

A NOTE ON LOG-PERIODIC OSCILLATIONS OF SPECTRAL QUANTITIES AND THE TRANSMISSION COEFFICIENT OF CANTOR SETS AND FIBONACCI SEQUENCES

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The purpose of this note is to study some specific features of fractal spectra. The exact way we define and calculate those spectra will be defined later, but we can already notice that a fractal spectrum can result from the eigenvalue spectrum of an operator (e.g. a Laplacian) defined on a spatially self-similar (fractal) structure, such as a Sierpinski gasket, but not necessarily. For instance, the spectrum obtained from the product of transfer matrices on a Fibonacci chain and calculated from the transmission coefficient, is fractal (a Cantor set) but the spatial structure is not, it is a chain embedded in the 1D line.

We first start with the fractal triadic Cantor set spectrum, in order to establish on a simple and well known example, the basic and fundamental scaling behavior which leads to log-periodic oscillations. We then consider the more complicated but still straightforward case of the fractal spectrum of the Laplacian defined on a Sierpinski gasket and show that it bears the same behavior. Finally, we consider the general case of a function which obeys scaling behavior with a finite characteristic scale and deduce the characteristic behavior of the resulting log-periodic oscillations and the information they provide on the system.

1. CANTOR SET SPECTRUM AND ASSOCIATED HEAT KERNEL

We start by considering a bounded energy spectrum, say the unit interval $[0, 1]$ and call it the step $n = 0$ of the iteration. The triadic Cantor set is generated iteratively as follows. At step $n = 1$, we divide the initial unit interval into 3 segments of equal length ($1/3$), and remove the middle segment, thus leaving the two extreme ones. For the sake of further simplicity, we consider the so-called discrete case, for which we keep 2 states at $n = 0$: $\epsilon_0^{(0)} = 0$ and $\epsilon_1^{(0)} = 1$. Then, for $n = 1$, we have 4 energy states : $\epsilon_0^{(1)} = 0$, $\epsilon_1^{(1)} = 1/3$, $\epsilon_2^{(1)} = 2/3$ and $\epsilon_3^{(1)} = 1$. We thus generate iteratively an ensemble of two-level systems. The heat Kernel $Z_n(t) = \sum_k e^{-E_k^{(n)}t}$ at the iteration n , can thus be obtained from the iterated spectrum and it is given by,

$$(1) \quad Z_n(t) = \prod_{k=1}^n \left(1 + e^{-2l^k t}\right)$$

where we have given a name to the basic scaling factor $l = 1/3$. It is easy to see that this calculation can be generalized to include a large class of Cantor sets, with different scales

etc. From (1), we notice that the infinitely iterated heat kernel $Z(t) = \lim_{n \rightarrow \infty} Z_n(t)$ fulfills the scaling relation:

$$(2) \quad Z(t/l) = 2Z(t)$$

At this point, it is essential to pay attention to the fact that, unlike continuous phase transitions where the scaling factor l is unconstrained and can thus take any value down to 1, for fractals, the scaling factor l corresponds to the structure of the fractal, here $l = 1/3$ for the triadic Cantor set, and (2) is valid only for this specific characteristic value.

2. ENERGY SPECTRUM OF THE DIAMOND FRACTAL

In the previous section, we have generated a discrete spectrum (at each step of the iteration) using the iteration of the triadic Cantor set and obtained as a result a scaling behavior for the associated heat kernel. This scaling is not restricted to the case of a discrete spectrum. Consider now the standard real space self similar (fractal) diamond obtained by iterating the initial unit interval. The spectrum of the Laplacian defined on such a structure (there is a large literature on this problem) is obtained from the spectrum of the unit interval with Dirichlet (vanishing) boundary conditions, $\epsilon_k = \pi^2 k^2$, k being a positive integer. The corresponding heat kernel is thus

$$(3) \quad Z_0(t) = \sum_{k=1}^{\infty} e^{-\pi^2 k^2 t}$$

Unlike the previous case, we have at the $n = 0$ step of the iteration, an infinite number of energy states. The iterated spectrum is obtained along similar lines as for the Cantor set, so that the resulting heat kernel is

$$(4) \quad Z(t) = Z_0(t) + B \sum_{n=0}^{\infty} l^{d_h n} Z_0(l^{d_w n} t)$$

where, as before, l is the scaling factor. $d_h = 2$ is the fractal dimension of the fractal itself and $d_w = 2$ is the so-called walk dimension, and $B = l^{d_h - 1} - 1$ is a constant. It is easy to check that $Z(t)$ follows a scaling relation as in (2),

$$(5) \quad Z(t) = Z_0(t) + l^{d_h} Z(l^{d_w} t)$$

To check this, we iterate (5), and defining $p = l^{d_h}$ and $q = l^{d_w}$, we have

$$(6) \quad Z(t) = \sum_{n=0}^{\infty} p^n Z_0(q^n t)$$

which is (4) (+ boundary conditions).

3. LOG-PERIODIC OSCILLATIONS OF SPECTRAL QUANTITIES

The aim of this section is to show that scaling behavior which is the characteristic of a fractal spectrum implies necessarily that $Z(t)$ is of the form

$$(7) \quad Z(t) = t^{-d_h/d_w} F(t)$$

where the function $F(t)$ fulfills $F(qt) = F(t)$, namely, $F(t)$ is a periodic function of $\ln t$ of period $\ln q = d_w \ln l$.

There are different ways to show this, which are more or less formal. One of them, we propose here results from a variational calculation.

Consider the heat kernel previously defined in (6), but now without specifying the function $f(t)$:

$$(8) \quad Z(t) = \sum_{n=0}^{\infty} p^n f(q^n t)$$

in the limit $t \rightarrow 0$. To estimate Z , we use a saddle point approximation to find the value of n which dominates Z . By differentiating with respect to n , we obtain the relation

$$(9) \quad (\ln p)f + (\ln q)q^n t f' = 0$$

Define $u = q^n t$, so that the previous equation becomes,

$$(10) \quad (\ln p)f(u) + (\ln q)u f'(u) = 0$$

This equation in the variable u has a solution (just by considering explicitly the expression of f), so that there exists a solution N to (9), such that $u = q^N t$ and then

$$(11) \quad N = \frac{\ln u - \ln t}{\ln q}$$

N is not necessarily an integer and it can be written as $N = n_0 + x$, where n_0 is an integer and $0 \leq x \leq 1$. Then, we can rewrite Z in (8) under the equivalent form,

$$(12) \quad \begin{aligned} Z(t) &= \sum_{n=0}^{\infty} p^n f(q^n t) = \sum_{m=-n_0+1}^{\infty} p^{n_0+m} f(q^{n_0+m} t) \\ &= \sum_{m=-n_0+1}^{\infty} p^{N+m-x} f(q^{N+m-x} t) \end{aligned}$$

With the notations that have been used and (11), $p^N = \left(\frac{u}{t}\right)^{d_h/d_w}$, and

$$(13) \quad Z(t) = \left(\frac{u}{t}\right)^{d_h/d_w} \sum_m p^{m-x} f(q^{m-x} u)$$

Notice that in Eq.(11), u and q are fixed, so that when $t \rightarrow 0$, then $n_0 \rightarrow -\infty$, so that in the previous relation we have,

$$(14) \quad Z(t) = \left(\frac{u}{t}\right)^{d_h/d_w} \sum_{m=-\infty}^{\infty} p^{m-x} f(q^{m-x} u)$$

where the series is now a periodic function of x of period 1, namely using (11), a periodic function of $\frac{\ln t}{\ln q} = \frac{\ln t}{d_w \ln l} = \ln t^{1/d_w \ln l}$ of period 1.

We have considered in this note the behavior of the heat kernel. But all other spectral quantities are derived from it using well known expression and as thus, other spectral quantities, such as the density of states of the counting function (its integral), also display the oscillating behavior of $Z(t)$.