEL REPRESENTATION

Averaged two-particle Green's functions, such as Eq. (3), can be obtained from an effective field theory of interacting matrices governed by a Lagrangian^{3-6,14}

$$L = -\frac{1}{t} \int d^d r \text{Tr} \nabla_\mu Q_{ab}^{pp'} \nabla_\mu Q_{ba}^{p'p} + \eta p(E) \int d^d r \text{Tr} (-)^p Q_{aa}^{pp}(r), \quad (14)$$

where Q in this case belongs to the coset space $U(m+n)/U(m) \times U(n)$ and where the trace stands for the sum over a, b (replica indices) which run from 1 to m, n for $p=1, 2$. The quantity $1/t$ plays the role of "bare" conductance and the results for the disordered electronic system are obtained in the limit $m, n \rightarrow 0$.

In this representation, Eq. (3) can be written as

$$K = \sum_{pp'=1,2} (-)^{p+p'} [A \langle Q_{ab}^{pp'}(r) Q_{ba}^{p'p}(r) \rangle + B \langle Q_{aa}^{pp}(r) Q_{bb}^{p'p'}(r) \rangle]_{a \neq b}. \quad (15)$$

Equation (15) expresses the most singular contribution to K as the expectation of a local operator. (There are other contributions involving operators of higher dimensions but these can be neglected as one approaches the mobility edge E_c .) The coefficients A, B are determined by an underlying noncritical theory and are not universal.

According to the theory of nonlinear σ model,^{8,9} a critical index is associated with each irreducible representation of such operators with respect to the symmetry group, which is in this case $U(m+n)$. Notice that the operator (15) can be written in more general form

$$K = \left\langle \sum_{\alpha\beta\gamma\delta} v_{\beta\delta}^{\alpha\gamma} Q_{\alpha\beta}(r) Q_{\gamma\delta}(r) \right\rangle, \quad (16)$$

where $\alpha \dots \delta$ stand for the pair of indices (p, a) , etc. Obviously, v is invariant under interchange of the pairs $(\alpha\beta), (\gamma\delta)$. Furthermore, v , according to (13), is traceless:

$$\sum_i v_{i\delta}^{i\gamma} = \sum_i v_{\beta i}^{i\gamma} = 0. \quad (17)$$

There are two irreducible representations for these operators, a symmetric one

$$v_{\beta\delta}^{\alpha\gamma} = v_{\beta\delta}^{\gamma\alpha} \quad (18)$$

and an antisymmetric one

$$v_{\beta\delta}^{\alpha\gamma} = -v_{\beta\delta}^{\gamma\alpha}. \quad (19)$$

The quantity K will, in general, pick up a contribution from both representations.

The easiest way to calculate the anomalous dimension of these operators is to add a term to the Lagrangian, proportional to these operators, thus providing the necessary infrared regularization. A gauge invariant [i.e., $U(m) \times U(n)$ invariant] regularization is provided by using the representation

$$O_{\pm}(r) = \sum_{\substack{p_1 p_2 \\ ab}} k_{\pm}^{p_1 p_2} [Q_{aa}^{p_1 p_1}(r) Q_{bb}^{p_2 p_2}(r) \pm Q_{ab}^{p_1 p_2}(r) Q_{ba}^{p_2 p_1}(r)], \quad (20)$$

$$k_{\pm}^{p_1 p_2} = \begin{bmatrix} -\frac{m}{n \pm 1} & 1 \\ 1 & -\frac{n}{m \pm 1} \end{bmatrix}. \quad (21)$$

The addition of a term $t^{-1} h_{\pm} \int O_{\pm}(r)$:

$$L \rightarrow \frac{1}{t} \int d^d r [\text{Tr} \nabla_{\mu} Q \nabla_{\mu} Q + h_{\pm} O_{\pm}(r)] \quad (22)$$

leads to a renormalizable theory by power counting in two dimensions.⁸ There are only two renormalization constants needed, namely, the coupling-constant renormalization $t = t_R Z_1$ and a multiplicative renormalization Z_2^{\pm} of the traceless operator O_{\pm} . The final renormalized action reads

$$L \rightarrow \frac{1}{t Z_1} \int d^d r [\text{Tr} \nabla_{\mu} Q \nabla_{\mu} Q + h_{\pm} Z_2^{\pm} O_{\pm}(r)]. \quad (23)$$

The renormalization-group functions are obtained in the standard fashion

$$\beta(t) = \frac{\epsilon t}{1 + t \partial \ln(Z_1) / \partial t}, \quad \epsilon = d - 2 \quad (24)$$

$$\gamma_{\pm}(t) = \beta(t) \frac{\partial}{\partial t} \ln(Z_2^{\pm} / Z_1), \quad (25)$$

whereas the renormalization constants Z_1 and Z_2^{\pm} are determined by requiring that the free energy be finite.⁷ This method is advantageous, because it gives γ_{\pm} one order higher than would be normally obtained in a one-loop computation. The same is true in a calculation of the invariant two-point function which has been discussed and

exploited in Ref. 19.

The parametrization of $Q \in [U(m+n)/U(m) \times U(n)]$ is most conveniently done in terms of the projectors $P = \frac{1}{2} Q + \frac{1}{2}$ leading to $n \times m$ complex components V_{α}^i as independent field components.⁷ In terms of V , the operators $O_{\pm}(r)$ can be written, up to a normalization factor

$$O_{\pm} = -\frac{mn}{2(m+n \pm 1)} + V_{\alpha}^{i*} V_{\alpha}^i - \frac{m+n \pm 2}{2[nm \pm (m+n) + 1]} \times (V_{\alpha}^{i*} V_{\alpha}^i V_{\beta}^{j*} V_{\beta}^j \pm V_{\alpha}^{i*} V_{\beta}^j V_{\beta}^{j*} V_{\alpha}^i) \quad (26)$$

using the convention of summation over repeated indices. On the other hand, the action becomes an infinite series⁷

$$A = \frac{1}{t} \int d^d r \nabla_{\mu} V_{\alpha}^{i*} \nabla_{\mu} V_{\alpha}^i + \frac{1}{2} \nabla_{\mu} V_{\alpha}^{i*} V_{\alpha}^j \nabla_{\mu} V_{\beta}^{j*} V_{\beta}^i + \frac{1}{2} V_{\alpha}^{i*} \nabla_{\mu} V_{\alpha}^j V_{\beta}^{j*} \nabla_{\mu} V_{\beta}^i + O(V^6). \quad (27)$$

The free energy is obtained by treating all but the Gaussian fluctuations as a perturbation. Using dimensional regularization, one gets up to one-loop order

$$f_{\pm} = h_{\pm} Z_2^{\pm} / t Z_1 \frac{1}{2(m+n \pm 1)} - \int_q \ln(1 + h_{\pm} Z_2^{\pm} / q^2) + \frac{1}{2} t h_{\pm} (m+n \pm 2) \times \left[\int_q \frac{1}{q^2 + h_{\pm}} \right]^2 \quad (28)$$

from which one deduces the renormalization constants via minimal subtraction

$$\frac{Z_2^{\pm}}{Z_1} = 1 - 2(m+n \pm 1) \frac{t}{\epsilon} + (m+n \pm 2)(m+n \pm 1) \left[\frac{t}{\epsilon} \right]^2 + O(t^3), \quad (29)$$

$$Z_2^{\pm} = 1 - (m+n \pm 2) \frac{t}{\epsilon} + O(t^2). \quad (30)$$

A factor of 2π has been absorbed in t . Equations (24), (25), (29), and (30) imply for the anomalous dimension

$$\gamma_{\pm}(t) = -2(m+n \pm 1)t + O(t^3). \quad (31)$$

On the other hand, the β function is known to fourth order²⁰

$$\beta(t) = \epsilon t - (m+n)t^2 - 2(mn+1)t^3 - \frac{1}{2}(3nm+7)(m+n)t^4 + O(t^5). \quad (32)$$

The critical exponents ν and x_{\pm} are obtained as

$$1/\nu = -\beta'(t_c) = \epsilon + 2 \frac{nm+1}{(n+m)^2} \epsilon^2 + O(\epsilon^3), \quad (33)$$

$$x_{\pm} = -\gamma_{\pm}(t_c) = 2 \frac{m+n\pm 1}{m+n} \epsilon - 4 \frac{(mn+1)(m+n\pm 1)}{(m+n)^3} \times \epsilon^2 + O(\epsilon^3),$$

(34)

$$t_c = \frac{\epsilon}{m+n} - 2 \frac{mn+1}{(m+n)^3} \epsilon^2 + O(\epsilon^3).$$

In the limit of zero field components we obtain

$$1/\nu = -\beta'(t_c) = 2\epsilon + O(\epsilon^2),$$

(35)

$$x_{\pm} = -\gamma_{\pm}(t_c) = \pm \sqrt{2\epsilon} + O(\epsilon^{3/2}),$$

$$t_2 = \left[\frac{\epsilon}{2} \right]^{1/2} + O(\epsilon).$$

(36)

For $m=n=1$, the unitary model reduces to the $O(3)$ nonlinear σ model. The O_{\pm} functions correspond to a composite operator in the $O(3)$ model and γ_{+}, γ_{-} have been calculated in Ref. 9 as ξ_2 , respectively, ξ_1 .²¹ The critical exponent x in Eqs. (6) and (10) correspond with x_{-} in Eq. (36) since the antisymmetric one yields the dominant contribution.

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²¹Note that Eqs. (20) and (21) are seemingly singular for $m=n=1$. This artifact, however, is cured for after parametrization [Eq. (26)] such that comparison with the $O(3)$ model operator can be made.