Spectral operators on the Sierpinski gasket I

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March 27, 2008

Abstract

We study spectral operators for the Kigami Laplacian on the Sierpinski gasket (SG). These operators may be expressed as functions of the Laplacian (Dirichlet or Neumann), or as Fourier multipliers for the associated eigenfunction expansions. They include the heat operator, the wave propagator, and spectral projections onto various families of eigenspaces. Our approach is both theoretical and computational. Our main result is a technical lemma, extending the method of spectral decimation of

*Research supported by the National Science Foundation through the Research Experiences for Undergraduates (REU) Program at Cornell University.

†Research supported by the National Science Foundation through the Research Experiences for Undergraduates (REU) Program at Cornell University.

‡Research supported in part by the National Science Foundation, grant DMS–0652440.

Fukushima and Shima to certain eigenfunctions corresponding to "forbidden" eigenvalues. This enables us to compute the kernel of a spectral operator (Neumann) when one of the variables is a boundary point. We present the results of these computations in various cases, and formulate conjectures based on this experimental evidence. We also prove a new result about the trace of the heat kernel as $t \to 0$: not only does it blow up as a power of t (known from the standard on–diagonal heat kernel estimates), but after division by this power of t it exhibits an oscillating behavior that is asymptotically periodic in $\log t$. Our experimental evidence suggests that the same oscillating behavior holds for the heat kernel on the diagonal.

1 Introduction

Let Δ denote the standard Kigami Laplacian on the Sierpinski gasket (SG), regarded as a self–adjoint operator, with either Dirichlet or Neumann boundary conditions, on $L^2(\mu)$ where μ is the standard measure on SG. Let $\{u_j\}$ denote an orthonormal basis of eigenfunctions,

$$-\Delta u_j = \lambda_j u_j \text{with } \lambda_1 \le \lambda_2 \le \cdots \to \infty. \tag{1.1}$$

Then for any function f we can define the *spectral operator* $f(-\Delta)$ by

$$f(-\Delta)u = \sum_{j=1}^{\infty} f(\lambda_j)\langle u, u_j\rangle u_j. \tag{1.2}$$

This class of operators includes the heat kernel ($f(x) = e^{-tx}$) and solution operators for other space–time equations, and spectral projection operators onto various families of eigenspaces. Spectral operators play a vital role in smooth analysis, where Δ is a classical Laplacian. So a better understanding of spectral operators on SG and other fractals is a major goal in the developmenet of analysis on fractals.

There are reasons to hope that SG is a good model space to work out a detailed understanding of spectral operators. The method of spectral decimation described by Fukushima and Shima ([Sh1], [FS]) gives a complete description of all eigenvalues and eigenfunctions. This method has been elaborated in many subsequent papers, including [DSV], [GRS], [OSS], [Sh2], [S2], [T]. In particular, a method of computing inner products of eigenfunctions is given in [OSS]. However, there is an obstacle in using spectral decimation: there are many eigenvalues with high multiplicity, and within the associated eigenspaces there appears to be no natural orthonormal bases. Since there exist natural bases, it is possible to use ad hoc procedures such as Gram–Schmidt to produce orthonormal bases for moderate size multiplicities, and this procedure was used in [OSS] for some numerical examples. Other numerical methods were used in [DSV], [GRS], [CDS] and [BS] to study spectral operators related to differential equations. Also, probabilistic methods have been very effective in obtaining

heat kernel estimates [Ba], but we will see that these estimates, despite being sharp, do not tell the entire story.

Because of the high multiplicities, it makes sense to lump together the terms in (1.2) corresponding to the same eigenvalue, and write

$$f(-\Delta)u(x) = \sum_{\lambda} f(\lambda) \int P_\lambda(x,y)u(y)d\mu(y) \tag{1.3}$$

where the sum is over all distinct eigenvalues, and

$$P_\lambda(x,y) = \sum_j \varphi_j(x)\varphi_j(y) \tag{1.4}$$

where $\{\varphi_j\}$ is an orthonormal basis of the λ–eigenspace E_λ. Our main idea is to choose the orthonormal basis in such a way as to simplify the sum (1.4). Fix a point x, and let E_λ^x denote the subspace of E_λ of functions vanishing at x. Typically this subspace has codimension one in E_λ (there are some cases when $E_\lambda^x = E_\lambda$). Let $(E_\lambda^x)^\perp$ denote the orthogonal complement in E_λ. If we choose the orthonormal basis so that the first element φ_1 is in $(E_\lambda^x)^\perp$ and the rest belong to E_λ^x, then (1.4) reduces to a single term,

$$P_\lambda(x,y) = \varphi_1(x)\varphi_1(y). \tag{1.5}$$

In Section 3 we study the case of Neumann boundary conditions when x is a point on the boundary. We show that it is quite feasible to compute (1.5) in this case. It turns out that the orthogonality condition $u_1 \in (E_\lambda^x)^\perp$ may be interpreted as a discrete eigenvalue equation, and so the function φ_1 satisfies an extended version of spectral decimation that is not valid for all functions in E_λ. (This result is also implicit in Theorem 3.6 of [T].) Moreover, we are able to identify φ_1 as a constant multiple of the "level eigenfunction" first descrbed in [BSSY]. Identifying the constant in order to achieve the required normalization is a bit complicated, but is not computationally difficult. Thus we have all the numerical tools to approximate the kernel of any spectral operator when one of the variables lies on the boundary. In part II [PRRS] different methods will be used to approximate the full kernel. In Sections 4 we use the methods from Section 3 to compute approximations to the kernels of various spectral operators which solve space-time differential equations, including the heat equation, the wave equation, and the Laplace equation.

In our numerical approximations we will truncate the infinite sum (1.3) after a finite number of terms. In the case of ordinary Fourier expansions this procedure is unreliable due to well-known convergence problems. But on SG it was shown in [S4] that, as long as we take natural truncation points s_m that sum up to a gap in the spectrum, that the partial sums

$$\sum_{\lambda \le s_m} \int P_\lambda(x,y)u(y)d\mu(y) \tag{1.6}$$

converge uniformly to u if u is continuous. It follows that doing a similar truncation to (1.3) should yield a good approximation of the spectral operator.

The known estimates for the heat kernel imply that on the diagonal it is bounded above and below by $t^{-\alpha}$ for $\alpha = \log 3/\log 5$ as $t \to 0$. However, the numerical evidence presented in Section 4 is that $t^\alpha h_t(x,x)$ for x a point on the boundary behaves asymptotically like a periodic function of $\log t$ with period $\log 5$. Similar evidence, but this time for the normal derivatives of the heat kernel, was noted in [BSSY]. In Section 2 we actually prove the analogous statement for the traces of the heat kernel

$$\int h_t(x,x)d\mu(x). \tag{1.7}$$

Note that this is expressible entirely in terms of the eigenvalues, namely

$$\sum_{j=1}^{\infty} e^{-t\lambda_j} \quad (\{\lambda_j\} \text{ as in (1.1)}). \tag{1.8}$$

It is not hard to relate this to the eigenvalue counting function

$$N(s) = \#\{j : \lambda_j \le s\}, \tag{1.9}$$

and the proof is based on the known asymptotics of $N(s)$. In particular, we are able to show that the periodic function in question is nonconstant and is close to, but not exactly equal to a pure sine function.

We now present a brief summary of the pertinent facts about the Laplacian on SG and its spectrum. For more details the reader may consult the books [Ba], [Ki] and [S5], or the expository paper [S1]. We realize SG as the limit of graphs Γ_m with vertices V_m and edge relation $x \underset{m}{\sim} y$. Γ_0 is simply the complete graph on three vertices $V_0 = \{q_0, q_1, q_2\}$ which we identify with the vertices of an equilateral traingle in the plane. Let F_i denote the plane homothety with contraction ratio 1/2 and fixed–point q_i. Then we define inductively $V_m = \bigcup_i F_i V_{m-1}$. Let $w = (w_1, \dots, w_m)$ denote a word of length $|w| = m$ with each w_j taking on the values 0, 1, 2, and define $F_w = F_{w_1} \circ F_{w_2} \circ \cdots \circ F_{w_m}$. Then

$$V_m = \bigcup_{|w|=m} F_w V_0. \tag{1.10}$$

The points in V_0 will be regarded as boundary points in each of the graphs Γ_m and SG. Every nonboundary point $x \in V_m \setminus V_0$ can be written in two distinct ways $x = F_w q_i = F_{w'} q_{i'}$ in (1.10), and will be called a junction point. It is easy to see that $V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots$. We define $V_* = \bigcup_m V_m$. Then SG is the closure of V_* in the plane. This agrees with the usual definition as the unique nonempty compact set satisfying

$$K = \bigcup_{i=0}^{2} F_i K.$$

Since all the functions with which we are concerned are continuous, it suffices to understand their restriction to V_*.

We call F_wK an m–cell ($m = |w|$) and regard F_wV_0 as its boundary. Note that each point $x \in V_m \setminus V_0$ has exactly four neighbors in Γ_m, two in each of the m–cells that contain x as a boundary point. (Of course x also belongs to $V_{m'} \setminus V_0$ for any $m' \geq m$, but its neighbors in $\Gamma_{m'}$ will vary with m'.) We define the Laplacian by

$$\Delta u(x) = \lim_{m\to\infty} \frac{3}{2}5^m \sum_{y \underset{m}{\sim} x} (u(y) - u(x)) \tag{1.11}$$

for $x \in V_* \setminus V_0$. Note that this very closely resembles the definition of the second derivative on the line as a limit of difference quotients. Also the summation on the right side of (1.11) is exactly the graph Laplacian on Γ_m. The precise definition is that $u \in \text{dom } \Delta$ and $\Delta u = f$ provided u and f are continuous and f is given by the right side of (1.11), with a uniform limit. This Laplacian has a self–adjoint closure on $L^2(\mu)$, here μ is the symmetric self–similar probability measure on SG, with either Dirichlet or Neumann boundary conditions. Dirichlet conditions require the vanishing of u on the boundary, while Neumann conditions require the vanishing of the normal derivative $\partial_n u$ on the boundary, defined by

$$\partial_n u(q_i) = \lim_{m\to\infty} \left(\frac{5}{3}\right)^m \sum_{y \underset{m}{\sim} q_i} (u(q_i) - u(y)) \tag{1.12}$$

(there are two terms in the sum). The normal derivatives exist for every $u \in \text{dom } \Delta$. Perhaps the best way to think about these boundary conditions is in terms of the *double* $\widetilde{SG}$, obtained by gluing together two copies of SG at the three corresponding boundary points. (This is a special case of the *fractafold* construction in [S2], and is analogous to constructing a circle by gluing together two intervals.) The double $\widetilde{SG}$ has no boundary, and we can define a Laplacian $\widetilde{\Delta}$ on it exactly as before. Then Dirichlet boundary conditions correspond to functions that have odd extensions to $\widetilde{SG}$ in dom $\widetilde{\Delta}$, and Neumann boundary conditions correspond to functions that have even extensions to $\widetilde{SG}$ in dom $\widetilde{\Delta}$.

Since both Dirichlet and Neumann Laplacians have compact resolvants, there is a discrete orthonormal basis $\{u_j\}$ of eigenfunctions

$$-\Delta u_j = \lambda_j u,\ 0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty. \tag{1.13}$$

Moreover, there is an exact decription of the eigenfunctions and eigenvalues via the method of spectral decimation [Sh1], [FS]. Consider any eigenfunction, $-\Delta u = \lambda u$, without imposing any boundary condition. Then there exists a *generation of birth* $m_0 \geq 0$, such that $u|_{V_m}$ is an eigenfunction of the graph Laplacians,

$$\Delta_m u(x) = \sum_{y \underset{m}{\sim} x} (u(y) - u(x)) = -\lambda_m u(x) \tag{1.14}$$

for $x \in V_m \setminus V_0$, for all $m \geq m_0$ (when $m_0 = 0$ we only require $m \geq 1$ in (1.14)), where the sequence $\{\lambda_m\}$ of discrete eigenvalues (not related to the eigenvalues in (1.13)) is determined by

$$\lambda = \lim_{m\to\infty} \frac{3}{2} 5^m \lambda_m \tag{1.15}$$

$$\lambda_{m-1} = \lambda_m(5 - \lambda_m) \text{ for } m > m_0. \tag{1.16}$$

Conversely, given any functon u on V_{m_0} satisfying the λ_{m_0}–eigenvalue equation, it is possible to extend u to all V_* to satisfy the λ_m–eigenvalue equation on Γ_m for all $m \geq m_0$, and hence the λ–eigenvalue equation on SG, provided (1.15) and (1.16) hold, and provided λ_m does not equal one of the "forbidden" eigenvalues 2, 5 or 6 for $m > m_0$. Note that we do not exclude $\lambda_{m_0} = 2$, 5 or 6, but if $\lambda_{m_0} = 6$ then we do exclude $\lambda_{m_0+1} = 2$ (this does satisfy (1.16)), but allow $\lambda_{m_0+1} = 3$. In general we can find two solutions of (1.16), namely

$$\lambda_m = \frac{5 + \varepsilon_m \sqrt{25 - 4\lambda_{m-1}}}{2} \text{ for } \varepsilon_m = \pm 1, \tag{1.17}$$

and if we start with any value of λ_{m_0} we may use (1.17) to define subsequent values of λ_m, provided that we make all but a finite number of choices $\varepsilon_m = -1$ to make the limit (1.15) exist.

The extension of u from V_{m-1} to V_m is given by the following explicit local algorithm. Let $|w| = m - 1$, and consider the $(m-1)$–cell $F_w K$. We assume that the values $u(F_w q_i)$ are known, since $F_w q_i \in V_{m-1}$. Consider a point in $V_m \cap F_w K$, say $F_w F_0 q_1$ to be specific. Then

$$\begin{aligned} u(F_w F_0 q_1) = \; & \tfrac{4-\lambda_m}{(2-\lambda_m)(5-\lambda_m)} (u(F_w q_0) + u(F_w q_1)) \\ & + \tfrac{2}{(2-\lambda_m)(5-\lambda_m)} u(F_w q_2). \end{aligned} \tag{1.18}$$

(By the way, it is clear from (1.18) why $\lambda_m = 2$ or 5 are forbidden; that 6 is also forbidden has a subtler explanation.)

The above description is valid for all eigenfunctions, but if we impose Dirichlet or Neumann boundary conditions we can say more. (In the Neumann case we exclude the constant function with eigenvalue zero in what follows.) Namely λ_{m_0} must be a forbidden eigenvalue. We then refer to the eigenvalues as belonging to the 2–series, 5–series or 6–series. The value 2 occurs only in the Dirichlet case and $m_0 = 1$, with multiplicity one. Otherwise $\lambda_{m_0} = 5$ or 6 (in the Neumann case $\lambda_{m_0} = 5$ requires $m_0 \geq 2$, while in the Dirichlet case $\lambda_{m_0} = 5$ requires $m_0 \geq 1$ and $\lambda_{m_0} = 6$ requires $m_0 \geq 2$) and the multiplicity grows exponentially with m_0. For example, in the Neumann case the multiplicity for $\lambda_{m_0} = 6$ is $\#(V_{m_0-1}) = \frac{3^{m_0}+3}{2}$, and we can take essentially any values for $u|_{V_{m_0}-1}$ to determine an eigenfunction (using (1.18) to extend u to V_{m_0}). This is the case that we will examine in detail in Section 3.

2 Trace of the heat kernel

We begin with a rather generic observation relating the trace of the heat kernel and the eigenvalue counting function. Consider any "Laplacian" with eigenvalues

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty \tag{2.1}$$

repeated according to multiplicities. The eigenvalue counting function is

$$N(s) = \#\{j : \lambda_j \leq s\} \text{ for } s > 0 \tag{2.2}$$

and the trace of the heat kernel is

$$h(t) = \sum_{j=1}^{\infty} e^{-t\lambda_j}, \text{ for } t > 0. \tag{2.3}$$

Then clearly

$$h(t) = \int_0^{\infty} te^{-ts} N(s) ds. \tag{2.4}$$

We are interested in the consequences for the trace of heat kernel of estimates of the form

$$N(s) = s^{\beta}\psi(s) + o(s^{\beta}) \text{ as } s \to \infty \tag{2.5}$$

for some $\beta > 0$, where ψ is bounded away from zero, bounded, and multiplicitively periodic,

$$\psi(\tau s) = \psi(s) \text{ for some } \tau > 1. \tag{2.6}$$

Note that we do not want to assume that ψ is continuous, but we certainly should assume that it is measurable.

Theorem 2.1. *Assume (2.5) holds as above. Then $h(t)$ satisfies*

$$h(t) = t^{-\beta} g(t) + o(t^{-\beta}) \text{ as } t \to 0^+ \tag{2.7}$$

where $g(t)$ is bounded away from zero, bounded, and multiplicatively periodic as in (2.6) for the same period τ. Moreover g is continuous on $(0, \infty)$ and is given by

$$g(t) = \int_0^{\infty} e^{-s} s^{\beta} \psi(s/t) ds. \tag{2.8}$$

Proof. Substituting (2.5) in (2.4) we obtain

$$h(t) = \int_0^{\infty} te^{-ts} s^{\beta} \psi(s) ds + R(t) \tag{2.9}$$

for

$$R(t) = \int_0^{\infty} te^{-ts} o(s^{\beta}) ds. \tag{2.10}$$

It is clear by a change of variable that the first term on the right side of (2.9) is equal to $t^{-\beta} g(t)$. Also by a change of variable, $g(t)$ satisfies the multiplicative

periodicity condition. Clearly $g(t)$ is positive, and the continuity follows by standard convolution–type arguments. So it suffices to show that $R(t) = o(t^{-\beta})$. Given $\varepsilon > 0$, choose T so that $o(s^\beta) \le \varepsilon s^\beta$ for $s \ge T$. Then, since $\beta > 0$ (so $o(s^\beta)$ is bounded above),

$$R(t) \le c \int_0^T t e^{-ts} ds + \varepsilon \int_T^\infty t e^{-ts} s^\beta ds$$

hence

$$t^\beta R(t) \le cTt^{\beta+1} + \varepsilon \int_{Tt}^\infty e^{-s} s^\beta ds,$$

and this can be made less than a fixed multiple of ε by taking t small enough. □

Theorem 2.2. *If ψ is nonconstant then g is nonconstant.*

Proof. Let $\Psi(x) = \psi(e^x)$ and $G(x) = g(e^x)$. It suffices to show that if Ψ is nonconstant then so is G. But Ψ is a periodic function of period $\log \tau$, so it can be expanded in a Fourier series

$$\Psi(x) = \sum_{-\infty}^{\infty} c_n e^{2\pi i n x / \log \tau}. \tag{2.11}$$

To say that Ψ is nonconstant is to say that some $c_n \ne 0$ for $n \ne 0$. By substituting (2.11) in (2.8) we can compute the Fourier series expansion of G,

$$\begin{aligned} G(x) &= \int_0^\infty e^{-s} s^\beta \Psi(\log s - x) ds \\ &= \sum_{-\infty}^{\infty} c_n \Big(\int_0^\infty e^{-s} s^{(\beta + \frac{2\pi i n}{\log \tau})} ds \Big) e^{-2\pi i n x / \log \tau} \\ &= \sum_{-\infty}^{\infty} c_n \Gamma\Big(1 + \beta + \frac{2\pi i n}{\log \tau}\Big) e^{-2\pi i n x / \log \tau}. \end{aligned} \tag{2.12}$$

Since the Γ-function never vanishes, it follows that G is nonconstant. (We've subtly exchanged a sum and an integral above, using convergence assumptions justified elsewhere.) □

For the standard Laplacian on SG (with Dirichlet or Neumann boundary conditions), the estimate (2.5) holds for $\beta = \log 3 / \log 5$ and $\tau = 5$ [KL]. The function ψ (it is the same for either boundary condition) is known to be discontinuous, so it is certainly not constant. So the estimate (2.7) holds with nonconstant function g. Since the heat kernel is multiplicative for products

[S3], it follows that for SG_k (the k–fold product) the trace of the heat kernel satisfies

$$h_k(t) = t^{-k \log 3/ \log 5} g_k(t) + o(t^{-k \log 3/ \log 5}) \text{ as } t \to 0^+ \tag{2.13}$$

for $g_k(t) = (g_1(t))^k$.

We conjecture that the spectral counting function for SG_k satisfies

$$N_k(s) = s^{k \log 3/ \log 5} \psi(s) + o(s^{k \log 3/ \log 5}) \text{ as } s \to \infty. \tag{2.14}$$

It is not clear how to run the argument of Theorem 1 backward to deduce (2.14) from (2.13). Another approach might be to try to deduce (2.14) from the $k = 1$ case. In the meantime we present experimental evidence.

Figures 1, 2, and 3 show the graph of $s^{-k \log 3/ \log 5} N_k(s)$ on a logarithmic scale ($x = e^s$) for $k = 1, 2, 3$. We see clearly a rapid convergence to the periodic function $\Psi_k(x)$. We see an improvement in the behavior of this function, both in the relative range of values and the "smoothness" of the graph (although the functions are all discontinuous) as k increases. In Figure 4 we show the graph of $t^{\log 3/ \log 5} h_1(t)$ on a logarithmic scale. Note the strong resemblence to a sine curve. In Figure 5 we show the remainder after subtracting off the best fitting function of the form $a \sin(bx + c) + d$. The values we found are

$$\begin{aligned} a &= 0.00063199 \\ b &= 8.98969426 \\ c &= 1.55780347 \\ d &= 0.13356482. \end{aligned}$$

Note that the predicted period is $\log_{10} 5$ which yields the value $b = 2\pi/ \log_{10} 5 = 8.9892059$, rather close to the fitted value. The relative amplitude of the remainder to the value of a (the amplitude of the sine curve) is about 1/60. This is easy to understand using the Fourier series (2.12). While we don't have much information about the coefficients c_n, the gamma factors decay exponentially with n. So only the terms corresponding to small values of n contribute significantly to the sum. In fact

$$\left| \Gamma\Big(1 + \beta + \frac{2\pi i}{\log \tau}\Big) \right| = 0.02765160$$

and

$$\left| \Gamma\Big(1 + \beta + \frac{4\pi i}{\log \tau}\Big) \right| = 0.00013481,$$

so on this basis we would expect the remainder to have a relative amplitude of about 1/200.

3 Spectral decimation

One of the big technical obstacles to computing kernels of spectral operators is the existence of eigenspaces of high multiplicity with no obvious orthonormal

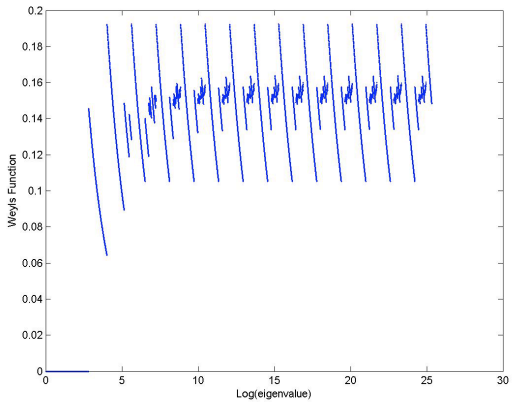

Figure 1: Weyl's Ratio for SG^1

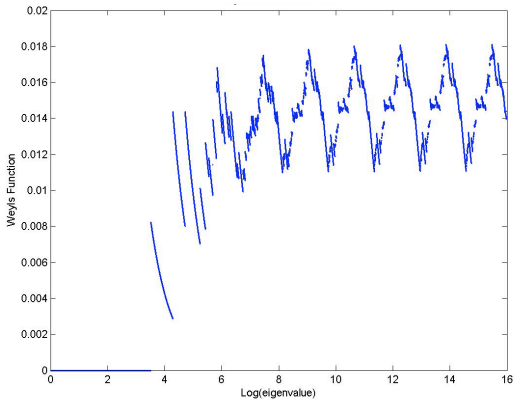

Figure 2: Weyl's Ratio for SG^2

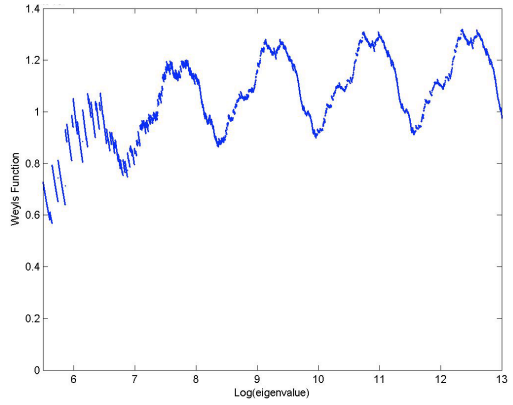

Figure 3: Weyl's Ratio for SG^3 (vertical scale in 1000's)

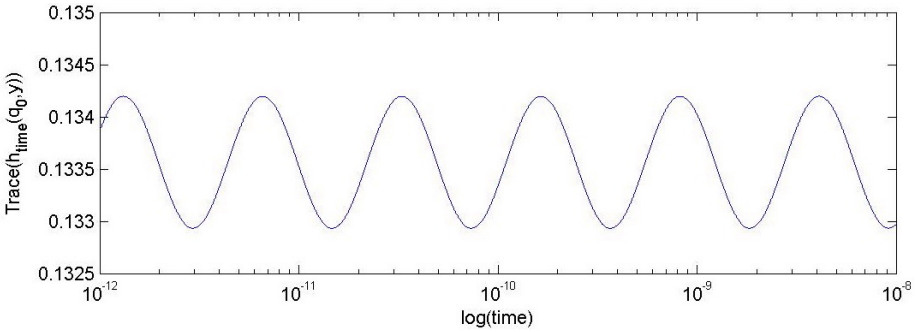

Figure 4: $t^{\log 3/\log 5} h_1(t)$

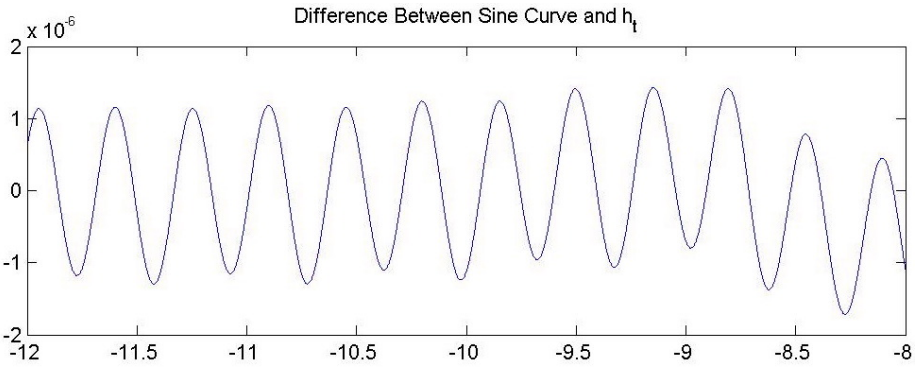

Figure 5: Difference from pure sine curve

basis. To make matters worse, these eigenspaces correspond to the discrete eigenvalues 5 and 6 which are among the "forbidden" values for the method of spectral decimation. In this section we present a "magic cure" for both problems in the special case that we fix one of the variables to be a boundary point. We believe that a similar, but more complicated, method might work for other junction points.

Let E_λ denote an eigenspace of dimension N, and let $\varphi_1, \ldots, \varphi_N$ be any real–valued orthonormal basis. Then the kernel of the projection onto E_λ is

$$P_\lambda(x,y) = \sum_{j=1}^{N} \varphi_j(x)\varphi_j(y), \tag{3.1}$$

independent of the orthonormal basis. If we fix a value of x, we might try to choose the orthonormal basis to simplify the computation of (3.1). In particular, let E_λ^x denote the subspace of E_λ of functions vanishing at x. Typically this has codimension one. Then if we choose $\varphi_2, \ldots, \varphi_N$ to be an orthonormal basis of E_λ^x, we have

$$P_\lambda(x,y) = \varphi_1(x)\varphi_1(y) \tag{3.2}$$

where φ_1 is a normalized function in $(E_\lambda^x)^\perp$. This is a drastic simplification, especially if we have some method to compute a function in $(E_\lambda^x)^\perp$.

Now we specialize to the case $x = q_0$ and Neumann boundary conditions. We note that all eigenfunctions for λ in the 5–series vanish on the boundary, so the only nonzero $P_\lambda(q_0, y)$ functions correspond to the 6–series. That means there is a generation of birth m_0 ($m_0 \geq 0$), with $\lambda_{m_0} = 6$, $\lambda_{m_0+1} = 3$, and inductively

$$\lambda_{k+1} = \frac{5 + \varepsilon_k\sqrt{25 - 4\lambda_k}}{2} \text{ for } k \geq m_0 + 1, \tag{3.3}$$

for $\varepsilon_k = \pm 1$, with all but a finite number $= -1$. The eigenvalue λ is given by

$$\lambda = \frac{3}{2} \lim_{k\to\infty} 5^k \lambda_k, \tag{3.4}$$

and we can solve (3.3) to obtain

$$\lambda_k = \lambda_{k+1}(5 - \lambda_{k+1}). \tag{3.5}$$

(This is the process described in equations 1.16 and 1.17 in reverse.) The space of eigenfunctions has dimension $\#(V_{m_0-1})$, and may be described as follows: choose arbitrary values on V_{m_0-1}, and then extend to V_k for $k \geq m_0$ inductively by spectral decimation. (The case $m_0 = 0$ will be treated separately.) This space has an obvious basis, namely $\{u_x\}$ for $x \in V_{m_0-1}$ defined by

$$u_x(y) = \delta_{xy} \text{ for } y \in V_{m_0-1}, \tag{3.6}$$

but these functions are not orthogonal. However, it is clear that $E_\lambda^{q_0}$ is spanned by $\{u_x\}$ as x varies over $V_{m_0-1} \setminus \{q_0\}$. So $(E_\lambda^{q_0})^\perp$ is defined by orthogonality

to u_x for all $x \in V_{m_0-1} \setminus \{q_0\}$, and it is not difficult to understand what this means.

Note that (3.5) enables us to define λ_k for all values of k. In particular $\lambda_{m_0-1} = -6$. Ordinarily this is meaningless. The restriction to V_{m_0-1} of a function in E_λ is arbitrary, so it will not in general satisfy a λ_{m_0-1}–eigenvalue equation, and neither will the restriction to V_k for any $k < m_0$ satisfy a λ_k–eigenvalue equation. (This is symptomatic of the fact that 6 is a forbidden eigenvalue for the spectral decimation method.) Miraculously, functions in $(E_\lambda^{q_0})^\perp$ will have these properties on restrictions to coarser graphs. In fact we will show that the orthogonality condition to u_x for $x \in V_{m_0-1} \setminus V_0$ is exactly the (-6)–eigenvalue equation at x (for $x = q_1$ or q_2 it is a Neumann type boundary condition). Once we know this, then the spectral decimation method applied in reverse gives the analogous statements on V_k for $k < m_0 - 1$, since we never encounter a forbidden eigenvalue again.

The key to this approach is the following computational lemma:

Lemma 3.1. *For any 6-series eigenvalue with $m_0 = 1$,*

$$\langle u_{q_0}, u_{q_1} \rangle = -\frac{1}{5} \langle u_{q_0}, u_{q_0} \rangle. \tag{3.7}$$

Proof. Using spectral decimation, we compute the values of $2u_{q_0}$ and $2u_{q_1}$ on V_1 in Figure 3.1 (the factor 2 makes all values integers). By Corollary 2.4 of [OSS], the inner products are proportional to the discrete inner product

$$\langle f, g \rangle_1 = \sum_{x \in V_0} f(x)g(x) + 2 \sum_{x \in V_1 \setminus V_0} f(x)g(x). \tag{3.8}$$

We compute $\langle u_{q_0}, u_{q_0} \rangle_1 = 10$ and $\langle u_{q_0}, u_{q_1} \rangle = -2$, so (3.7) follows. □

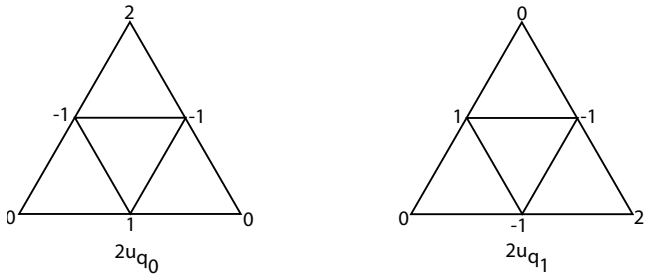

Figure 6: 6-series characteristic eigenfunctions with $m_0 = 1$

Theorem 3.2. *(cf. [T], Theorem 3.6) Consider any 6–series eigenvalue with $m_0 \geq 1$.*

(a) For any $x \in V_{m_0-1} \setminus V_0$, the condition $\langle u, u_x \rangle = 0$ is equivalent to

$$10u(x) = \sum_{y \underset{(m_0-1)}{\sim} x} u(y), \tag{3.9}$$

the (-6)–eigenvalue equation on level $m_0 - 1$ at x.

(b) For $x = q_1$ or q_2, the condition $\langle u, u_x\rangle = 0$ is equivalent to

$$5u(x) = \sum_{y \underset{(m_0-1)}{\sim} x} u(y), \tag{3.10}$$

the Neumann boundary condition at x (for a (-6)–eigenfunction on level m_0-1).

(c) If $u \in (E_\lambda^{q_0})^\perp$, then the restriction of u to V_{m_0-1} is a (-6)–eigenfunction satisfying Neumann boundary conditions at q_1 and q_2.

Proof. We can write

$$u = \sum_{V_{m_0-1}} u(y) u_y, \tag{3.11}$$

so

$$\langle u, u_x\rangle = \sum_{V_{m_0-1}} u(y)\langle u_y, u_x\rangle. \tag{3.12}$$

If x and y are not neighbors in V_{m_0-1}, then u_x and u_y have disjoint support so $\langle u_x, u_y\rangle = 0$. If $x \neq y$ are neighbors in V_{m_0-1} then the supports of u_x and u_y overlap in exactly one (m_0-1)-cell, namely the one containing x and y, and $\langle u_x, u_y\rangle = -c$ independent of the pair (x, y). If $x = y = q_1$ or q_2 then $\langle u_{q_i}, u_{q_i}\rangle = 5c$ by Lemma 3.1. On the other hand, if $x = y \in V_{m_0-1} \setminus V_0$ then the support of u_x is contained in two (m_0-1)–cells, so $\langle u_x, u_x\rangle = 10c$ by Lemma 3.1. Thus the equation (3.12) equals zero becomes (3.9) or (3.10) in the two cases. Since $10 = 4 - (-6)$, (3.9) is the (-6)-eigenvalue equation. At the boundary we need to extend the function by even reflection and impose the eigenvalue equation to check the Neumann boundary condition, and so (3.10) verifies this. Then (c) follows by applying (a) and (b) at all points $x \in V_{m_0-1} \setminus \{q_0\}$. □

The case $m_0 = 0$ is slightly different. Here the space E_λ has dimension 2, and is obtained by assigning values on V_0 satisfying $u(q_0) + u(q_1) + u(q_2) = 0$, and extending to V_1 with $\lambda_1 = 3$, etc. There is an obvious basis consisting of an even function $(u(q_0) = 2, u(q_1) = u(q_2) = -1)$ and an odd function $(u(q_0) = 0, u(q_1) = -u(q_2) = 1)$ and these are orthogonal. Thus the even function generates $(E_\lambda^{q_0})^\perp$. Figure 3.2 shows its values on V_1. The fact that this function takes the same value at the point $F_1 q_2 = F_2 q_1$ as it does at q_1 and q_2 is no coincidence. In fact it is constant along the whole line segment joining q_1 and q_2, as can be shown by symmetry consideration. In fact the same is true for all the functions in $(E_\lambda^{q_0})^\perp$, and this enables us to identify them, up to a constant, with the *level eigenfunctions* described in Section 7 of [BSSY].

We will denote this function $\mathcal{L}_\lambda$ (slightly different notation was used in [BSSY]). It is described by setting

$$\mathcal{L}_\lambda(q_1) = \mathcal{L}_\lambda(q_2) = 1, \quad \mathcal{L}_\lambda(q_0) = 1 - \frac{\lambda_0}{2} \tag{3.13}$$

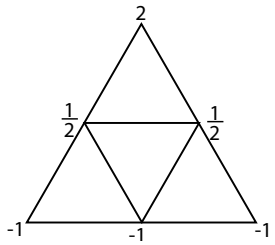

Figure 7: $(E_\lambda^{q_0})^\perp$

(we are using (3.5) to define λ_j) and applying spectral decimation. In particular

$$\mathcal{L}_\lambda(F_w q_0) = 1 - \frac{\lambda_m}{2} \text{ for all } w \text{ with } |w| = m \text{ and all } w_i = 1 \text{ or } 2 \tag{3.14}$$

and by continuity

$$\mathcal{L}_\lambda(y) = 1 \text{ for } y \text{ on the line segment between } q_1 \text{ and } q_2. \tag{3.15}$$

Figure 8 shows the values of $\mathcal{L}_\lambda$ on V_1 and V_2 for $m_0 = 1$ and $m_0 = 2$. It is easy to see that $\partial_n \mathcal{L}_\lambda(q_j) = 0$ for $j = 1, 2$ because this equation is identical to the level m eigenfunction equation at any nonboundary vertex along the line segment. It follows by Theorem 3.2 (c) that $\mathcal{L}_\lambda$ belongs to $(E_\lambda^{q_0})^\perp$, and hence generates the space because it is 1–dimensional.

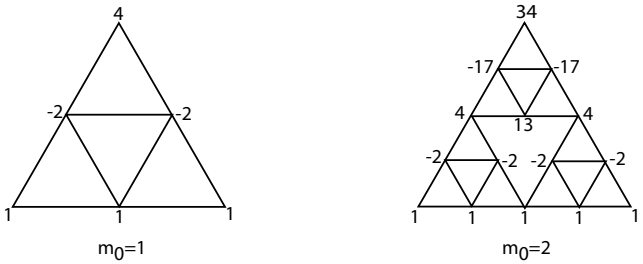

Figure 8: $\mathcal{L}_\lambda$ on V_1 and V_2 for $m_0 = 1$ and $m_0 = 2$

Next we consider the normalization factor. We need to compute

$$\langle \mathcal{L}_\lambda, \mathcal{L}_\lambda \rangle, \tag{3.16}$$

for then we should take $\varphi_1 = \langle \mathcal{L}_\lambda, \mathcal{L}_\lambda \rangle^{-1/2} \mathcal{L}_\lambda$ in (3.2), so

$$P_\lambda(x, y) = \langle \mathcal{L}_\lambda, \mathcal{L}_\lambda \rangle^{-1} \mathcal{L}_\lambda(x) \mathcal{L}_\lambda(y). \tag{3.17}$$

We can express (3.16) as a product of two factors, one which depends only on m_0, and one which depends on the sequence $\{\varepsilon_j\}_{j \geq m_0+1}$. We denote this

sequence by ε, and let u_{q_0} be the function in Lemma 3.1 with $m_0 = 1$ and this choice of ε (so this determines λ). Write

$$c(\varepsilon) = \langle u_{q_0}, u_{q_0} \rangle. \tag{3.18}$$

Then Lemma 3.1 implies

$$\begin{aligned}\langle u, u\rangle &= \frac{1}{3^{m_0-1}}\Big(5\sum_{x\in V_0} u(x)^2 + 10\sum_{x\in V_{m-1}\setminus V_0} u(x)^2 \\ &\qquad - 2\sum_{x,y\in V_{m-1};x\neq y} u(x)u(y)\Big)c(\varepsilon) \\ &= \frac{1}{3^{m_0-1}}\sum_{|w|=m_0-1}\Big(5\sum_i u(F_w q_i)^2 - 2\sum u(F_w q_i)u(F_w q_{i+1})\Big)c(\varepsilon)\end{aligned} \tag{3.19}$$

for any $u \in E_\lambda$.

Now the computation of $c(\varepsilon)$ has been done in Corollary 2.4 of [OSS], namely

$$c(\varepsilon) = \frac{2}{9}\prod_{j=m_0+1}^{\infty} b(\lambda_j) \tag{3.20}$$

for

$$b(t) = \frac{\big(1-\frac{1}{6}t\big)\big(1-\frac{2}{5}t\big)}{\big(1-\frac{1}{5}t\big)\big(1-\frac{1}{2}t\big)} \tag{3.21}$$

(the proof given in [OSS] assumed Dirichlet boundary conditions, but it can easily be modified for all eigenfunctions).

Table 3 displays the values of $c(\varepsilon)$ for different choices of ε. Thus it remains to compute the other factor in (3.19) for $u = \mathcal{L}_\lambda$. To do this we introduce the notation

$$Q_{A,B}(f) = A\Big(\sum_i f(q_i)^2\Big) - B\Big(\sum_i f(q_i)f(q_{i+1})\Big) \tag{3.22}$$

for f a function on V_0, where A and B are positive parameters. Then (3.19) can be expressed as

$$\langle u, u\rangle = \frac{1}{3^{m_0-1}}\sum_{|w|=m_0-1} Q_{5,2}(u\circ F_w)c(\varepsilon). \tag{3.23}$$

We would like to recursively determine values of A_k, B_k for $k \le m_0 - 1$ such that

$$K_{m_0} = \sum_{|w|=k} Q_{A_k,B_k}(u\circ F_w) \tag{3.24}$$

is independent of k, with $A_{m_0-1} = 5$, $B_{m_0-1} = 2$. If we can do this, then for $u = \mathcal{L}_\lambda$ we have

$$K_{m_0} = A_0\Big(\Big(1-\frac{\lambda_0}{2}\Big)^2 + 2\Big) - B_0\Big(1 + 2\Big(1-\frac{\lambda_0}{2}\Big)\Big) \tag{3.25}$$

ε	λ (10^5)	$c(\varepsilon)$	ε	λ (10^5)	$c(\varepsilon)$
$----$	0.000271	2.7784	$---++$	5.38742	0.12786
$+----$	0.002197	2.2811	$+--++$	5.39469	0.29747
$++---$	0.009528	2.4335	$++-++$	5.42217	0.64963
$-+---$	0.013290	1.0803	$-+--++$	5.43615	0.33772
$-++--$	0.044090	1.3275	$-++-++$	5.54747	0.70901
$+++--$	0.050039	2.3823	$+++-++$	5.56838	1.3403
$+-+--$	0.064462	1.1126	$+-+-++$	5.61833	0.69226
$--+--$	0.069155	0.49415	$--+-++$	5.63437	0.31588
$--++-$	0.216476	0.62958	$--+++$	6.09296	0.59942
$+-++-$	0.223418	1.3326	$+-+++$	6.11277	1.2832
$++++-$	0.246180	2.3993	$++++++$	6.17677	2.3955
$-+++-$	0.256271	1.2452	$-+++++$	6.20469	1.2631
$-+-+-$	0.317041	0.57545	$-+-+++$	6.36740	0.64503
$++-+-$	0.325709	1.1108	$++-+++$	6.38989	1.2647
$+--+-$	0.343623	0.51455	$+--+++$	6.43584	0.60632
$---+-$	0.348858	0.22210	$---+++$	6.44843	0.26436
$---++$	1.07830	0.28517	$---+-+$	7.88927	0.12095
$+--++$	1.08552	0.65173	$+--+-+$	7.90041	0.27836
$++-++$	1.11200	1.3419	$++-+-+$	7.94086	0.58626
$-+-++$	1.12503	0.68919	$-+-+-+$	7.96054	0.29997
$-+++-$	1.22086	1.2721	$-+++-+$	8.10044	0.58808
$++++-$	1.23758	2.3936	$++++-+$	8.12401	1.1121
$+-++-$	1.27622	1.2590	$+-++-+$	8.17754	0.59028
$--++-$	1.28826	0.58481	$--++-+$	8.19397	0.27478
$--+-+$	1.57911	0.26809	$--+--+$	8.55450	0.12475
$+-+-+$	1.58971	0.58442	$+-+--+$	8.56638	0.27142
$+++-+$	1.62292	1.1126	$+++--+$	8.60304	0.51332
$-++-+$	1.63691	0.58435	$-++--+$	8.61823	0.26878
$-+--+$	1.71227	0.26748	$-+---+$	8.69757	0.12077
$++--+$	1.72183	0.51186	$++---+$	8.70734	0.23051
$+---+$	1.74071	0.23196	$+----+$	8.72643	0.10391
$----+$	1.74572	0.09943	$-----+$	8.73145	0.44475

Table 1: $c(\varepsilon)$ and associated eigenvalues

and finally

$$\langle \mathcal{L}_\lambda, \mathcal{L}_\lambda \rangle = \frac{1}{3^{m_0-1}} K_{m_0} c(\varepsilon). \tag{3.26}$$

To determine the recursion relation we want the identity

$$Q_{A_{k-1},B_{k-1}}(u \circ F_w) = \sum_j Q_{A_k,B_k}(u \circ F_w \circ F_j) \tag{3.27}$$

to hold for all $|w| = k-1$. This is clearly independent of w, and involves the λ_k–eigenfunction extension algorithm. If we denote x_0, x_1, x_2 the values of u on the boundary of the $(k-1)$–cell $F_w K$, then

$$Q_{A_{k-1},B_{k-1}}(u \circ F_w) = A_{k-1}(x_0^2 + x_1^2 + x_2^2) - B_{k-1}(x_0x_1 + x_1x_2 + x_2x_0). \tag{3.28}$$

The values of u on the points of $(V_k \setminus V_{k-1}) \cap F_w K$ are given by

$$\frac{(4-\lambda_k)(x_1+x_2)+2x_0}{(2-\lambda_k)(5-\lambda_k)} \ , \text{ etc.} \tag{3.29}$$

We use these on the right side of (3.27), expand out, and equate the coefficients of $x_0^2 + x_1^2 + x_2^2$ and $x_0x_1 + x_1x_1 + x_2x_0$ on both sides to obtain

$$\begin{cases} A_{k-1} = \Big(1 + \frac{4(4-\lambda_k)^2+8}{(2-\lambda_k)^2(5-\lambda_k)^2}\Big)A_k - \Big(\frac{2(4-\lambda_k)}{(2-\lambda_k)(5-\lambda_k)} + \frac{(4-\lambda_k)(8-\lambda_k)}{(2-\lambda_k)^2(5-\lambda_k)^2}\Big)B_k \\ B_{k-1} = -\Big(\frac{4(4-\lambda_k)(6-\lambda_k)}{(2-\lambda_k)^2(5-\lambda_k)^2}\Big)A_k + \Big(\frac{2(6-\lambda_k)}{(2-\lambda_k)(5-\lambda_k)} + \frac{3(4-\lambda_k)^2+4(5-\lambda_k)}{(2-\lambda_k)^2(5-\lambda_k)^2}\Big)B_k. \end{cases} \tag{3.30}$$

In Table 2 we display the values of K_{m_0}. In Figure 9 we show graphs of $\mathcal{L}_\lambda(x)$ for various choices of m_0 and ε. Figure 10 shows the restrictions of several such graphs to the line segment joining q_0 to q_1.

m_0	K_{m_0}	m_0	K_{m_0}	m_0	K_{m_0}
1	2.5×10^{-1}	5	3.9×10^{-8}	9	2.9×10^{-118}
2	9.6×10^{-2}	6	1.8×10^{-15}	10	9.6×10^{-236}
3	1.2×10^{-2}	7	3.6×10^{-30}		
4	1.8×10^{-4}	8	1.6×10^{-59}		

Table 2: K_{m_0}

4 Space-time equations

The heat equation

$$\frac{\partial u(x,t)}{\partial t} = \Delta_x u(x,t) \text{ for } x \in SG, \ t > 0, \tag{4.1}$$

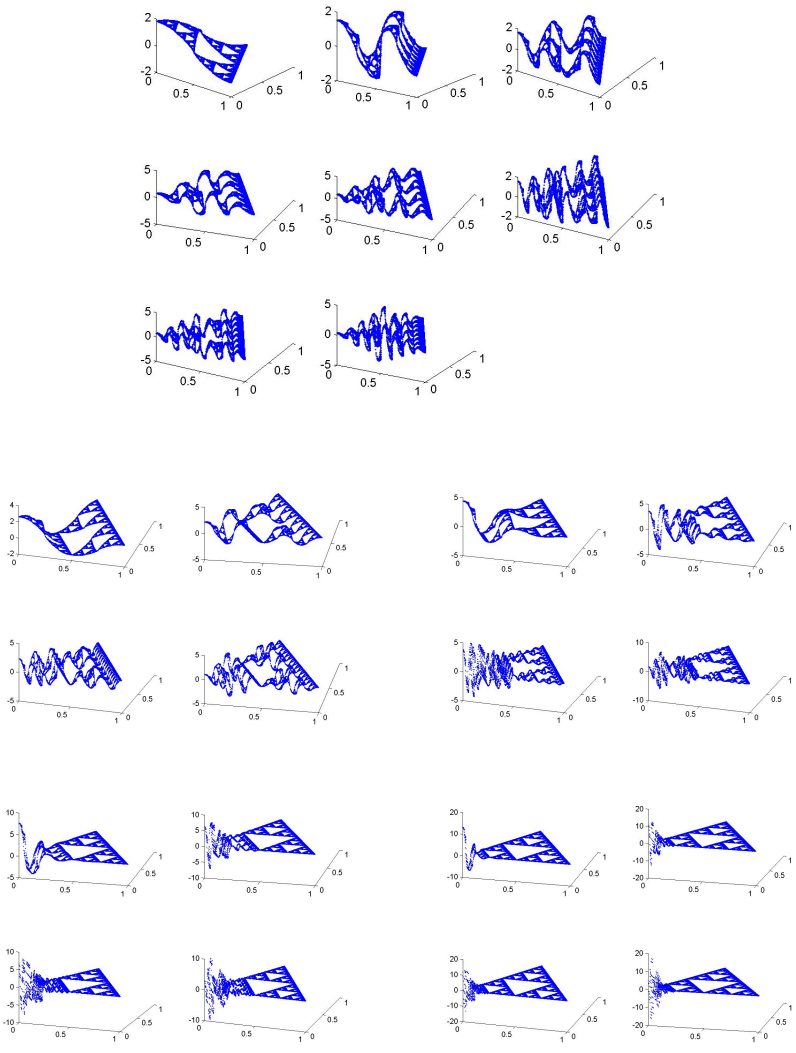

Figure 9: $m_0 = 0, 1, 2, 3, 4$

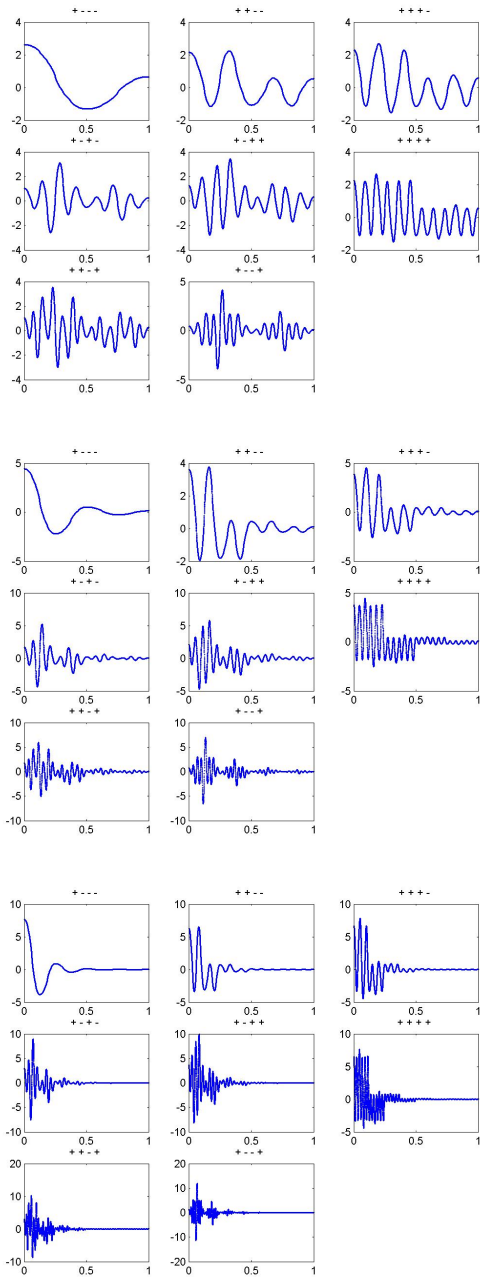

Figure 10: Restrictions of some graphs from figure 9 to a line segment

subject to the initial conditions

$$u(x,0) = f(x) \tag{4.2}$$

and either Dirichlet or Neumann boundary condtions, is solved by the spectral operator

$$u(x,t) = \sum_{\lambda} e^{-\lambda t} \int P_\lambda(x,y) f(y) d\mu(y) \tag{4.3}$$

whose kernel is the heat kernel

$$h_t(x,y) = \sum_{\lambda} e^{-\lambda t} P_\lambda(x,y). \tag{4.4}$$

Choosing Neumann boundary conditions and $x = q_0$, we may use the methods of Section 3 to compute good approximations to $h_t(q_0, y)$. Clearly, the closer t is to zero, the more terms we need to take in the sum. When $y = q_0$ we are on the diagonal. If we plot $t^\alpha h_t(q_0, q_0)$ on a logarithmic scale and the remainder after subtracting the best fitting sine curve, the results are nearly identical to the corresponding graphs for the trace of the heat kernel presented in Section 2 (figures 4 and 5). Figure 11 gives a plot of $t^\alpha h_t(q_0, q_0)$ on a logarithmic scale.

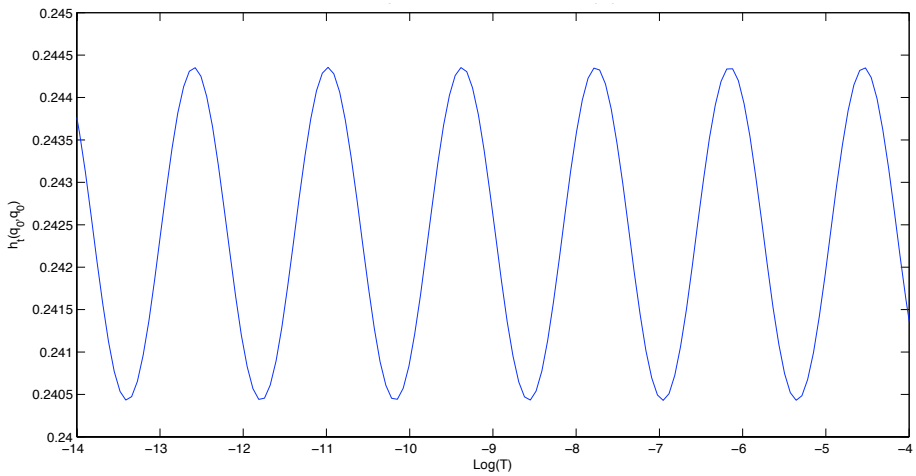

Figure 11: $t^{\log 3/\log 5} h_t(q_0, q_0)$

Another important space–time equation is the wave equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \Delta_x u(x,t) \tag{4.5}$$

with initial conditions

$$\begin{cases} u(x,0) = f_0(x) \\ \frac{\partial u}{\partial t}(x,0) = f_1(x) \end{cases} \tag{4.6}$$

and either Dirichlet or Neumann boundary conditions. It is conventional to define the *wave propagator* by setting $f_0 = 0$ and considering the map $W_t : f_1 \to u(\cdot, t)$. This is a spectral operator given by

$$u(x,t) = \sum_\lambda \frac{\sin t\sqrt{\lambda}}{\sqrt{\lambda}} \int P_\lambda(x,y) f_1(y) d\mu(y) \tag{4.7}$$

with kernel

$$W_t(x,y) = \sum_\lambda \frac{\sin t\sqrt{\lambda}}{\sqrt{\lambda}} P_\lambda(x,y). \tag{4.8}$$

(The general solution is easily obtained from W_t, and even the solution to an inhomogeneous version of (4.5) may be expressed using a Duhamel type integral.) Previous attempts to compute W_t ([DSV] and [CDS]) have been rather frustrating because it appears to be highly oscillatory, and we were no more successful using the method of Section 3. However, we found it was possible to compute the time integral

$$\int_0^t W_s(x,y) ds \tag{4.9}$$

of the wave propagator, also a spectral operator with kernel

$$\sum_\lambda \Big(\frac{1 - \cos t\sqrt{\lambda}}{\lambda} \Big) P_\lambda(x,y), \tag{4.10}$$

as this is now an absolutely convergent sum. This is still a highly oscillatory function, but the partial sums appear to be converging. The wave propagator itself is the derivataive of this function. Figure 12 shows the trace, which is just

$$\sum_{j=1}^\infty \frac{1 - \cos t\sqrt{\lambda_j}}{\lambda_j}. \tag{4.11}$$

The last space–time equation we consider is the Laplace equation

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \Delta_\lambda u(x,t) = 0 \text{ in } t > 0 \tag{4.12}$$

subject to the same initial and boundary conditions as the heat equation, as well as the condition that u is bounded as $t \to \infty$. This leads to a spectral operator whose kernel is the Poisson kernel

$$p_t(x,y) = \sum_\lambda e^{-t\sqrt{\lambda}} P_\lambda(x,y). \tag{4.13}$$

The well-known Bochner formula

$$P_t(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} h_{t^2/4u}(x,y) du \tag{4.14}$$

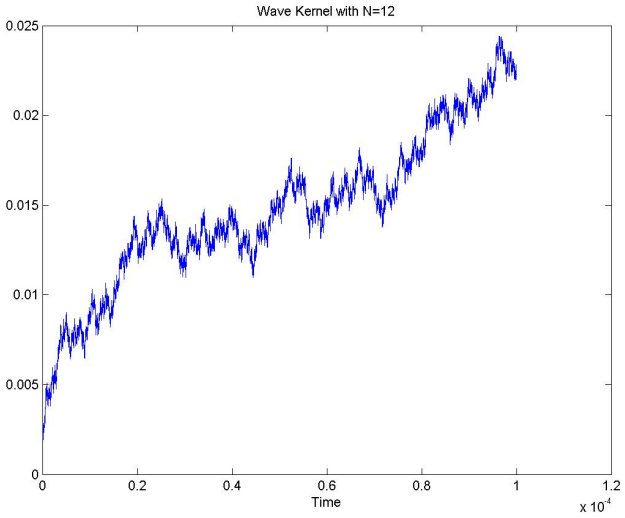

Figure 12: Trace of the integral of the wave propagator

relates the heat and Poisson kernels, and allows us to transfer asymptotic statements, such as those discussed in section 2. For example, we can substitute equation 2.7 into 4.14. If we write the Fourier series expansion

$$g(t) = \sum_{-\infty}^{\infty} b_n e^{-2\pi i n \log t/\log \tau} \tag{4.15}$$

then the trace $p(t) = \int p_t(x,x) d\mu(x)$ satisfies

$$p(t) = t^{-2\beta} \sum_{-\infty}^{\infty} c_n b_n e^{-2\pi i n \log t/\log \tau} + R(t) \tag{4.16}$$

for

$$c_n = \frac{1}{\pi} \int_0^{\infty} e^{-u} u^{\beta - \frac{1}{2}} e^{2\pi i n \log(tu)/\log \tau} du, \tag{4.17}$$

with remainder

$$R(t) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{e^{-u}}{\sqrt{u}} o\left(\frac{4u}{t^2}\right)^{\beta} du = o(t^{-2\beta}) \tag{4.18}$$

Figure 13 shows the graph of $t^{2\beta} p(t)$. We can also evaluate the Poisson kernel at any junction point, and 14 shows the kernel at (q_0, q_0).

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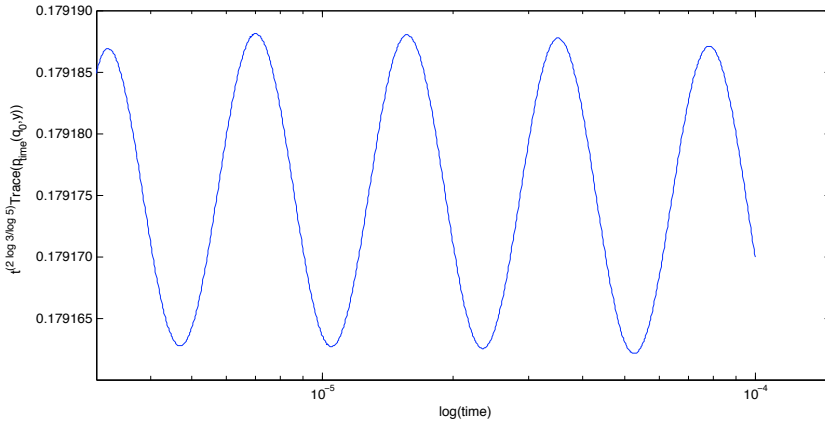

Figure 13: Trace of the Poisson kernel.

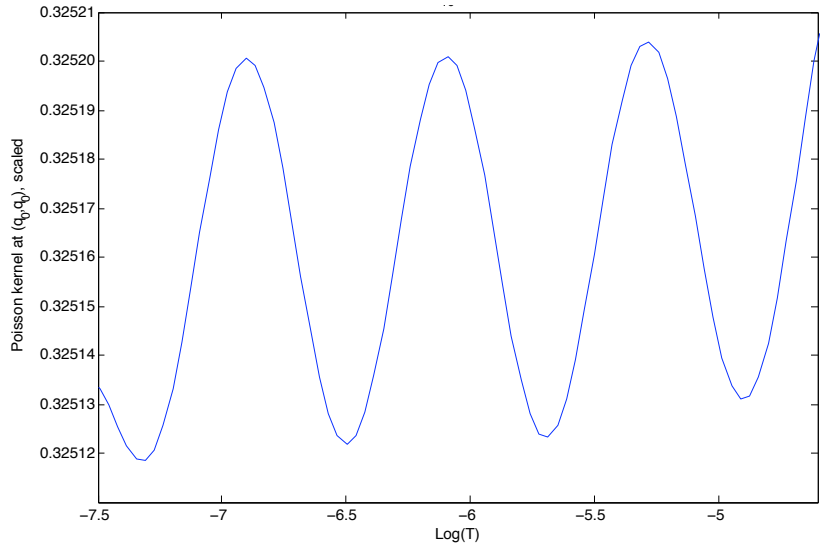

Figure 14: The Poisson kernel at (q_0, q_0).

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