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# Spontaneous emission from a fractal vacuum

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**Abstract** – Spontaneous emission of a quantum emitter coupled to a QED vacuum with a deterministic fractal structure of its spectrum is considered. We show that the decay probability does not follow a Wigner-Weisskopf exponential decrease but rather an overall power law behavior with a rich oscillatory structure, both depending on the local fractal properties of the vacuum spectrum. These results are obtained by giving first a general perturbative derivation for short times. Then we propose a simplified model which retains the main features of a fractal spectrum to establish analytic expressions valid for all time scales. Finally, we discuss the case of a Fibonacci cavity and its experimental relevance to observe these results.

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Spontaneous emission results from the coupling of a quantum system (an “atom”) to a quantum vacuum. This is an important and widely studied phenomenon both from fundamental and applied points of view [1], which allows to probe properties of the quantum vacuum, its dynamics and correlations. The wide zoology of behaviors depends on spectral properties of the vacuum and on its coupling to the atom [2]. A standard textbook description [1] considers the coupling to a vacuum having a smooth and non-singular density of photon modes, in which case, the probability for spontaneous emission follows the well-known Wigner-Weisskopf decay law  $|U_e(t)|^2 = e^{-\Gamma_e(\omega_e)t}$ . Relevant definitions of the quantum amplitude  $U_e(t)$  and of the inverse lifetime  $\Gamma_e(\omega_e)$  will be given below. This description has been further developed towards quantum emitters coupled to more complicated environments such as semiconductors, QED cavities, photonic crystals and micro-cavities [2]. The existence of singularities in the spectrum of the vacuum leads to a qualitatively different behavior which has been studied in various cases [2,3].

In this letter, we address the problem of spontaneous emission from an atom coupled to a vacuum whose spectrum is characterized by a *discrete scaling symmetry* expressed by the property

$$\mu(\omega + \Delta\omega) - \mu(\omega) = \frac{\mu(T(\omega + \Delta\omega)) - \mu(T(\omega))}{a} \quad (1)$$

where  $\mu(\omega)$  is the integrated density of modes (IDOM). The dimensionless scaling parameter  $a$  and the map  $T(\omega)$  provide a full characterization of the specific discrete

scaling symmetry. Introducing  $\mathcal{N}_{\omega_u}(\omega) \equiv \mu(\omega) - \mu(\omega_u)$ , the scaling relation (1) can be written more concisely as  $\mathcal{N}_{\omega_u}(\omega) = \frac{1}{a} \mathcal{N}_{T(\omega_u)}(T(\omega))$ . A spectrum described by (1) is often called fractal [4], a denomination that we shall retain all over but which covers a broad class of systems that extends beyond the conventional self-similar character usually associated to fractals. Relevant examples include cavities made out of quasi-periodic heterostructures [5], or cavities generated from deterministic self-similar fractal structures like a Sierpinski gasket. The prominent feature of fractal spectra, resulting from their discrete scaling symmetry (1), is that they are highly lacunar, possessing an infinity of gaps appearing at all scales<sup>1</sup>.

Singularities of a spectrum satisfying (1) correspond to fixed points of the map  $T(\omega)$ . Around a fixed point  $\omega_u$  the map can be linearized,  $T(\omega) \simeq \omega_u + T'(\omega_u)(\omega - \omega_u)$ . Thus, the IDOM obeys the equation  $\mathcal{N}_{\omega_u}(\omega) = \frac{1}{a} \mathcal{N}_{\omega_u}(\omega_u + T'(\omega_u)(\omega - \omega_u))$ , whose general solution is

$$\mathcal{N}_{\omega_u}(\omega) = |\omega - \omega_u|^\alpha \mathcal{F}\left(\frac{\ln|\omega - \omega_u|}{\ln|T'(\omega_u)|}\right), \quad (2)$$

where  $\alpha = \frac{\ln a}{\ln|T'(\omega_u)|}$  is the local ( $\omega_u$ -dependent) spectral exponent and  $\mathcal{F}(x)$  is a periodic function of period unity<sup>2</sup>. Generically,  $\alpha$  changes between zero and unity

<sup>1</sup>It is worth noticing that for a singular spectrum, the IDOM  $\mu(\omega)$  is usually well defined, while its derivative, the density of modes  $\rho(\omega) = \frac{d\mu(\omega)}{d\omega}$ , may not be.

<sup>2</sup>Note that frequencies are dimensionless and expressed in units of a coarse-graining frequency characteristic of the fractal spectrum. In some cases,  $\alpha$  is related to the non-local spectral dimension  $d_s$  [6].

spanning the range between smooth continuous and point spectra [7].

We show below that the coupling of a two-level atom whose resonance frequency is close to  $\omega_u$  leads to a similar scaling behavior of the time-dependent decay amplitude  $U_e(t)$ , given in the long-time limit by

$$|U_e(t)|^2 = t^{-2\gamma} \mathcal{G}\left(\frac{\ln t}{\lambda}\right), \quad (3)$$

where  $\mathcal{G}(x+1) = \mathcal{G}(x)$  is another periodic function and  $\gamma$  is a function of the spectral exponent  $\alpha$ . The exponent  $\gamma$ , the real parameter  $\lambda \equiv \ln |T'(\omega_u)|$  and the expression of  $\mathcal{G}(x)$  are direct consequences of the specific scaling relation (2). Thus, for a fractal spectrum, spontaneous emission does not follow the Wigner-Weisskopf exponential decay, a result which could be partly anticipated, since it is indeed known that the existence of spectral singularities leads to an algebraic time decrease of the decay probability [3]. But the existence of log-periodic fluctuations described by the function  $\mathcal{G}(x)$  is a direct consequence of the discrete scaling symmetry (1).

Expression (3) constitutes the main result of this letter. To establish it, we shall first recall some basic definitions and results. Then, we will give a general perturbative derivation starting from the Fermi golden rule, essentially limited to small times. To go beyond this limit, we will consider a model general enough to include all relevant characteristics of a fractal spectrum, yet simple enough to allow for a thorough analytical derivation.

A two-level atom ( $|g\rangle, |e\rangle$ ), whose Hamiltonian is  $H_e = \hbar\omega_e|e\rangle\langle e|$ , is coupled to the EM field described by  $H_F = \hbar\sum_k \omega_k a_k^\dagger a_k$ , where  $k$  stands for an appropriate set of quantum numbers. The atom-photon interaction is described within the rotating wave approximation by the Hamiltonian  $H_{int} = \sum_k (V_k^* a_k^\dagger |g\rangle\langle e| + \text{h.c.})$ . The matrix element  $V_k$ , which accounts for the strength of the coupling, depends generally on the atom's position. In the initial state  $|e, 0_k\rangle$ , the atom is in the excited state and no photon is present. The probability amplitude  $U_e(t)$  to find the quantum system in the initial state a time  $t$  after it evolves with the total Hamiltonian  $H = H_e + H_F + H_{int}$  is defined by  $U_e(t) = \langle e, 0_k | \hat{U}(t, 0) | e, 0_k \rangle$ . The evolution operator  $\hat{U}(t, 0)$ , written in terms of the resolvent operator  $\hat{G}(z) = 1/(z - H)$ , is [1]

$$\hat{U}(t, 0) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dE e^{-iEt/\hbar} (\hat{G}_-(E) - \hat{G}_+(E)). \quad (4)$$

The matrix element  $G_e(z) \equiv \langle e, 0_k | \hat{G}(z) | e, 0_k \rangle = 1/(z - \hbar\omega_e - \Sigma_e(z))$  of the resolvent defines the self-energy  $\Sigma_e(\omega \pm i0^+) = \Delta_e(\omega) \mp \frac{i\hbar}{2}\Gamma_e(\omega)$  in terms of two spectral functions,  $\Delta_e(\omega)$  and  $\Gamma_e(\omega)$ , which respectively account for the shift and the spectral width of the atomic energy.

The spectral function  $\Gamma_e(\omega)$  is related to the vacuum response function

$$\Phi_e(t) = \int \frac{d\omega}{2\pi} \Gamma_e(\omega) e^{-i\omega t} = \int d\mu(\omega_k) \frac{|V_k|^2}{\hbar^2} e^{-i\omega_k t}, \quad (5)$$

by using the IDOM  $\mu(\omega)$  in the second equality. Within the dipole approximation [1], the response  $\Phi_e(t)$  is the time correlation function

$$\Phi_e(t) = \hbar^{-2} |d_{ge}|^2 \langle 0_k | \hat{E}_z(\mathbf{r}, t) \hat{E}_z^\dagger(\mathbf{r}, 0) | 0_k \rangle \quad (6)$$

of the electric-field component  $\hat{E}_z(\mathbf{r}, t)$  along the polarization direction  $\hat{z}$  of the atom. Here,  $d_{ge}$  and  $\mathbf{r}$  are, respectively, the dipole matrix element and the position of the atom. A convenient form of the probability amplitude  $U_e(t)$  is given in terms of the Laplace transform  $\tilde{\Phi}_e(s)$  of the response function [1]:

$$U_e(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \frac{e^{(s-i\omega_e)t}}{s + \tilde{\Phi}_e(s - i\omega_e)}. \quad (7)$$

From the previous definitions and results we identify two relevant energy scales for the problem of spontaneous emission. One,  $\Gamma_e(\omega_e)$ , is the strength of the coupling between the emitter and the vacuum. The second energy scale  $\Delta$  is given by the spectral width of  $\Gamma_e(\omega)$ . Their ratio defines the dimensionless coupling parameter

$$g = \Gamma_e(\omega_e)/\Delta. \quad (8)$$

In the weak-coupling limit,  $g \ll 1$ , the quantum amplitude  $U_e(t)$  is determined by the pole in (7), given by the approximate solution  $s \approx -\tilde{\Phi}_e(-i\omega_e) = -i\hbar\Sigma_e(\omega_e + i0^+)$ . This leads straightforwardly to the well-known Wigner-Weisskopf exponential decay. At very long times,  $t \gg \Gamma_e^{-1}(\omega_e)$ , this pole approximation breaks down, even in free space, and both the probability amplitude  $U_e(t)$  and the correlation function  $\Phi_e(t)$ , are dominated by the singularity at the edge  $\omega = 0$  of the vacuum spectrum [1]. For an atom coupled to the  $d$ -dimensional scalar QED vacuum  $\Gamma_e(\omega) \sim \rho(\omega)|\mathcal{E}(\omega)|^2$ , where the density of modes  $\rho(\omega) \sim \omega^{d-1}$  and the amplitude of the electric field  $\mathcal{E}(\omega) \sim \sqrt{\omega}$ , so that  $\Gamma_e \sim \omega^d$  and  $\Phi_e(t) \sim 1/t^{d+1}$  and, according to (7),  $U_e(t) \sim 1/t^{d+1}$ . In more structured vacuum spectra such as in photonic crystals, the spectral function  $\Gamma_e(\omega)$  exhibits singularities of the type  $\Gamma_e(\omega) = C|\omega - \omega_u|^{\alpha-1}\theta(\omega - \omega_u)$  around certain frequencies  $\omega_u$  [2]. In that case the coupling is strong and the pole approximation is not valid. Instead, one obtains a *generalized* exponential decay limited to small times well accounted for by the Fermi golden rule,  $U_e(t) - 1 \simeq Ct^{2-\alpha}$ . At large times, it turns into an algebraic decrease  $U_e(t) \simeq t^{-(2-\alpha)}$ , possibly coexisting with a non-decaying component [3].

We now consider a fractal vacuum spectrum obeying the scaling (2). In this case the weak-coupling limit becomes ill-defined, since the spectral function  $\Gamma_e(\omega) \sim \rho(\omega)$  is singular with a vanishing width  $\Delta$ . Thus, according to (8), we are effectively in a strong-coupling regime  $g \gg 1$ , even for a finite and small  $V_k$ . On the other hand, the short-time perturbative limit remains applicable. We start with this limit to present an intuitive explanation of the effect of the vacuum spectrum fractality on the decay dynamics.

At short times one obtains from (7) and (5)

$$|U_e(t)|^2 \simeq 1 - \int_0^t dt' \Gamma_e(t'), \quad (9)$$

where

$$\Gamma_e(t) = \frac{2t}{\hbar^2} \int d\mu(\omega_k) |V_k|^2 \frac{\text{sinc}(\omega_k - \omega_e)t}{(\omega_k - \omega_e)t}. \quad (10)$$

While this expression constitutes textbook materials, we wish to re-examine it in the context of a fractal spectrum. The sinc function in (10) indicates that  $\Gamma_e(t)$  is a wavelet (rather than a Fourier) transform of the spectral function  $\Gamma_e(\omega)$ . Generally, the wavelet transform  $S_w(a, b)$  of a function  $s(x)$  is defined by [8]

$$S_w(a, b) \equiv \frac{1}{a} \int dx s(x) w\left(\frac{x-b}{a}\right). \quad (11)$$

It can be viewed as a mathematical microscope which probes the function  $s(x)$  at a point  $b$  with a magnification  $1/a$  and an optics specified by the choice of the specific wavelet  $w(x)$ . Thus,  $\Gamma_e(t)$  in (10) is the wavelet transform of the IDOM  $\mu(\omega)$  at a frequency  $\omega_e$ , with a magnification  $t$  and  $w(x) = \text{sinc}(x)$  as a probe. An important property of the wavelet transform is that it preserves the discrete scaling symmetry (1) of the probed function, here the IDOM  $\mu(\omega)$  weighted by  $|V_k|^2$ . For a smooth and continuous spectrum, the sinc function probes energy scales of the order of  $t^{-1}$  and in the long-time limit it goes to  $\delta(\omega_k - \omega_e)$ , so that  $\Gamma_e(t)$  becomes  $t$ -independent. However there is no such well-defined limit for a fractal spectrum and inserting (2) into (10) for  $\omega_e = \omega_u$ , we obtain instead<sup>3</sup>

$$\Gamma_e(t) = t^{1-\alpha} \tilde{\mathcal{F}}_1\left(\frac{\ln t}{\ln |T'(\omega_u)|}\right) \quad (12)$$

and from (9)

$$|U_e(t)|^2 = 1 - t^{2-\alpha} \tilde{\mathcal{F}}_2\left(\frac{\ln t}{\ln |T'(\omega_u)|}\right), \quad (13)$$

where  $\tilde{\mathcal{F}}_{1,2}(x)$  are periodic functions of period unity. This constitutes a short-time counterpart to the asymptotic result (3) [9]. The behavior of  $\Gamma_e(t)$  is illustrated in fig. 1 for the case of a Fibonacci quasi-periodic dielectric cavity (to be discussed later on), whose IDOM is of the form (2). We observe, as predicted by (12), an overall power law behavior with  $\alpha \approx 0.8$ , explainable by the renormalization group analysis of [10], and also log-periodic oscillations around it, which are the fingerprint of the underlying fractal structure of the spectrum. Note that these log-periodic oscillations are already noticeable for systems of finite size.

In order to go beyond the previous, short-time regime, we consider the following model for the spectral function:

$$\Gamma_e(\omega) = \frac{C}{\pi|\omega - \omega_u|^{1-\alpha}} \left[ 1 + A \cos\left(\frac{2\pi}{\lambda} \ln \frac{|\omega - \omega_u|}{\Omega}\right) \right], \quad (14)$$

<sup>3</sup>We assume that  $|V_k|^2 d\mu(\omega_k)$  also obeys the scaling form (2).

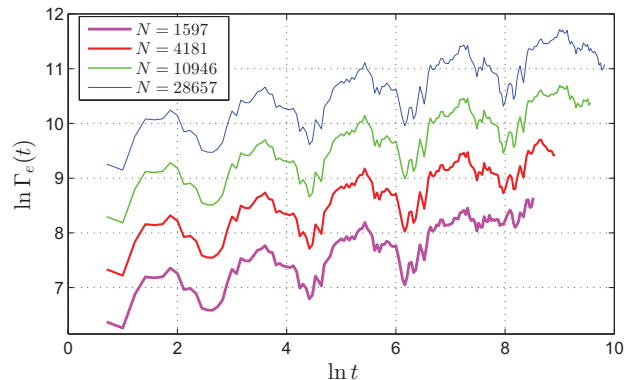


Fig. 1: (Color online) Numerical results for  $\Gamma_e(t)$ , given by (10), for Fibonacci dielectric cavities  $S_j$  (defined in text) of different lengths  $N$ . The photonic spectrum has been obtained from a diagonal tight-binding description of the Fibonacci potential. A constant matrix element  $|V_k|^2 = V^2$  was assumed in (10), which amounts to averaging  $\Gamma_e(t)$  over the atom's position in the cavity. The dimensionless time  $t$  is in units of the tight-binding hopping constant.

where  $0 < \alpha < 1$  is the local spectral exponent introduced in (2),  $C$  is the coupling strength, and  $0 \leq A \leq 1$  and  $\Omega$  define respectively the modulation amplitude and phase. For simplicity, we let  $-\infty < \omega < \infty$ . This expression exhibits the basic features required to describe a fractal structure of the vacuum, around  $\omega_u$ , as defined in (1) and (2) with  $\lambda = \ln |T'(\omega_u)|$ . We have approximated the log-periodic function by its first harmonic, which happens to be a good approximation as shown in related situations [6,11]. In the absence of log-periodic modulation, *i.e.* for  $A = 0$ , we recover the known case of a singularity in the spectrum [3]. The important point here is that the modulation results from the scaling properties of the spectrum defined in (1) and discussed subsequently. The model (14) allows to obtain closed analytical expressions of the quantum amplitude  $U_e(t)$  and therefore to recover previous results in the short-time limit and to study the long-time limit which was not possible using the Fermi golden rule. From (5), we obtain

$$\Phi_e(t) = C \frac{2e^{-i\omega_u t}}{\pi t^\alpha} \left[ \Gamma(\alpha) \cos \frac{\pi\alpha}{2} - A \text{Im} F(t) \right], \quad (15)$$

where  $\Gamma(x)$  is the Euler Gamma function and we have defined  $F(t) \equiv (\Omega t)^{2i\pi/\lambda} \cosh\left(\frac{\pi^2}{\lambda} + \frac{i\pi\alpha}{2}\right) \Gamma\left(\alpha - \frac{2i\pi}{\lambda}\right)$ . Thus,  $\Phi_e(t)$  is not short-ranged in time. Consequently, there is no Wigner-Weisskopf exponential decay. While this is true already for  $A = 0$ , note that the log-periodic modulation of  $\Gamma_e(\omega)$  adds to  $\Phi_e(t)$  an oscillatory log-periodic term, which further modifies the behavior of  $U_e(t)$ . To investigate this point in more detail, we study the pole structure in (7). The Laplace transform of  $\Phi_e(t)$  is

$$\tilde{\Phi}_e(s - i\omega_e) = \frac{C}{z^{1-\alpha}} \left[ \csc \frac{\pi\alpha}{2} + \frac{A}{2i} (\tilde{F}(z) - \tilde{F}^*(z)) \right], \quad (16)$$

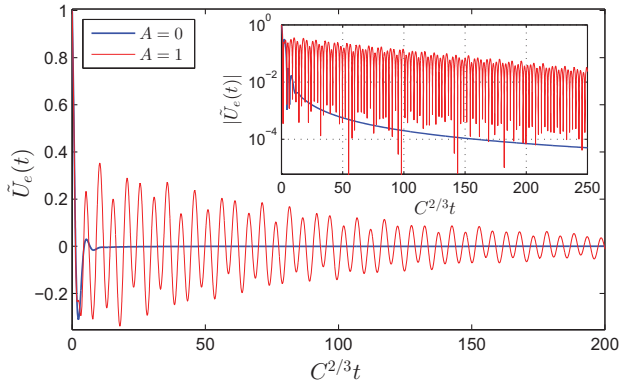


Fig. 2: (Color online) Amplitude  $\tilde{U}_e(t)$  calculated with the model (14) for  $A = 0$  (blue line) and  $A = 1$  (red line). In both cases  $\alpha = \frac{1}{2}$  and  $\lambda = 1$ . The inset shows  $|\tilde{U}_e(t)|$  in a semi-log scale.

where  $z \equiv s + i\Delta\omega$ ,  $\Delta\omega \equiv \omega_u - \omega_e$  and  $\tilde{F}(z) \equiv \left(\frac{z}{\Omega}\right)^{2\pi i/\lambda} \sinh^{-1}\left(\frac{\pi^2}{\lambda} - \frac{i\pi\alpha}{2}\right)$ . The poles  $s_n$  in (7) are solutions of  $s = -\tilde{\Phi}_e(s - i\omega_e)$ . For simplicity, we consider  $\Delta\omega = 0$ , in which case all the poles come in complex conjugate pairs. For  $A > \csc\left(\frac{\pi\alpha}{2}\right) e^{-\pi^2/\lambda}$ , the poles accumulate near the origin and are distributed log-periodically with the distance to it. In particular, for  $|s_n|^{2-\alpha} \ll C$ , their expression in the lower half-plane is [12]

$$s_n \approx -i\Omega e^{\lambda\left(\frac{3-\alpha}{4} + n\right) + i\theta_0}, \quad (17)$$

with  $\theta_0 \equiv \frac{\lambda}{2\pi} \ln(A \sin \frac{\pi\alpha}{2})$  and integer  $n$ . The corresponding residues acquire a rather simple form,  $\text{res}(s_n) \approx -\frac{\lambda \sin(\pi\alpha/2)}{2\pi i C} s_n^{2-\alpha} e^{s_n t}$ . Then, assuming  $e^{2\pi^2/\lambda} \gg 1$ , and for large times  $Ct^{2-\alpha} \gg 1$ , the probability amplitude  $\tilde{U}_e(t) \equiv e^{i\omega_e t} U_e(t)$  can be written as [12]

$$\tilde{U}_e(t) = -\frac{\lambda \sin \frac{\pi\alpha}{2}}{\pi C} \text{Im} \sum_{n=-\infty}^{+\infty} s_n^{2-\alpha} e^{s_n t}, \quad (18)$$

where the summation was extended to  $+\infty$  exploiting the condition on  $t$  (see footnote <sup>4</sup>). Using (17), one can show that only a few terms in the sum give the main contribution at a given time. Therefore, locally, the decay envelope appears as a slow exponential, while it is algebraic  $|U_e(t)| \sim 1/Ct^{2-\alpha}$  over large time scales [12]. More interesting is the discrete scaling symmetry displayed in (18), namely

$$\tilde{U}_e(\beta t) = \beta^{\alpha-2} \tilde{U}_e(t) \quad \text{with} \quad \beta = e^\lambda, \quad (19)$$

which is straightforward when using (17) in (18). More generally, if for large  $t$  one can neglect the free term  $s$  in the denominator in (7), then (19) follows immediately from scaling properties of  $\tilde{\Phi}_e(s - i\omega_e)$ . Thus, as a result

<sup>4</sup>For  $\Delta\omega = 0$ , the amplitude  $\tilde{U}_e(t)$  is real-valued owing to the symmetry of the spectral function  $\Gamma_e(\omega)$  in (14). Also, in certain cases, the decaying part (18) coexists with a non-decaying term corresponding to the incomplete decay of the excited state [12].

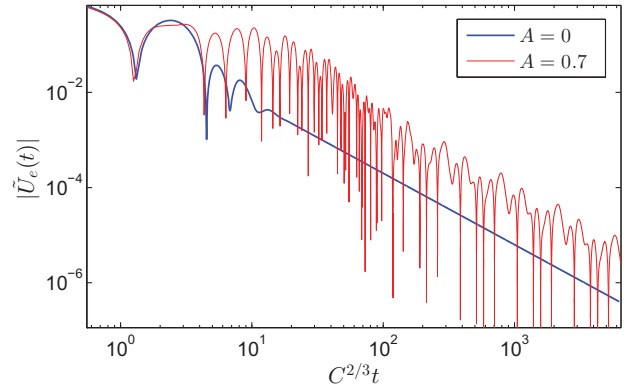


Fig. 3: (Color online) Absolute value of the amplitude,  $|\tilde{U}_e(t)|$ , calculated with the model (14) for  $A = 0$  (blue line) and  $A = 0.7$  (red line). In both cases  $\alpha = \frac{1}{2}$  and  $\lambda = 1$ . The asymptotic log-periodic structure is clearly observed for  $A = 0.7$ .

of (1),  $\tilde{U}_e(t)$  takes precisely the form (3) with  $\gamma = 2 - \alpha$ . The log-periodic function  $\mathcal{G}$  can be calculated from (18). The decay amplitude  $U_e(t)$ , plotted in fig. 2, has a much slower decay (more than three orders of magnitude, see inset) for  $A = 1$ , than for  $A = 0$ , *i.e.* without the log-periodic modulation. As noted above, the decay envelope appears as exponential since the power law in (3) shows up at time scales larger than displayed in fig. 2. The oscillating structure of  $U_e(t)$  shows beats resulting from interferences between the dominant terms in (18). The long-time power law decay with the log-periodic modulation is shown in fig. 3, where the cases  $A = 0.7$  and  $A = 0$  are compared. The log-periodic structure reflects the discrete scaling symmetry expressed by (19).

A physical realization of the previous results is provided by a quantum emitter coupled to the vacuum of a quasi-periodic, *e.g.* Fibonacci, dielectric cavity [13], made of a sequence of slabs of two types,  $A$  and  $B$ , of width and refractive index denoted, respectively, by  $d_A, d_B$  and  $n_A, n_B$ . Fibonacci sequences  $S_j$  of such slabs are constructed by recursion  $S_{j \geq 3} = [S_{j-2} S_{j-1}]$ , with  $S_1 = B, S_2 = A$ . The spectral properties and the spatial behavior of the corresponding eigenmodes in Fibonacci and similar quasi-periodic structures have been extensively studied [5]. The spectrum of the Fibonacci system  $S_j$  is highly fragmented, with the degree of fragmentation increasing with  $j$  and the contrast  $n_A/n_B$  [10,14]. More specifically, the IDOM  $\mu(\omega)$  has a discrete scaling symmetry (1) governed by specific  $p$ -cycles of the associated renormalization group transformation, with  $a = \sigma^p$  and  $\sigma = \frac{1+\sqrt{5}}{2}$ . Near fixed points of a given  $p$ -cycle, the map  $T(\omega)$  can be linearized, and  $\mu(\omega)$  obeys (2) with  $\alpha = \frac{\ln a}{\lambda} = \frac{\ln \sigma^p}{\lambda}$  and  $\lambda = \ln |T'(\omega)|$ . For instance, for  $d_A n_A = d_B n_B = d$ , fixed points of a 6-cycle are determined by  $\frac{\omega_n d}{2\pi c} = \frac{2n+1}{4}$ , where  $c$  is the vacuum speed of light and  $n = 0, 1, \dots$ . Linearizing the map at these points, one obtains  $e^\lambda = |T'(\omega_n)| = 1 + 8\eta^4 + 4\eta^2 \sqrt{1 + 4\eta^4}$  with  $\eta = \frac{1}{2} \left( \frac{n_A}{n_B} + \frac{n_B}{n_A} \right)$  [14]. Experimentally, a reasonable contrast can be achieved with  $n_A = 1.45$  and

$n_B = 2.23$  [13], so that  $\alpha = 0.898$  and  $\lambda = 3.2$  (see footnote <sup>5</sup>).

In conclusion, we have argued that spontaneous emission of a quantum emitter coupled to a vacuum characterized by a discrete scaling symmetry exhibits unusual and measurable features. We have shown that the quantum probability amplitude never follows the Wigner-Weisskopf exponential decay, but rather an overall power law (whose exponent depends on the fractal properties of the vacuum) modulated by log-periodic fluctuations. These fluctuations constitute an unambiguous fingerprint of the underlying fractal structure. Underneath the slow algebraic decay and log-periodic modulation, the probability amplitude was shown to exhibit a rich dynamics with nearly periodic oscillations and beats resulting from interferences. Finally, we have discussed the example of a Fibonacci cavity, a non-fractal device but whose spectrum is characterized by a scaling symmetry (1), and we have argued that it may be a possible candidate to observe the unusual features of spontaneous emission described in this letter. Spontaneous emission constitutes a specific way to probe a fractal nature of spectrum. Thermodynamics [11] and spatial correlations [15] provide other examples of physical probes. Let us mention finally, that the approach developed here could be extended to related problems in quantum mesoscopic physics (Coulomb blockade) and in the study of quantum vacuum effects such as the dynamical Casimir effect and the Schwinger and Unruh effects.

\* \* \*

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<sup>5</sup>Actually, near the fixed points of the 6-cycle, IDOM  $\mu(\omega)$  scales with a  $\lambda$  equal to half this value [10].