GRADIENTS OF LAPLACIAN EIGENFUNCTIONS ON THE SIERPINSKI GASKET

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Abstract. We use spectral decimation to provide formulae for computing the harmonic gradients of Laplacian eigenfunctions on the Sierpinski Gasket. These formulae are given in terms of special functions that are defined as infinite products.

1. Introduction

There are few functions more ubiquitous in Euclidean analysis than the sine, cosine and exponential, which are the eigenfunctions of the Laplacian on an interval in $\mathbb{R}$. In the theory of analysis on fractals, the Laplacian eigenfunctions arguably have an even more prominent role, as the Laplacian is the fundamental differential operator on which the analysis is based. Despite this, there are a number of interesting open questions about the structure of such eigenfunctions. In this paper we consider the local behavior of Laplacian eigenfunctions on the Sierpinski Gasket (SG), in terms of the harmonic tangents and gradients introduced by Teplyaev in [6]. Using the spectral decimation method of Fukashima and Shima [1] (see also Chapter 3 of [5]) we give infinite product formulae for the tangents at boundary points, and use them to describe the one-sided tangents at junction points. These results may be seen as a Sierpinski Gasket version of the well known formulae for the derivatives of the sine and cosine functions on an interval, though the precise analogue on $[0,1]$ is more complicated (see Equations 2.2-2.4).

The Sierpinski Gasket is the simplest non-trivial example of a fractal to which the standard theory of analysis on fractals applies. We refer to the monographs [2, 5] for detailed proofs of all results we use from this theory. Recall that $SG \subset \mathbb{R}^2$ is the attractor of an iterated function system consisting of three maps $F_i(x) = (x+q_i)/2$, where the points q_0, q_1 and q_2 are the vertices of an equilateral triangle. This means that $SG = \cup_i F_i(SG)$, where the sets $F_i(SG)$ are usually referred to as 1-cells. For a length m word $w = w_1 \dots w_m$ with letters $w_j \in \{0,1,2\}$ we define $F_w = F_{w_1} \circ \cdots \circ F_{w_m}$ and call $F_w(SG)$ an m-cell. The points q_i, $i = 0,1,2$ are the boundary of SG; the set of boundary points is V_0, and we use V_m to denote points of the form $F_w(q_i)$ where w is a word of length m. These V_m are vertices of the usual graph approximation of SG at scale m, in which vertices x and y are joined by an edge (written $x \sim_m y$) if they belong to a common m-cell. Clearly $V_* = \cup_m V_m$ is dense in SG.

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The Laplacian Δ on SG is a renormalized limit of graph Laplacians Δ_m on the m-scale graphs:

$$\Delta u(x) = \frac{3}{2} \lim_{m\to\infty} 5^m \Delta_m u(x) \tag{1.1}$$

$$\Delta_m u(x) = \sum_{y\sim_m x} (u(y) - u(x)) \quad \text{for } x \in V_m \setminus V_0 \tag{1.2}$$

and we say $u \in \mathrm{dom}(\Delta)$ if the right side of (1.1) converges uniformly on $V_* \setminus V_0$ to a continuous function. The function is extended to SG by continuity and density of V_*. At a boundary point $q_i \in V_0$ there is an associated normal derivative defined (with $q_{i+3} = q_i$) by

$$\partial_n u(q_i) = \lim_{m\to\infty} \Big(\frac{5}{3}\Big)^m (2u(q_i) - u(F_i^m(q_{i+1})) - u(F_i^m(q_{i+2}))).$$

A harmonic function h is one for which $\Delta h = 0$, and for any assignment of values on V_0 there is a unique harmonic function with these boundary values. For this reason we identify the harmonic functions with the space of functions on V_0. The harmonic functions are also graph harmonic, so it is elementary to compute the values of h on V_1 from those on V_0, and recursively to obtain the values on V_m for any m. It will be useful to formalize this by defining the harmonic extension matrices A_i, which map the values of h on V_0 to those on $F_i(V_0)$, by

$$\begin{pmatrix} h\circ F_i(q_0) \\ h\circ F_i(q_1) \\ h\circ F_i(q_2) \end{pmatrix} = A_i \begin{pmatrix} h(q_0) \\ h(q_1) \\ h(q_2) \end{pmatrix}$$

and more generally $(h\circ F_w(q_0), h\circ F_w(q_1), h\circ F_w(q_2))^T = A_w(h(q_0), h(q_1), h(q_2))^T$, where $A_w = A_{w_m} A_{w_{m-1}} \cdots A_{w_1}$. We usually write this in the compact form

$$h\big|_{F_w V_0} = A_w h\big|_{V_0}.$$

The matrices are

$$A_0 = \frac{1}{5}\begin{pmatrix} 5 & 0 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix}, \quad A_1 = \frac{1}{5}\begin{pmatrix} 2 & 2 & 1 \\ 0 & 5 & 0 \\ 1 & 2 & 2 \end{pmatrix}, \quad A_2 = \frac{1}{5}\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 5 \end{pmatrix}.$$

The structure of eigenfunctions of the Laplacian is similar to that of the harmonic functions. Specifically, it is true on SG that if m is sufficiently large then the restriction of a function u satisfying $-\Delta u = \lambda u$ from SG to V_m gives an eigenfunction of the graph Laplacian $-\Delta_m u = \lambda_m u$, with

$$\lambda_m(5 - \lambda_m) = \lambda_{m-1} \tag{1.3}$$

$$\lambda = \frac{3}{2} \lim_{m\to\infty} 5^m \lambda_m. \tag{1.4}$$

Note that (1.3) implies that λ_m is one of $\frac{1}{2}(5 \pm \sqrt{25 - 4\lambda_{m-1}})$, but the positive root is only permitted to occur for finitely many values of m in order that the limit in (1.4) exists. This *spectral decimation* property was first recognized by Fukashima and Shima [1]. It is not true on all fractals, but on those where it is valid, it gives both a method for computing the

spectrum and a recursion for the eigenfunctions [3]. Let us define

$$A_0(\lambda) = \frac{1}{(5-\lambda)(2-\lambda)}\begin{pmatrix}(5-\lambda)(2-\lambda) & 0 & 0\\ (4-\lambda) & (4-\lambda) & 2\\ (4-\lambda) & 2 & (4-\lambda)\end{pmatrix} \tag{1.5}$$

$$A_1(\lambda) = \frac{1}{(5-\lambda)(2-\lambda)}\begin{pmatrix}(4-\lambda) & (4-\lambda) & 2\\ 0 & (5-\lambda)(2-\lambda) & 0\\ 2 & (4-\lambda) & (4-\lambda)\end{pmatrix} \tag{1.6}$$

$$A_2(\lambda) = \frac{1}{(5-\lambda)(2-\lambda)}\begin{pmatrix}(4-\lambda) & 2 & (4-\lambda)\\ 2 & (4-\lambda) & (4-\lambda)\\ 0 & 0 & (5-\lambda)(2-\lambda)\end{pmatrix} \tag{1.7}$$

provided $\lambda \neq 2, 5$. The essence of the spectral decimation method on SG may then be summarized in the following theorem, which we have taken from Sections 3.2 and 3.3 of [5].

Theorem 1.1 (Spectral Decimation Method)**.** *If* $(\Delta + \lambda)u = 0$ *then there is a sequence* $\{\lambda_m\}_{m\geq m_0}$ *satisfying* (1.3) *and* (1.4)*, and such that* $\lambda_m \neq 2, 5, 6$ *for* $m > m_0$*, with the property that* $(\Delta_m + \lambda_m)u\big|_{V_m} = 0$ *for all* $m \geq m_0$*. The values of* u *on* V_m *for* $m > m_0$ *can be constructed recursively using the matrices from* (1.5)*,* (1.6)*, and* (1.7) *as follows. If* $w = w_1 w_2 \dots w_m$ *then*

$$u\big|_{F_w(V_0)} = A_{w_m}(\lambda_m)A_{w_{m-1}}(\lambda_{m-1})\cdots A_{w_{m_0+1}}(\lambda_{m_0+1})u\big|_{F_{w_1\dots w_{m_0}}(V_0)}\,, \tag{1.8}$$

and we call this the spectral decimation relation.

If λ *is not a Dirichlet eigenvalue then we may assume* $m_0 = 0$*, at which point the condition* $(\Delta_{m_0} + \lambda_{m_0})u\big|_{V_{m_0}} = 0$ *is taken to be vacuous. The corresponding eigenspace is* 3*-dimensional and parametrized by the values of* u *on* V_0*.*

If λ *is Dirichlet several possibilities occur. We indicate the initial configurations, all of which may then be continued by the spectral decimation formula. A spanning set for the configurations when* $m_0 = 1$ *are shown in Figure 1. The one on the left has* $\lambda_1 = 2$ *while those on the right have* $\lambda_1 = 5$*. If* $m_0 \geq 2$ *then* $\lambda_{m_0} = 5$ *or* $\lambda_{m_0} = 6$*, and in the latter case* $\lambda_{m_0+1} = 3$*. Those with* $\lambda_{m_0} = 5$ *are formed from scaled and rotated copies of the functions on the right in Figure 1, arranged so that their normal derivatives cancel. A basis of chains for* $m_0 = 2$ *is shown in Figure 2; those for general* m *are naturally indexed by the loops in* V_m*, plus two strands connecting points of* V_0*. In the case* $\lambda_{m_0} = 6$ *the eigenfunctions are indexed by points in* $V_{m_0-1} \setminus V_0$*; a basis is obtained by scaling and rotating two copies of the function on the left in Figure 3 and gluing them at the chosen point, as shown on the right in Figure 3 for the case* $m_0 = 2$ *and a point in* V_1*.*

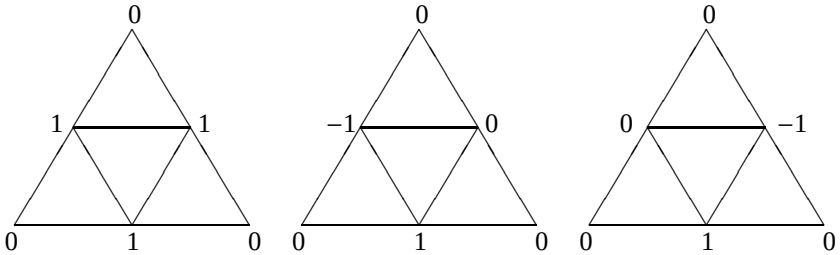

FIGURE 1. Dirichlet eigenfunctions with $m_0 = 1$.

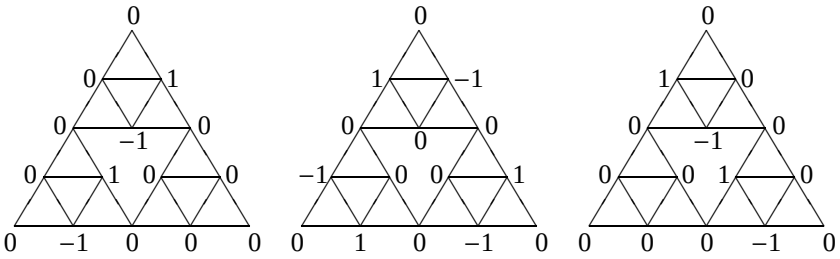

FIGURE 2. A basis of chains for $m_0 = 2$ and $\lambda_{m_0} = 5$

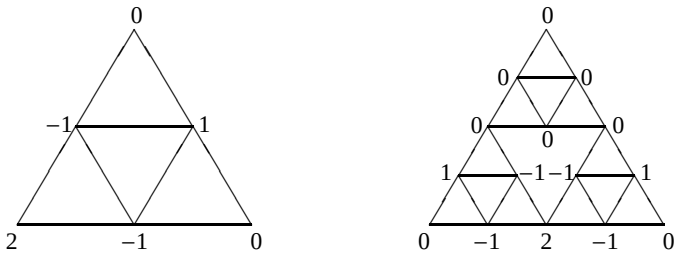

FIGURE 3. Eigenfunction construction in the case $\lambda_{m_0} = 6$.

2. TANGENTS TO EIGENFUNCTIONS

In [6], Teplyaev introduced the notion of a harmonic gradient and harmonic tangent for functions on the Sierpinski Gasket. He also proved that functions in the domain of the Laplacian have harmonic gradients at all points in a set of full measure, and that functions with Hölder continuous Laplacian have harmonic gradients at all points. Eigenfunctions of the Laplacian fall into the latter category because of the well-known fact that continuity of $\Delta^2 u$ implies Hölder continuity of Δu ([2], Lemma 2.2.5).

Defintion 2.1 ([6], Section 3)**.** Let $w = w_1 w_2 \dots$ be an infinite word, and $[w]_m = w_1 w_2 \dots w_m$ be the length m truncation of w. If u is a function on SG we let $H_{[w]_m} u$ be the harmonic function on SG which coincides with u on $F_{[w]_m}(V_0)$, so

$$H_{[w]_m} u = A^{-1}_{[w]_m} u\big|_{F_{[w]m}(V_0)} = A^{-1}_{w_1} \cdots A^{-1}_{w_m} u\big|_{F_{[w]_m}(V_0)} . \tag{2.1}$$

Define the harmonic tangent $T_w u$ of u at w to be $\lim_{m\to\infty} H_{[w]_m} u$, if the limit exists. It should be remarked that there can be two words w and w' such that $F_w(SG) = F_{w'}(SG)$. We will nonetheless treat the tangents T_w and $T_{w'}$ separately, as they are rarely equal.

The harmonic gradient is defined in the same way, but using the space of harmonic functions with average zero and the projection of the action of the matrices A_i to this subspace. It is evident that if u is continuous then the gradient exists whenever the tangent exists, and conversely. See [4] for details.

If we consider the interval $[0, 1]$ rather than the Sierpinski Gasket then it is clear that the harmonic tangent at x_0 is the vector $(L(0), L(1))^T$ characterizing the unique linear function $L(x)$ having the properties $L(x_0) = f(x_0)$ and $L'(x_0) = f'(x_0)$. For an eigenvalue $\lambda \in \mathbb{R}$ which is not equal to $\pi^2 k^2$ for any $k \in \mathbb{N}$, it may readily be verified that the harmonic tangent is given by

$$M(x_0) \begin{pmatrix} f(0) \\ f(1) \end{pmatrix} \tag{2.2}$$

where

$$M(x_0) = \frac{1}{\sin\sqrt{\lambda}}\begin{pmatrix} a+b & a-b \\ a-b & a+b \end{pmatrix} \tag{2.3}$$

and

$$a = \sin\left((1-x_0)\sqrt{\lambda}\right), \quad b = \sqrt{\lambda}x_0\cos\left((1-x_0)\sqrt{\lambda}\right). \tag{2.4}$$

We cannot obtain as explicit a description of the harmonic tangent on SG; however, we produce formulae that permit its computation at any point of V_*. The key observation is that when u is an eigenfunction, the computation of the gradient has a particularly elegant structure. Recall from Theorem 1.1 that the values of u on $F_{[w]_m}(V_0)$ may be computed using the spectral decimation method, meaning that starting from a scale m_0 they can be obtained as in (1.8). Combining this with (2.1) we see that

$$\begin{aligned} T_w u &= \lim_{m\to\infty} A_{w_1}^{-1}\cdots A_{w_m}^{-1}\cdot A_{w_m}(\lambda_m)\cdots A_{w_{m_0+1}}(\lambda_{m_0+1})\,u\big|_{F_{w_1\dots w_{m_0}}(V_0)} \\ &= \left(A_{w_1}^{-1}\cdots A_{w_{m_0}}^{-1}\right)\left(\lim_{m\to\infty} A_{w_{m_0+1}}^{-1}\cdots A_{w_m}^{-1}\cdot A_{w_m}(\lambda_m)\cdots A_{w_{m_0+1}}(\lambda_{m_0+1})\right)u\big|_{F_{w_1\dots w_{m_0}}(V_0)} \end{aligned} \tag{2.5}$$

in which we know the limit exists by Theorem 3 of [6]. A special case occurs when $F_w(SG)$ is a point in V_*, because in this case, all but finitely many letters in the word w are equal to a single letter i. By taking m_0 to be sufficiently large we see that it is useful to understand the limit

$$\lim_{k\to\infty} A_i^{-k}\cdot A_i(\lambda_{m_0+k})\cdots A_i(\lambda_{m_0+1})$$

and it is evident from the symmetry of the matrices A_i and $A_i(\lambda)$ that it suffices to deal with the case $i = 0$.

Theorem 2.2. *Let $\alpha = (0,1,1)^T$, $\beta = (0,1,-1)^T$, $\gamma_m = (4, 4-\lambda_m, 4-\lambda_m)^T$. If neither of the values 2 or 5 occur in the sequence $\{\lambda_m\}_{m>m_0}$, then*

$$\lim_{k\to\infty} A_0^{-k}\cdot A_0(\lambda_{m_0+k})\cdots A_0(\lambda_{m_0+1})\,\alpha = \frac{4\lambda}{3\cdot 5^{m_0}\lambda_{m_0}(2-\lambda_{m_0+1})}\prod_{j=2}^{\infty}\left(1-\frac{\lambda_{m_0+j}}{3}\right)\alpha$$

$$\lim_{k\to\infty} A_0^{-k}\cdot A_0(\lambda_{m_0+k})\cdots A_0(\lambda_{m_0+1})\beta = \frac{2\lambda}{3\cdot 5^{m_0}\lambda_{m_0}}\beta$$

$$\lim_{k\to\infty} A_0^{-k}\cdot A_0(\lambda_{m_0+k})\cdots A_0(\lambda_{m_0+1})\,\gamma_{m_0} = (4,4,4)^T$$

Proof. Observe that α and β are eigenvectors of $A_0(\lambda)$, with eigenvalues $(6-\lambda)(2-\lambda)^{-1}(5-\lambda)^{-1}$ and $(5-\lambda)^{-1}$ respectively, provided $\lambda \neq 2, 5$. As $A_0 = A_0(0)$ is a special case, we compute immediately that

$$\lim_{k\to\infty} A_0^{-k}\cdot A_0(\lambda_{m_0+k})\cdots A_0(\lambda_{m_0+1})\beta = \lim_{k\to\infty} 5^k\prod_{j=1}^{k}(5-\lambda_{m_0+j})^{-1}\beta.$$

However induction on (1.3) implies that $\lambda_{m_0} = \lambda_{m_0+k}\prod_{j=1}^{k}(5-\lambda_m)$ and therefore

$$\lim_{k\to\infty} A_0^{-k}\cdot A_0(\lambda_{m_0+k})\cdots A_0(\lambda_{m_0+1})\beta = \lim_{k\to\infty} 5^k\lambda_{m_0+k}\lambda_{m_0}^{-1}\beta = \frac{2\lambda}{3\cdot 5^{m_0}\lambda_{m_0}}\beta$$

The corresponding computation for α can be simplified by observing from (1.3) that $(6-\lambda_m) = (3-\lambda_{m+1})(2-\lambda_{m+1})$, whereupon

$$\begin{aligned}
&\lim_{k\to\infty} A_0^{-k}\cdot A_0(\lambda_{m_0+k})\cdots A_0(\lambda_{m_0+1})\alpha\\
&\quad= \lim_{k\to\infty}\Big(\frac{5}{3}\Big)^k \prod_{j=1}^{k}\frac{(6-\lambda_{m_0+j})}{(5-\lambda_{m_0+j})(2-\lambda_{m_0+j})}\alpha\\
&\quad= \lim_{k\to\infty}\Big(\frac{2-\lambda_{m_0+k}}{2-\lambda_{m_0+1}}\Big)\Big(5^k\prod_{j=1}^{k}\frac{1}{5-\lambda_{m_0+j}}\Big)\Big(3^{-k}\prod_{j=1}^{k}(3-\lambda_{m_0+j+1})\Big)\alpha\\
&\quad= \Big(\frac{2}{2-\lambda_{m_0+1}}\Big)\Big(\frac{2\lambda}{3\cdot 5^{m_0}\lambda_{m_0}}\Big)\prod_{j=2}^{k+1}\Big(1-\frac{\lambda_{m_0+j}}{3}\Big)\alpha
\end{aligned}$$

where we used the previously computed limit for the middle factor, and the fact that $\lambda_m = O(5^{-m})$ as $m\to\infty$ from (1.4).

For γ_m the situation is a little different. Observe that

$$A_0(\lambda_{m+1})\gamma_m = (2-\lambda_{m+1})^{-1}(5-\lambda_{m+1})^{-1}\begin{pmatrix} 4(2-\lambda_{m+1})(5-\lambda_{m+1})\\ 4(4-\lambda_{m+1})+(4-\lambda_m)(6-\lambda_{m+1})\\ 4(4-\lambda_{m+1})+(4-\lambda_m)(6-\lambda_{m+1})\end{pmatrix}$$

however we can perform the following simplification from (1.3):

$$\begin{aligned}
4(4-\lambda_{m+1})+(4-\lambda_m)(6-\lambda_{m+1}) &= 4(4-\lambda_{m+1})+(4-5\lambda_{m+1}+\lambda_{m+1}^2)(6-\lambda_{m+1})\\
&= 4(4-\lambda_{m+1})+(4-\lambda_{m+1})(1-\lambda_{m+1})(6-\lambda_{m+1})\\
&= (4-\lambda_{m+1})(10-7\lambda_{m+1}+\lambda_{m+1}^2)\\
&= (4-\lambda_{m+1})(5-\lambda_{m+1})(2-\lambda_{m+1}).
\end{aligned}$$

Inserting this into the previous computation shows that $A_0(\lambda_{m+1})\gamma_m = \gamma_{m+1}$ provided $\lambda_m \neq 2, 5$, and therefore $A_0(\lambda_{m_0+k})\cdots A_0(\lambda_{m_0+1})\gamma_{m_0} = \gamma_{m_0+k}$. To proceed we must apply A_0^{-k} to γ_{m_0+k}, which is most easily done by writing it in terms of eigenvectors as $\gamma_{m_0+k} = (4,4,4)^T - \lambda_{m_0+k}\alpha$. The result is

$$\begin{aligned}
\lim_{k\to\infty} A_0^{-k}\cdot A_0(\lambda_{m_0+k})\cdots A_0(\lambda_{m_0+1})\gamma_{m_0} &= \lim_{k\to\infty} A_0^{-k}\Big((4,4,4)^T-\lambda_{m_0+k}\alpha\Big)\\
&= \lim_{k\to\infty}\Big((4,4,4)^T-(\frac{5}{3})^k\lambda_{m_0+k}\Big)\\
&= (4,4,4)^T
\end{aligned}$$

because $\lambda_m = O(5^{-m})$ as $m\to\infty$. □

Theorem 2.3. *Suppose w is a word of the form $w = [w]_{k_0}000\cdots$, that $(\Delta+\lambda)u = 0$, and that m_0 is chosen large enough that the spectral decimation formula holds with $\lambda_m \neq 2, 5, 6$ when $m > m_0$. If $k = \max\{k_0, m_0\}$ then*

$$T_w u = \Big(A_{w_1}^{-1}\cdots A_{w_k}^{-1}\Big)M_0(\lambda,k)\,A_{w_k}(\lambda_k)\cdots A_{w_{m_0+1}}(\lambda_{m_0+1})\,u\big|_{F_{w_1\dots w_{m_0}}(V_0)} \tag{2.6}$$

where

$$M_0(\lambda,k) = \begin{pmatrix} 1 & 0 & 0\\ 1-\frac{(4-\lambda_k)\lambda\tau_k(\lambda)}{3\cdot 5^k\lambda_k} & \frac{\lambda(2\tau_k(\lambda)+1)}{3\cdot 5^k\lambda_k} & \frac{\lambda(2\tau_k(\lambda)-1)}{3\cdot 5^k\lambda_k}\\ 1-\frac{(4-\lambda_k)\lambda\tau_k(\lambda)}{3\cdot 5^k\lambda_k} & \frac{\lambda(2\tau_k(\lambda)-1)}{3\cdot 5^k\lambda_k} & \frac{\lambda(2\tau_k(\lambda)+1)}{3\cdot 5^k\lambda_k}\end{pmatrix} \tag{2.7}$$

and

$$\tau_k(\lambda) = \frac{1}{(2-\lambda_{k+1})}\prod_{j=2}^{\infty}\Big(1-\frac{\lambda_{k+j}}{3}\Big).$$

Proof. From (2.5) we have

$$\begin{aligned} T_w u &= \Big(A_{w_1}^{-1}\cdots A_{w_k}^{-1}\Big)\Big(\lim_{m\to\infty} A_0^{-(m-k)}\cdot A_0(\lambda_m)\cdots A_0(\lambda_{k+1})\Big)\,u\big|_{F_{w_1\dots w_k}(V_0)} \\ &= \Big(A_{w_1}^{-1}\cdots A_{w_k}^{-1}\Big)\Big(\lim_{m\to\infty} A_0^{-(m-k)}\cdot A_0(\lambda_m)\cdots A_0(\lambda_{k+1})\Big)A_{w_k}(\lambda_k)\cdots A_{w_{m_0+1}}(\lambda_{m_0+1})\,u\big|_{F_{w_1\dots w_{m_0}}(V_0)} \end{aligned}$$

because $w_j = 0$ for $j \geq k$ and spectral decimation applies for $j \geq m_0$. The result is therefore equivalent to

$$M_0 = \lim_{m\to\infty} A_0^{-(m-k)}\cdot A_0(\lambda_m)\cdots A_0(\lambda_{k+1})$$

which follows from Theorem 2.2. □

Remark 2.4. The function $\tau_k(\lambda)$ may appear to depend on the sequence $\{\lambda_j\}$, but in fact this sequence is uniquely determined by λ. Indeed, there is an entire analytic function $\Psi(z)$ with the property that $\lambda_j = \Psi(5^{-j}\lambda)$. To see this, let $\psi(z) = z(5-z)$ and $\psi_m(z) = \psi^{\circ m}(\frac{2}{3}5^{-m}z)$. The sequence $\psi_m(z)$ consists of entire functions with $\psi_m(0) = 0$ and $\psi_m'(0) = \frac{2}{3}$, so is normal with limit $\Psi(z)$, a power series for which may be computed recursively. It is then clear that $\Psi(5^{-j}\lambda) = \lim_m \Psi(\frac{3}{2}5^{m-j}\lambda_m) = \lim_m \psi_{m-j}(\lambda_m) = \lambda_j$. Moreover, we may define

$$\Upsilon(\lambda) = \frac{1}{(2-\Psi(5^{-1}\lambda))}\prod_{j=2}^{\infty}\Big(1-\frac{\Psi(5^{-j}\lambda)}{3}\Big) \tag{2.8}$$

at which point $\tau_k(\lambda) = \Upsilon(5^{-k}\lambda)$. In the same way that there are special functions associated to differential equations in Euclidean analysis, we suggest that the functions $\Psi(\lambda)$ and $\Upsilon(\lambda)$ should be considered to be special functions in analysis on the Sierpinski Gasket. In terms of these functions, (2.7) has the form

$$M_0(\lambda,k) = \begin{pmatrix} 1 & 0 & 0 \\ 1-\frac{\lambda\left(4-\Psi(5^{-k}\lambda)\right)\Upsilon(5^{-k}\lambda)}{3\cdot 5^k\Psi(5^{-k}\lambda)} & \frac{\lambda\left(2\Upsilon(5^{-k}\lambda)+1\right)}{3\cdot 5^k\Psi(5^{-k}\lambda)} & \frac{\lambda\left(2\Upsilon(5^{-k}\lambda)-1\right)}{3\cdot 5^k\Psi(5^{-k}\lambda)} \\ 1-\frac{\lambda\left(4-\Psi(5^{-k}\lambda)\right)\Upsilon(5^{-k}\lambda)}{3\cdot 5^k\Psi(5^{-k}\lambda)} & \frac{\lambda\left(2\Upsilon(5^{-k}\lambda)-1\right)}{3\cdot 5^k\Psi(5^{-k}\lambda)} & \frac{\lambda\left(2\Upsilon(5^{-k}\lambda)+1\right)}{3\cdot 5^k\Psi(5^{-k}\lambda)} \end{pmatrix}$$

As a particular consequence we may compute the normal derivatives of the eigenfunctions at points of V_0, because they are the same as the normal derivatives of the tangent functions. We expect this observation to have applications in the construction of a resolvent for the Laplacian.

Corollary 2.5. *If $(\Delta+\lambda)u = 0$ and the spectral decimation formula holds with $\lambda_m \neq 2, 5, 6$ for $m > 0$, then the normal derivative of u at q_0 is*

$$\partial_n u(q_0) = \Big((4-\lambda_0)u(q_0) - 2u(q_1) - 2u(q_2)\Big)\frac{2\lambda\Upsilon(\lambda)}{3\lambda_0}.$$

Excluding the values 2,5 and 6 from the sequence $\{\lambda_m\}$ in Theorems 2.2 and 2.3 is necessary because they occur precisely in the Dirichlet case, where the boundary data vanishes and cannot be used to determine the tangent. Nonetheless, Theorem 2.3 may be applied to find tangents to Dirichlet eigenfunctions in a simple fashion. The reason is that the description of the Dirichlet eigenfunctions given in Theorem 1.1 ensures that we need only compute the harmonic tangents of the functions in Figure 1 and the left of Figure 3. All harmonic tangents to Dirichlet eigenfunctions are then obtained from these by scaling and taking suitable linear combinations.

For the basic element used to construct the 6-series (Figure 3) we can directly apply Theorem 2.3 with $\lambda_1 = 6$ and $\lambda_2 = 3$. For example, if the top vertex in Figure 3 is q_0 and $w = 0\cdots$ we have

$$T_w u = \frac{\lambda}{9}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

To calculate the harmonic tangents of the basic element of the 2-series (on the left in Figure 1) we apply Theorem 2.3 to the function shown at left in Figure 4, starting the spectral decimation at each of the values $\lambda_1 = \frac{5\pm\sqrt{17}}{2}$. For the 5-series there are two basic elements (shown at right in Figure 1) and we proceed by calculating harmonic tangents for the initial configurations shown in the center and on the right of Figure 4, starting with $\lambda_1 = \frac{5\pm\sqrt{5}}{2}$. The harmonic tangents of all 2 and 5-series eigenfunctions then coincide with scaled and rotated copies of these pieces and their negatives, assembled in the obvious manner.

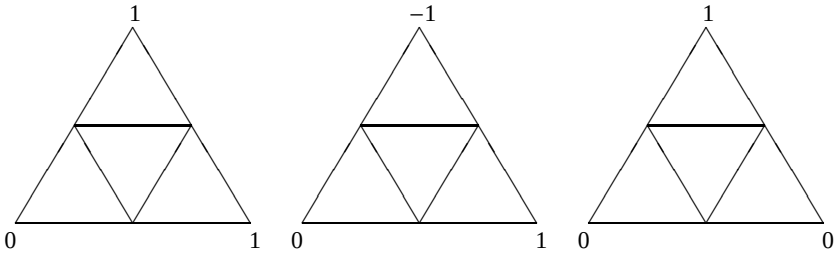

FIGURE 4. Computing harmonic tangents of 2 and 5-series eigenfunctions.

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