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Slow Decay of Temporal Correlations in Quantum Systems with Cantor Spectra

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We prove that the temporal autocorrelation function $C(t)$ for quantum systems with Cantor spectra has an algebraic decay $C(t) \sim t^{-\delta}$, where δ equals the generalized dimension D_2 of the spectral measure and is bounded by the Hausdorff dimension $D_0 \geq \delta$. We study various incommensurate systems with singular continuous and absolutely continuous Cantor spectra and find extremely slow correlation decays in singular continuous cases ($\delta = 0.14$ for the critical Harper model and $0 < \delta \leq 0.84$ for the Fibonacci chains). In the kicked Harper model we demonstrate that the quantum mechanical decay is unrelated to the existence of classical chaos.

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The temporal decay of correlation functions plays an important role in classical physics, as it can be used in ergodic theory to define mixing, a somewhat weaker property than chaotic behavior. As quantum mechanics precludes sensitive dependence on initial conditions, the possibility that quantum systems might exhibit mixing behavior through their decay of correlations has attracted considerable attention in recent years. The situation is far more complex than in classical physics and some of the investigations have arrived at controversial conclusions [1]. The complication is due to the absence of one-to-one relations between the nature of the decay and the spectral type (absolutely continuous, singular continuous, pure point, or any mixture). Only decays faster than any power law can be uniquely related to an absolutely continuous spectrum. Slow power-law decays, however, can be compatible with a singular continuous spectrum, as well as an absolutely continuous spectrum. Therefore, in order to relate the decay to the spectral type, Avron and Simon [2] had to introduce a distinction between "transient" and "recurrent" absolutely continuous spectra. Many problems, however, still remain to be solved. In this situation it is useful to investigate the correlation decay of various systems for which the spectral types are known and to develop a new general concept for a *quantitative* determination of the correlation decay.

In this Letter we first analyze the decay of the correlation function $C(t)$ in the unkicked and kicked Harper model for localized, critical, and extended states, as well as in the Fibonacci chains. Numerically we find slow algebraic decays $C(t) \sim t^{-\delta}$ with $0 < \delta \leq 0.84$. This is the first quantitative determination of the correlation decay in these systems and confirms the conjecture of anomalous transport. For the regime of extended states of the Harper model the power-law decay has an exponent $\delta = 0.84 \pm 0.01$ reflecting a recurrent absolutely continuous spectrum, whereas the singular continuous spectrum in the critical case gives rise to an extremely slow decay with $\delta = 0.14 \pm 0.01$. The singular continuous spectrum of the Fibonacci chains shows variable exponents $0 < \delta \leq 0.84$ with δ approaching 0.84 as $V \to 0$, where the spectrum becomes absolutely continuous. In the kicked Harper model, which is classically chaotic, we demonstrate that the decay of the quantum correlation function is unaffected and cannot be interpreted as a signature for the existence of classical chaos. We then show analytically that for Cantor set spectra (singular continuous or absolutely continuous) the correlation function decays algebraically,

$$C(t) \sim t^{-\delta}, \tag{1}$$

that δ is given by the generalized dimension D_2 of the spectral measure

$$\delta = D_2 , \tag{2}$$

and that the Hausdorff dimension D_0 gives an upper bound $D_0 \geq \delta$. Thus, while there are no one-to-one relations between spectral types and the decay of correlations, there does exist a relation determining the correlation decay from multifractal properties of the spectrum. This relation is simple and of general validity for fractal spectra of all types. In other words, the specific spectral type is rather irrelevant in the present context, whereas the multifractal concept is relevant.

Our first example is Harper's equation as a model for Bloch electrons in a magnetic field B in the framework of the Peierls substitution [3,4]. In this system we have recently found a new class of level statistics [5], where the level spacing distribution follows an inverse power law $p(s) \sim s^{-3/2}$. This behavior is related to an unbounded diffusive spread $\sim 2Dt$ of wave packets and persists under classically chaotic perturbations [6]. The model is defined by a discrete Schrödinger equation

$$\psi_{n+1} + \psi_{n-1} + \lambda \cos(2\pi n \sigma - \varphi_0) \psi_n = \omega \psi_n , \tag{3}$$

where ψ_n is the wave function at site n. The dimensionless parameter $\sigma = a^2 eB/hc$ gives the number of flux quanta per unit cell of area a^2 and determines the incommensurability of the system. It is known that $\lambda = 2$ is a critical case [7] separating a regime of extended states ($\lambda < 2$) from a regime of localized states ($\lambda > 2$) for irrational σ [8]. For $\lambda = 2$ the states are neither localized nor extended, but are called critical. The spectrum is a Cantor set for a dense set of parameter pairs (λ, σ) [9].

For rational values of $\sigma = r/q$ we use the transfer matrix method [10] to obtain the q eigenenergies and q eigenfunctions $\psi_{n,k}$ $(k = 1, 2, \ldots, q)$ of the system [11]. In this case the probability to be in the initial state $|\phi(t=0)\rangle$ at time t can be written as

$$p(t) = |\langle \phi(0) | \phi(t) \rangle|^2 = \left| \sum_k |a_k|^2 e^{-i\omega_k t} \right|^2 , \tag{4}$$

where we used the spectral decomposition $\phi_n(t=0) = \sum_k a_k \psi_{n,k}$. We define a temporal autocorrelation function $C(t)$ by the smoothened probability to be in the initial state at time t,

$$C(t) = \frac{1}{t} \int_0^t |\langle \phi(0) | \phi(t') \rangle|^2 dt' , \tag{5}$$

which is shown in Fig. 1 for σ an approximant of the golden mean and three different values of λ. All numerical simulations were started with localized initial wave packets. For $\lambda = 2$ we find an extremely slow decay of the correlation function following a power law $C(t) \sim t^{-\delta}$ with $\delta = 0.14 \pm 0.01$. For $\lambda < 2$ the correlation decays with $\delta = 0.84 \pm 0.01$, while the extended eigenstates and the absolutely continuous spectrum might have suggested an exponential decay. The slow power-law decay with $\delta = 0.84$ thus must be a manifestation of a recurrent absolutely continuous spectrum [2]. The singular continuous spectrum for $\lambda = 2$, on the other hand, has an even slower decay ($\delta = 0.14$). For $\lambda > 2$ there is no decay, as expected for localized states.

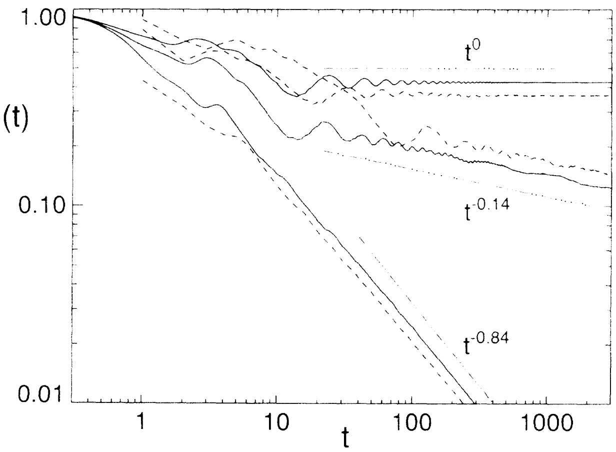

FIG. 1. Correlation function in the Harper model (solid lines) for $\sigma = 1597/2584$, an approximant of the golden mean, displaying power laws $C(t) \sim t^{-\delta}$ with $\delta = 0.84 \pm 0.01$, $\delta = 0.14 \pm 0.01$, and $\delta = 0$ for $\lambda = 1$, 2, and 3, respectively. For the kicked Harper model (dashed lines) with σ the golden mean, the same asymptotic behavior is obtained for $K = 6$ and $L = 3$, 6, and 9 corresponding to the extended, critical, and localized regimes, respectively.

In order to study the influence of classical chaos on the decay of $C(t)$ we numerically investigate the kicked Harper system, which has a classically chaotic phase space [6,12,13]. It is described by

$$\hat{H} = L \cos(\hat{p}) + K \cos(\hat{x}) \delta_1(t) , \tag{6}$$

where $\delta_1(t)$ is a periodic delta function of period one and $\hat{p} = -i\hbar\, \partial/\partial x$ plays the role of a momentum operator with an effective $\hbar = 2\pi\sigma$. We have shown that in the limit $L/\hbar, K/\hbar \to 0$ the unkicked Harper model with $\lambda = 2L/K$ is recovered [6]. The wave packet at integer time t is obtained conveniently by t iterative applications of the time evolution operator $\hat{U}$ for one period,

$$\hat{U} = e^{-i(L/\hbar)\cos(\hat{p})} e^{-i(K/\hbar)\cos(x)} . \tag{7}$$

The results for the correlation function Eq. (5) are shown by the dashed lines in Fig. 1 for parameters in the strongly chaotic regime. The asymptotic behavior of $C(t)$ is the same as for the integrable Harper model and thus the decay of the quantum correlation function does *not* reflect the existence of classical chaos.

As another quasiperiodic system we study the Fibonacci-chain model, which has attracted much attention since the discovery of the quasicrystalline phase in AlMn [14]. Here the potential only takes the two values $+V$ and $-V$ arranged in a Fibonacci sequence and replaces the cosine potential in Eq. (3) [7]. The eigenenergies can be determined by a very efficient method, using the trace map [7,15]. The decay of the correlation function also shows power-law behavior and the exponent δ depends on the parameter V. This dependence is shown in Fig. 2 where we find values of δ ranging from 0 to 0.84. Here the spectrum is singular continuous for all $V>0$ and becomes absolutely continuous for $V=0$ [16].

Summarizing the above results we find asymptotic power-law decays $C(t)\sim t^{-\delta}$ with $\delta<1$ for all three quasiperiodic systems. This asymptotic behavior calls for a general explanation. We give an analytic derivation of the correlation function $C(t)$ for spectra (singular continuous or absolutely continuous) characterized by generalized dimensions D_q [17]. First we specify the Hamiltonian $\hat{H}$ through its spectral projections $\hat{P}_\omega$ [18], where $\omega \in \mathrm{spec}(H)$. The spectral measure $\mu_\phi(\omega)$ with respect to an initial state vector $|\phi\rangle$ is defined by $\mu_\phi(\omega) = \langle\phi|\hat{P}_\omega|\phi\rangle$. The probability $p(t)=|\langle\phi(0)|\phi(t)\rangle|^2$ generally can be written as [2]

$$p(t)=\left|\int_{-\infty}^{+\infty} e^{-i\omega t}d\mu_\phi(\omega)\right|^2=2\pi\tilde{\mu}_\phi(t)\tilde{\mu}_\phi(t)^* , \qquad (8)$$

where $\tilde{\mu}_\phi(t)$ denotes the Fourier-Stieltjes transform of $\mu_\phi(\omega)$. Among the generalized dimensions D_q in particular we need D_2, which is defined by the scaling behavior

$$\gamma(l)=\int d\mu_\phi(\omega)\int_{\omega-l/2}^{\omega+l/2} d\mu_\phi(\omega')\sim l^{D_2} \quad (l\to 0) . \qquad (9)$$

The function $\gamma(l)$ gives the probability that two eigenfunctions picked (with the according probability) from the spectral decomposition of $|\phi\rangle$ have an energy difference less than l [19]. We now show that $\gamma(l)$ is related to $p(t)$. To this end we introduce the characteristic function

$$\chi_A(x)=\begin{cases}1 & \text{if } x\in A,\\ 0 & \text{if } x\notin A,\end{cases} \qquad (10)$$

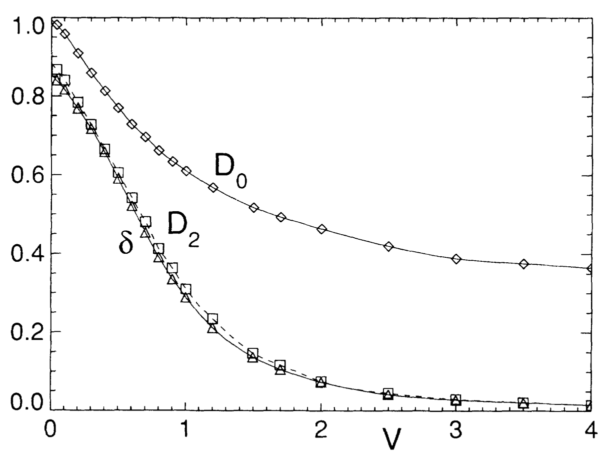

FIG. 2. Power-law exponents δ (triangles) of the correlation function decay for an initially localized wave packet and various potential strengths V in the Fibonacci model. Theory predicts the equality of δ and the generalized dimension D_2 (squares) according to Eq. (13) and gives an upper bound $D_0 \geq \delta$ [see Eq. (14)], where D_0 (diamonds) is the global Hausdorff dimension.

with $A=[-l/2,l/2]$ and rewrite Eq. (9) as

$$\gamma(l)=\int d\mu_\phi(\omega)\int \chi_A(\omega-\omega')d\mu_\phi(\omega') . \qquad (11)$$

Using the convolution theorem we write the second integral as the inverse Fourier transform of the product of the Fourier transforms of $\mu_\phi(\omega)$ and $\chi_A(\omega)$, yielding

$$\begin{aligned}\gamma(l)&=\frac{1}{\pi}\int_{-\infty}^{+\infty} d\mu_\phi(\omega)\int_{-\infty}^{+\infty} e^{i\omega\tau}\tilde{\mu}_\phi(\tau)\frac{\sin(l\tau/2)}{\tau}d\tau\\ &=\left(\frac{2}{\pi}\right)^{1/2}\int_{-\infty}^{+\infty}\tilde{\mu}_\phi(\tau)^*\tilde{\mu}_\phi(\tau)\frac{\sin(l\tau/2)}{\tau}d\tau\\ &=\left(\frac{2}{\pi^3}\right)^{1/2}\int_0^{+\infty} p(\tau)\frac{\sin(l\tau/2)}{\tau}d\tau , \end{aligned} \qquad (12)$$

where we made use of Eq. (8). We thus have a simple relation between the temporal probability $p(t)$ and the spectral probability $\gamma(l)$ [20]. Assuming a pure power-law behavior either for $\gamma(l)$ or for $C(t)$ we derive from Eq. (12) the relation

$$\gamma(l)\sim l^\alpha \ (l\to 0) \Longleftrightarrow C(t)\sim t^{-\alpha} \ (t\to\infty) \qquad (13)$$

which holds for $0<\alpha<1$ and where $\alpha=D_2$ according to Eq. (9) [21]. A similar relation with $C(t)$ replaced by $p(t)$ (i.e., without smoothing) does not hold in general, e.g., for the standard middle third Cantor set $C(t)$ decays like $C(t)\sim t^{-D_2}$ whereas $p(t)$ does not decay to zero.

This main result connects the algebraic decay of the correlation function with the multifractal structure of the spectral measure $\mu_\phi(\omega)$. The value of the exponent, i.e., the generalized dimension D_2, can only be computed if the spectrum, the eigenfunctions, and $\phi(t=0)$ are known. If only the spectrum is known, it is possible to given an upper bound for D_2 as follows. On the one hand, from $\int d\mu_\phi(\omega)=1$ it follows that $D_2(\mu_\phi(\omega))\leq D_0(\mu_\phi(\omega))$, where $D_0(\mu_\phi(\omega))$ is the Hausdorff dimension of the spectral measure [17]. On the other hand, $D_0(\mu_\phi(\omega))$ is the fractal dimension of the subset of spec(H) that is excited by the initial wave packet, whereby $D_0(\mu_\phi(\omega)) \leq D_0(\mathrm{spec}(H))$. Therefore the upper bound $D_0(\mathrm{spec}(H))$ gives a lower bound for the decay of the correlation function [22]

$$C(t)\geq ct^{-D_0(\mathrm{spec}(H))} \quad (t\to\infty) , \qquad (14)$$

where c is an appropriate constant.

To illustrate these analytical results we have determined the dimension D_2 for the Harper model and the Fibonacci model by dividing the energy range into boxes B_i of length l and computing the function $\gamma(l)$ as [19]

$$\gamma(l)=\sum_i\left(\sum_{\omega_k\in B_i}|a_k|^2\right)^2\sim l^{D_2} \quad (l\to 0) . \qquad (15)$$

For the Harper model this function is shown in Fig. 3 for three different values of λ corresponding to those of Fig. 1. The values of D_2 agree with those of δ in Fig. 1. In the same way we have computed the dimension D_2 for the

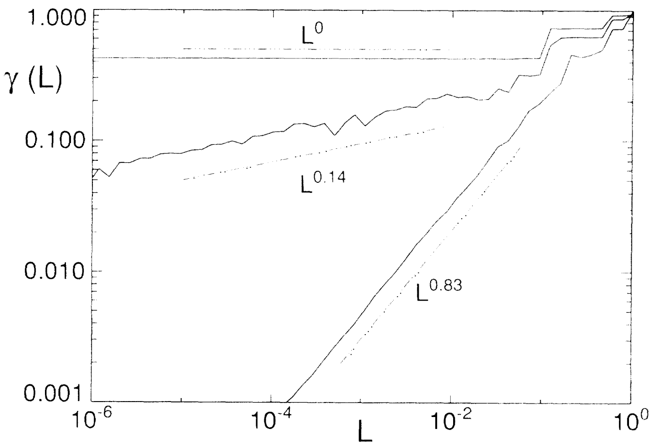

FIG. 3. Probability $\gamma(l)$ vs l [Eq. (15)] for Harper's equation, from which we deduce $D_2=0.83\pm0.01$, $D_2=0.14\pm0.01$, and $D_2=0$ for $\lambda=1$, 2, and 3, respectively. The values of D_2 equal those of δ in Fig. 1 within the numerical errors.

Fibonacci model as a function of the potential strength V. This result is shown by the dashed line in Fig. 2 where the values of δ and D_2 also coincide within the numerical errors and D_0 is recognized as an upper bound.

In conclusion, we have shown that the decay of the quantum autocorrelation function is determined by the generalized dimension D_2 of the spectral measure. In contrast, the growth of the variance of a wave packet is governed by the Hausdorff dimension, i.e., $\mathrm{var}(t)\sim t^{2D_0}$, as we previously demonstrated [5,23]. As a curiosity we note that in a different context, i.e., in dissipative dynamical systems, D_2 already goes by the name of correlation dimension as it describes the spatial correlation of an attractor.

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[1] M. Samuelides, R. Fleckinger, L. Touzillier, and J. Bellissard, Europhys. Lett. **1**, 203 (1986); Y. Pomeau, B. Dorizzi, and G. Grammaticos, Phys. Rev. Lett. **56**, 681 (1986); B. Sutherland, Phys. Rev. Lett. **57**, 770 (1986); R. Badii and P. F. Meier, Phys. Rev. Lett. **58**, 1045 (1987); J. M. Luck, H. Orland, and U. Smilansky, J. Stat. Phys. **53**, 551 (1988); R. Graham, Europhys. Lett. **8**, 717 (1989); T. Geisel, Phys. Rev. A **41**, 2989 (1990); G. Jona-Lasinio, C. Presilla, and F. Capasso, Phys. Rev. Lett. **68**, 2269 (1992).

[2] J. Avron and B. Simon, J. Funct. Anal. **43**, 1 (1981).

[3] For reviews, see e.g., B. Simon, Adv. Appl. Math. **3**, 463 (1982); J. B. Sokolov, Phys. Rep. **126**, 189 (1985).

[4] P. G. Harper, Proc. R. Soc. London A **68**, 874 (1955); M. Ya. Azbel', Zh. Eksp. Teor. Fiz. **46**, 429 (1964) [Sov. Phys. JETP **19**, 634 (1964)]; D. R. Hofstadter, Phys. Rev. B **14**, 2239 (1976).

[5] T. Geisel, R. Ketzmerick, and G. Petschel, Phys. Rev. Lett. **66**, 1651 (1991).

[6] T. Geisel, R. Ketzmerick, and G. Petschel, Phys. Rev. Lett. **67**, 3635 (1991).

[7] L. P. Kadanoff, M. Kohmoto, and C. Tang, Phys. Rev. Lett. **50**, 1870 (1983); S. Ostlund, R. Pandit, D. Rand, and H. J. Schellnhuber, Phys. Rev. Lett. **50**, 1873 (1983).

[8] S. Aubry and G. André, in *Proceedings of the Israel Physical Society,* edited by C. G. Kuper (Hilger, Bristol, 1979), Vol. 3, p. 133.

[9] J. Bellissard and B. Simon, J. Funct. Anal. **48**, 408 (1982).

[10] R. E. Borland, in *Mathematical Physics in One Dimension,* edited by E. H. Lieb and D. C. Mattis (Academic, New York, 1966), p. 929.

[11] We choose $\varphi_0=0$ and periodic eigenfunctions $\psi_{n,k}=\psi_{n+q,k}$. This is sufficient for the time evolution within a unit cell of the magnetic superlattice.

[12] P. Leboeuf, J. Kurchan, M. Feingold, and D. P. Arovas, Phys. Rev. Lett. **65**, 3076 (1990).

[13] R. Lima and D. L. Shepelyansky, Phys. Rev. Lett. **67**, 1377 (1991).

[14] D. Shechtman, I. Blech, D. Gratias, and J. V. Cahn, Phys. Rev. Lett. **53**, 1951 (1984).

[15] M. J. Feigenbaum and B. Hasslacher, Phys. Rev. Lett. **49**, 605 (1982).

[16] J. Bellissard, B. Iochum, E. Scoppola, and D. Testard, Commun. Math. Phys. **125**, 527 (1989).

[17] T. C. Halsey, M. H. Jensen, L. P. Kadanoff, I. Procaccia, and B. I. Shraiman, Phys. Rev. A **33**, 1141 (1986).

[18] M. Reed and B. Simon, *Methods of Modern Mathematical Physics,* Functional Analysis Vol. 1 (Academic, New York, 1978).

[19] Remember that in terms of fractal measures on attractors $\gamma(l)$ is the probability that two points on the attractor are separated by a distance smaller than l [P. Grassberger and I. Procaccia, Phys. Rev. A **28**, 2591 (1983)].

[20] T. Dittrich and U. Smilansky, Nonlinearity **4**, 59 (1991); **4**, 85 (1991), established a similar relation between the spectral two-point correlation function and a corresponding temporal correlation to obtain some rigorous results for the kicked quantum rotator.

[21] A scaling behavior in $\gamma(l)$ is not necessarily restricted to Cantor spectra, whereas a Cantor spectrum always implies the scaling behavior of Eq. (13). If the spectrum has continuous *and* pure point components, $C(t)$ will not decay even if $\gamma(l)$ shows scaling behavior.

[22] This bound is consistent with a bound given in R. Salem, Ark. Math. **1**, 353 (1950).

[23] T. Geisel, R. Ketzmerick, and G. Petschel, in *Quantum Chaos—Theory and Experiment,* edited by P. Cvitanovic, I. C. Percival, and A. Wirzba (Kluwer, Dordrecht, 1992), p. 43.