

# Quantum dynamics of a wave packet for a Cantor spectrum

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- 2 Analytical study in the case of a Cantor set
  - Appropriate measure
  - Time-averaged return probability
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Consider systems with a (spatial and/or spectral) **fractal feature**.

Motivation :

- **singular continuous** spectra are ubiquitous in physical systems, but not well known
- Explain experimental and numerical observations

Objectives :

- Identify typical behaviour of relevant physical quantities in the presence of a fractal feature (spatial and/or spectral) → power-law modulated by log-periodic oscillations
- Derive analytical expressions, and relate them to experimental results

Fractal :

- a set with a **discrete scaling symmetry**
- fractal dimension  $d_f$   
(length scale  $l$ ; volume scales as  $l^{d_f}$ )

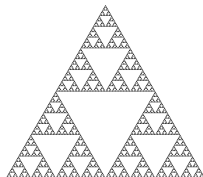


Figure: Sierpinski gasket ;  $d_f = \frac{\ln 3}{\ln 2}$

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Quantum dynamics for a triadic Cantor set spectrum.

$H = -\frac{d^2}{dx^2} + V(x)$  on  $L^2(\mathbb{R})$ , with associated triadic Cantor set spectrum.

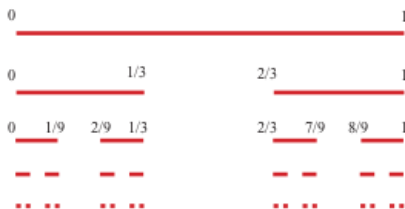
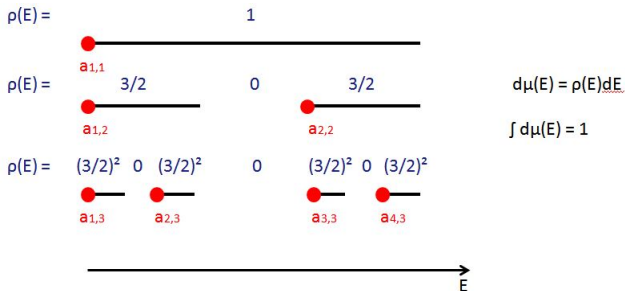


Figure: Triadic Cantor set. Fractal dimension  $d_f = \frac{\ln 2}{\ln 3}$

Triadic Cantor set  $K$  : zero (Lebesgue) measure. Degenerate density of states  $\rho(\epsilon)$ . Define measure :  $d\mu(\epsilon) \equiv \rho(\epsilon)d\epsilon$  :



Verifies :  $\int_K f(x)d\mu(x) = \frac{1}{2} \int_K f(\frac{x}{3})d\mu(x) + \frac{1}{2} \int_K f(\frac{x+2}{3})d\mu(x)$   
 (scaling property)

How does the fractal feature affect the dynamics of the wave packet  $|\psi(t)\rangle$  ?

- 1 RMS displacement  $\Delta x(t) = \sqrt{\int (x(t) - x_0)^2 |\psi(x, t)|^2 dx}$
- 2 return probability  $p(t) = |\langle \psi(0) | \psi(t) \rangle|^2$
- 3 ...

A relevant quantity : the **time-averaged return probability** :

$$C(t) = \frac{1}{t} \int_0^t |\langle \psi(0) | \psi(t') \rangle|^2 dt'$$

A relevant quantity : the **time-averaged return probability** :

$$C(t) = \frac{1}{t} \int_0^t |\langle \psi(0) | \psi(t') \rangle|^2 dt' = \iint_{K \times K} \text{sinc}((\epsilon' - \epsilon)t) d\mu(\epsilon) d\mu(\epsilon')$$

Note :  $\epsilon$  and  $t$  are coupled variables.

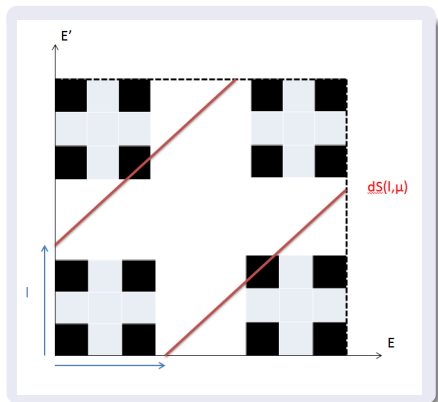
Method

- 1 use the scaling symmetry of the measure to obtain a functional equation
- 2 apply the Mellin transform

$$C(t) = \iint_{K \times K} \text{sinc}((\epsilon' - \epsilon)t) d\mu(\epsilon) d\mu(\epsilon')$$

Change of variable :  $l = |\epsilon' - \epsilon|$   
 $dS(l, \mu) = \text{area in the sense of } \mu$   
of the 2D Cantor set ("Cantor dust") within the red strips

$$C(t) = \int_0^1 \text{sinc}(lt) dS(l, \mu)$$



(cf. energy integrals : D. Bessis et al., Phys. Rev. A, Vol. 36, number 2, July 15, 1987)



The main steps :

- ① Mellin transform :  $\zeta(\alpha) = \int_0^1 l^{-\alpha} dS(l, \mu) = \iint_{K \times K} \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha}$
- ② scaling property :  

$$\iint_{K \times K} \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha} = \frac{1}{2} \iint_{K \times K} \frac{d\mu(x)d\mu(y)}{3^{-\alpha}|x-y|^\alpha} + \frac{1}{2} \iint_{K \times K} \frac{d\mu(x)d\mu(y)}{3^{-\alpha}|x-y+2|^\alpha}$$

$$= \frac{3^\alpha}{2} \zeta(\alpha) + \epsilon(\alpha)$$
- ③  $\zeta(\alpha) = \frac{\epsilon(\alpha)}{1 - \frac{3^\alpha}{2}}$
- ④ inverse Mellin transform  $\rightarrow dS(l, \mu) \rightarrow C(t)$

*Remind :*

- Mellin transform of  $f$  :  $M_f(s) = \int_0^\infty f(x)x^{s-1}dx$
- Inverse Mellin transform :  $M_\phi^{-1}(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \phi(s)x^{-s}ds$

## Time-averaged return probability for triadic Cantor set

$$C(t) = \frac{t^{-\ln 2 / \ln 3}}{\ln(3)} \sum_n \epsilon_n e^{-\frac{2i\pi n}{\ln(3)} \ln t} \sin\left(\frac{\pi}{2}(d_n - 1)\right) \Gamma(d_n - 1)$$

$d_n = \frac{\ln(2)}{\ln(3)} + \frac{2i\pi n}{\ln(3)}$ ,  $\epsilon_n = \int_{K \times K} \frac{d\mu(x)d\mu(y)}{|x-y+2|^{d+\frac{2i\pi n}{\ln(3)}}}$ ,  $\Gamma$  the Euler gamma function

$\Rightarrow C(t) = t^{-\ln 2 / \ln 3} G_p(\ln t / \ln 3)$ ,  $G_p(y+1) = G_p(y)$

- 1 power-law modulated by log-periodic oscillations.
- 2  $C(t) \propto t^{-d_f}$ , with  $d_f = \frac{\ln 2}{\ln 3}$  (fractal dimension) for a triadic Cantor set.

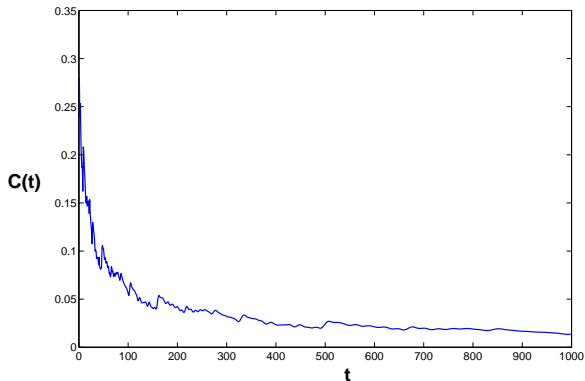


Figure: Return probability  $C(t)$  for a Cantor spectrum

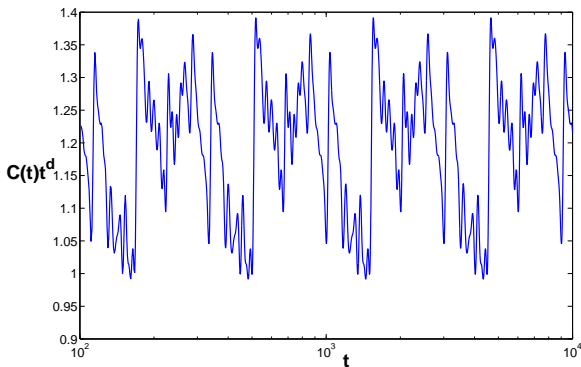


Figure: Return probability divided by  $t^{-\ln 2 / \ln 3}$ ,  $C(t)t^{\ln 2 / \ln 3}$  for a Cantor spectrum

## Partial summary :

- General method :
  - scaling property
    - functional equation  $f(x) = \frac{1}{b}f(ax) + g(x)$
    - Mellin transform
    - $f(x) = x^{\ln(b)/\ln(a)} G_p\left(\frac{\ln x}{\ln a}\right)$ ,  $G_p(y+1) = G_p(y)$
- Typical behaviour due to a fractal spectrum : power-law modulated by log-periodic oscillations
- The method gives an intuition of which are the relevant physical quantities to observe typical behaviour

Other relevant quantities :

- Participation ratio :  $P_r(t) = \frac{1}{t} \int_0^t \frac{dt'}{|\langle x | \psi(t') \rangle|^4}$
- RMS displacement :  $\Delta x(t) = \sqrt{\int (x(t) - x_0)^2 |\psi(x, t)|^2 dx}$

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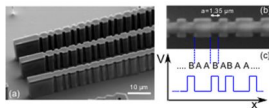
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Cavity polaritons (quasi-particles arising from the strong coupling between photons and excitons) placed in a quasi-periodic potential.

Mapping to a 1D problem with effective **quasi-periodic potential**  $V(x)$  with a **Cantor like spectrum** :

$$E\psi(x) = \frac{\hbar^2}{2m_{ph}} \left[ -\frac{d^2}{dx^2} + V(x) \right] \psi(x)$$

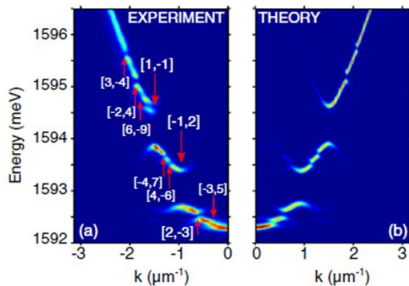


$S_1 = B$   
 $S_2 = A$   
 $S_3 = BA$   
 $S_4 = ABA$   
 $S_5 = BAABA$   
 $S_6 = ABABAABA$

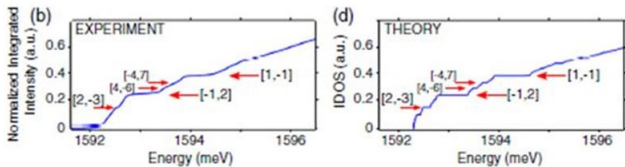
$$S_{j+1} = S_{j-1}S_j$$

Figure: Fibonacci cavity

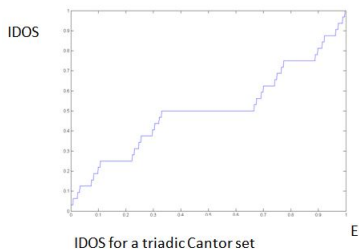
## Numerical and experimental results



Gaps of the energy spectrum for a Fibonacci potential (gap-labeling theorem)

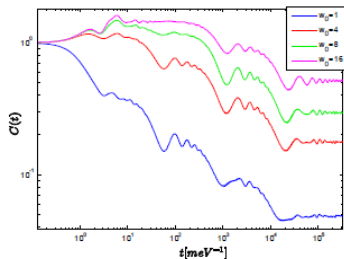


IDOS for a Fibonacci potential

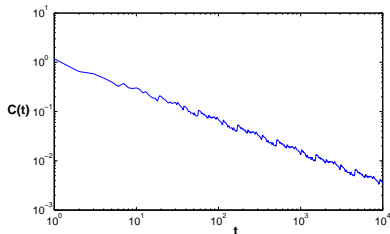


IDOS for a triadic Cantor set

Time-averaged return probability :  $C(t) = \frac{1}{t} \int_0^t |\langle \psi(0) | \psi(t') \rangle|^2 dt'$



**Figure:** Time-averaged return probability of a wave packet in a Fibonacci potential.



**Figure:** Time-averaged return probability for a triadic Cantor spectrum.

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In summary :

- We identified the signature of a fractal spectrum through measurable physical quantities : power-law modulated by log-periodic oscillation, for dynamical and spectral quantities ( $C(t)$ , integrated density of states)
- Oscillations difficult to observe ; easier with dynamical quantities
- Verified experimentally with polaritons in a Fibonacci potential (Cantor like spectrum)
- General method to study theoretically systems with a fractal feature
- applied to the return probability for a Cantor spectrum

Next : other relevant quantities in quantum mechanics

- Signature of the fractal spectral feature in the Thouless coefficient  $\gamma(\epsilon)$
- Spatial correlation of the eigenfunction ; generalization of Thouless formula :  $\overline{\phi(0, \epsilon) \phi(x, \epsilon)} \propto e^{-|x|\gamma(\epsilon)}$
- Spontaneous emission of a single atom coupled to a fractal spectrum
- Superradiance of a two atoms system coupled to a fractal spectrum

Questions ?