

Lecture 4

2. Manifolds

Outline: A. Real manifolds

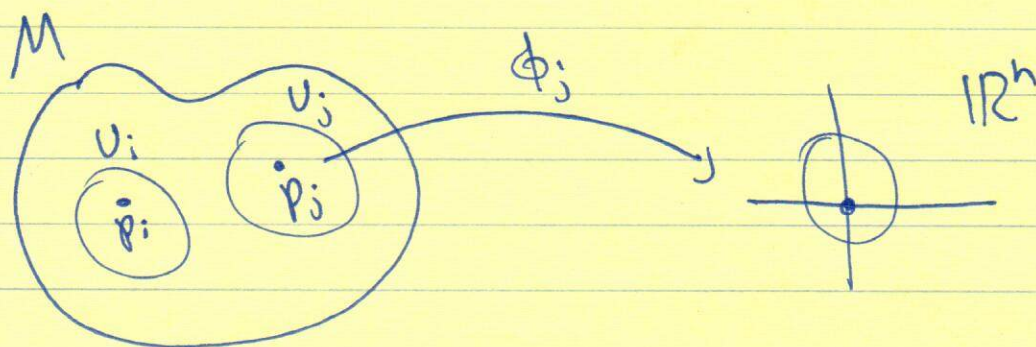
B. Differential forms

C. Homology and cohomology

D. Complex manifolds

A. Real manifolds

A (real) n -dimensional manifold M is a space that looks locally like \mathbb{R}^n around each point.



Local coordinates $\phi_j : U_j \rightarrow \mathbb{R}^n$

The transition functions at the intersections

$U_i \cap U_j$:

$$\phi_{ij} \equiv \phi_i \circ \phi_j^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

are required to be infinitely differentiable,
i.e. ϕ_{ij} are C^∞ functions.

The pair (U_i, ϕ_i) is called a chart.

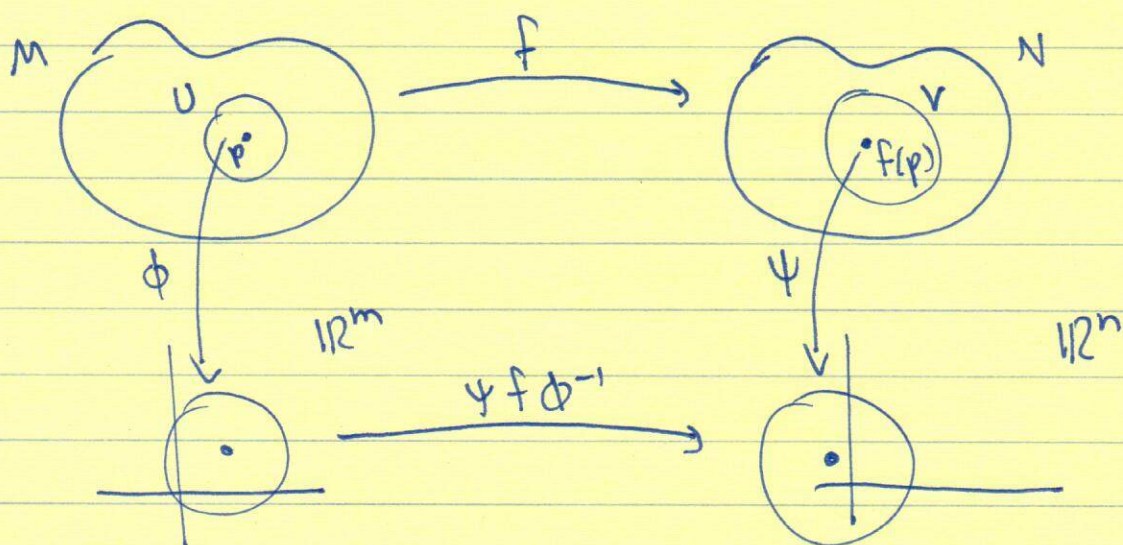
• Differentiable maps

Let M be an m -dim. manifold with chart (U, ϕ)

Let N " " n -dim. " " " " (V, ψ)

A map $f: M \rightarrow N$ is differentiable if the map

$$\psi \circ f \circ \phi^{-1}: \mathbb{R}^m \rightarrow \mathbb{R}^n \text{ is } C^\infty$$



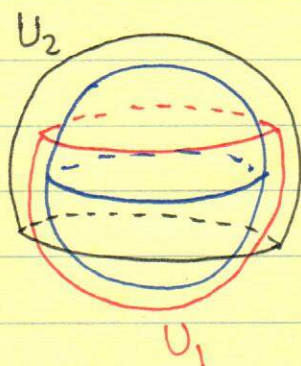
If $\psi \circ f \circ \phi^{-1}$ is ~~invertible~~ invertible, and both $y = \psi \circ f \circ \phi^{-1}(x)$ and $x = \phi \circ f^{-1} \circ \psi^{-1}(y)$ are C^∞ , then f is a diffeomorphism.

Diffeomorphic manifolds are regarded as the same manifold (in particular $\dim M = \dim N$).

Examples:

(1) $\underline{S^2}$

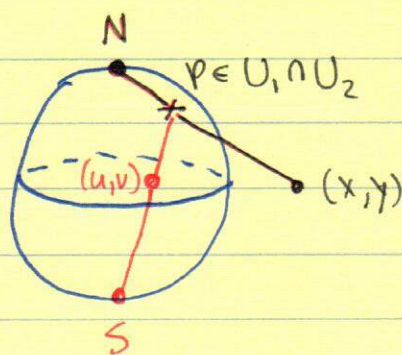
For the two sphere we need only two open sets U_1, U_2 , covering the southern and northern hemispheres, respectively:



Choose stereographic local coordinates

$$\phi_1 : U_1 \rightarrow \mathbb{R}^2_{(u,v)}$$

$$\phi_2 : U_2 \rightarrow \mathbb{R}^2_{(x,y)}$$



Then the transition function is given by

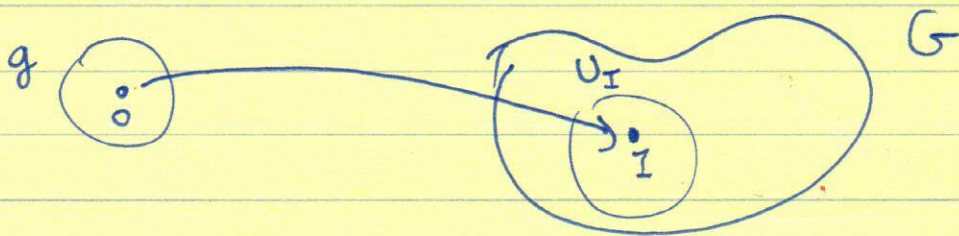
$$\phi_{12}(x,y) = \left(\frac{x}{x^2+y^2}, -\frac{y}{x^2+y^2} \right)$$

(2) Lie groups

Lie groups are also endowed with the structure of a manifold.

Let G be a Lie group ($U(n), O(n), GL(n, \mathbb{R}), \dots$) and \mathfrak{g} be the associated Lie algebra, i.e. a set of linear matrices that generate G .

This means that $e^{\mathfrak{g}}: \mathfrak{g} \rightarrow G$ is a diffeomorphism from a neighborhood of the origin in \mathfrak{g} to a neighborhood of I in G :



This defines a local coord. system near $I \in G$.

We can define a local coord. near any $g_0 \in G$ by $g_0 e^{\mathfrak{g}}$.

The transition functions are therefore given by left-multiplication in the group.

• Tangent and cotangent space

We need a more general definition of vectors on general manifolds.

Def. The tangent space to a manifold M at a point $P \in M$, $T_p(M)$, is the collection of all directional derivatives at P along any curve that goes through P .

In terms of local coords. x^i , any element can be expressed as $v^i \frac{\partial}{\partial x^i} \in T_p(M)$

$\left\{ \frac{\partial}{\partial x^i} \right\}$ forms a basis for $T_p(M)$.

A collection of directional derivatives at each point in M with smoothly varying coefficients $V^i(x)$ is called a vector field.

The cotangent space, $T_p^*(M)$, is just the dual vector space to the tangent space $T_p(M)$.

The basis elements of $T_p^*(M)$ in the local coords. x^i are the differential line elements dx^i .

The inner product is $dx^j \left(\frac{\partial}{\partial x^i} \right) = \left\langle \frac{\partial}{\partial x^i}, dx^j \right\rangle = \delta^j_i$

Let $V \in T_p(M)$ and $U \in T_p^*(M)$

$$V = v^i \frac{\partial}{\partial x^i} \quad U = u_i dx^i$$

The invariance of V, U under coord. trans.

$x \rightarrow x'(x)$ implies the trans. properties of the components:

$$v'^i = v^j \frac{\partial x'^i}{\partial x^j} \quad \text{"contravariant"}$$

$$u'_i = u_j \frac{\partial x^j}{\partial x'^i} \quad \text{"covariant"}$$

Of course the inner product is invariant:

$$\langle V, U \rangle = v^i u_i = v'^i u'_i$$

More generally, an element W of

$$\underbrace{T_p(M) \otimes \dots \otimes T_p(M)}_k \otimes \underbrace{T_p^*(M) \otimes \dots \otimes T_p^*(M)}_l$$

is called a tensor of type (k, l) , and

can be expanded in the basis of local coords. as:

$$W_{(k)}^{(l)} = W_{j_1 \dots j_k}^{i_1 \dots i_k} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_k}$$

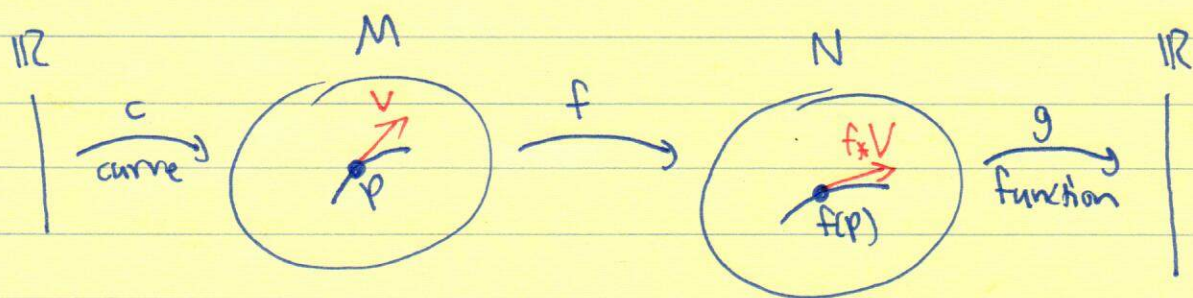
• Induced maps

Let's say we have a map between two manifolds $f: M \rightarrow N$.

How do vector fields, and more generally tensor fields map?

- Vectors map by push-forward

$$f_* : T_p M \rightarrow T_{f(p)} N$$



Let $V \in T_p(M) \Rightarrow V$ acts on gf to give a number $V[gf]$.

Define $f_*V \in T_{f(p)}(N)$ by $f_*V[g] \equiv V[gf]$.

In local coords. $V = V^i \frac{\partial}{\partial x^i}$, $f_*V = w^j \frac{\partial}{\partial y^j}$

Take $g = y^j$, then $w^j = V^i \frac{\partial y^j}{\partial x^i}$.

- Dual-vectors map by pull-back $f^* : T_{f(p)}^* N \rightarrow T_p^* M$

Let $V \in T_p M$ and $\omega \in T_{f(p)}^* N$

Define the pull-back of ω by f^* by: $\langle V, f^*\omega \rangle \equiv \langle f_*V, \omega \rangle$

Then in local coords: $\omega = \omega_j dy^j$, $f^*\omega = \xi_i dx^i \in T_p^* M$

where $\xi_i = \omega_j \frac{\partial y^j}{\partial x^i}$.

B. Differential forms

A h -form is a totally antisymmetric element of

$$\underbrace{T_p^*(M) \otimes T_p^*(M) \otimes \dots \otimes T_p^*(M)}_{h \text{ times}}$$

So a 1-form is just a dual vector $\in T_p^*(M)$, which is a differential line element $\omega_i dx^i$.

Local coord. expressions for h -forms are obtained using the wedge product, eg.

$$dx \wedge dy \equiv \frac{1}{2}(dx \otimes dy - dy \otimes dx) \quad \text{2-form} \\ (= -dy \wedge dx)$$

$$\Rightarrow dx \wedge dx = 0$$

So a h -form in local coords. x^1, \dots, x^h is given by:

$$\omega^{(h)} = \omega_{i_1 \dots i_h} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_h}$$

Only the totally antisymm. part of $\omega_{i_1 \dots i_h}$ contributes.

• $\Lambda^p(x) =$ set of p -forms at $x \in M$

This is a vector space of dimension $\binom{n}{p} = \frac{n!}{p!(n-p)!}$

Note that $\dim(\Lambda^p) = \dim(\Lambda^{n-p})$, and that

$\Lambda^p = \emptyset$ for $p > n$.

The set of all diff. forms on M is denoted:

$$\Lambda^* = \Lambda^0 \oplus \Lambda^1 \oplus \dots \oplus \Lambda^n$$

This is closed under the wedge operation \wedge ,

and so defines a graded algebra called

Cartan's exterior algebra of diff. forms:

If $\alpha_p \in \Lambda^p$ and $\beta_q \in \Lambda^q$, then

$$\alpha_p \wedge \beta_q = (-1)^{pq} \beta_q \wedge \alpha_p.$$

Odd forms anti-commute, and in particular $\alpha_i \wedge \alpha_i = 0$.

- Exterior derivative

$C^\infty(\Lambda^p)$ = set of smooth (infinitely differentiable) p -forms

$$f_{i_1, \dots, i_p}(x) dx^{i_1} \wedge \dots \wedge dx^{i_p}$$

The "top form" $\in C^\infty(\Lambda^n)$ is representable by a single function times the volume element:

$$f(x) dx^1 \wedge dx^2 \wedge \dots \wedge dx^n$$

The exterior derivative d maps $C^\infty(\Lambda^p) \rightarrow C^\infty(\Lambda^{p+1})$ as follows:

$$\omega_p = f_{i_1, \dots, i_p}(x) dx^{i_1} \wedge \dots \wedge dx^{i_p} \in C^\infty(\Lambda^p)$$

$$d\omega_p = \frac{\partial f_{i_1, \dots, i_p}(x)}{\partial x^i} dx^i \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p} \in C^\infty(\Lambda^{p+1})$$

Only the totally antisymmetric parts of the partial derivatives contribute.

In particular $d(\text{top form}) = 0$.

The exterior derivative is odd from the point of view of the exterior algebra:

$$d(\alpha_p \wedge \beta_q) = d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q$$

The exterior derivative is nilpotent:

$$\boxed{d^2 \omega_p = dd\omega_p = 0}$$

This follows from the equality of mixed partial derivatives,

e.g. $C^\infty(\Lambda^0) \xrightarrow{d} C^\infty(\Lambda^1) \xrightarrow{d} C^\infty(\Lambda^2)$

$$df = \partial_j f dx^j, \quad d^2 f = \partial_i \partial_j f dx^i \wedge dx^j = \frac{1}{2} (\partial_i \partial_j f - \partial_j \partial_i f) dx^i \wedge dx^j = 0$$

This leads to familiar statements in \mathbb{R}^3 :

$$\omega_0 = f(x)$$

$$\omega_1 = v_1(x) dx^1 + v_2(x) dx^2 + v_3(x) dx^3$$

$$\omega_2 = w_1(x) dx^2 \wedge dx^3 + w_2(x) dx^3 \wedge dx^1 + w_3(x) dx^1 \wedge dx^2$$

$$\omega_3 = \phi(x) dx^1 \wedge dx^2 \wedge dx^3 \quad (\text{top form})$$

$d\omega_0 = \partial_i f(x) dx^i$, the gradient of $f(x)$

$d\omega_1 = (\epsilon_{ijk} \partial_j v_k(x)) \frac{1}{2} \epsilon_{ilm} dx^l \wedge dx^m$, curl of $\vec{v}(x)$

$d\omega_2 = (\partial_1 w_1 + \partial_2 w_2 + \partial_3 w_3) dx^1 \wedge dx^2 \wedge dx^3$, divergence of $\vec{w}(x)$

Then $d^2\omega_0 = 0$ is $\vec{\nabla} \times (\vec{\nabla} f(x)) = 0$

and $d^2\omega_1 = 0$ is $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}(x)) = 0$

• Hodge star

Maps p -forms to $(n-p)$ -forms (recall $\dim(\Lambda^p) = \dim(\Lambda^{n-p})$):

$$*: C^\infty(\Lambda^p) \rightarrow C^\infty(\Lambda^{n-p})$$

as follows:

$$*(dx^{i_1} \wedge \dots \wedge dx^{i_p}) = \frac{1}{(n-p)!} \epsilon_{i_1 \dots i_p j_1 \dots j_{n-p}} dx^{j_1} \wedge \dots \wedge dx^{j_{n-p}}$$

[Note: When we introduce a metric, we'll have to be more careful. For now we don't care - we work in flat space \mathbb{R}^n .]

Hodge-starring twice gives back the same p -form up to a sign:

$$**\omega_p = (-1)^{p(n-p)} \omega_p$$

- Adjoint of exterior derivative

$$\delta: C^\infty(\Lambda^p) \rightarrow C^\infty(\Lambda^{p-1})$$

Defined as $\boxed{\delta = (-1)^{np+n+1} *d*}$

Like d , δ is also nilpotent: $\delta^2 \omega_p = \delta\delta\omega_p = 0$.

We can combine d and δ to define the Laplacian $\Delta: C^\infty(\Lambda^p) \rightarrow C^\infty(\Lambda^p)$, by

$$\boxed{\Delta \equiv (d + \delta)^2 = d\delta + \delta d}$$

For example on 1-forms

$$\Delta(\vec{V}(x) \cdot d\vec{x}) = - \frac{\partial^2 \vec{V}}{\partial x^k \partial x^k} \cdot d\vec{x}$$

A p -form satisfying:

- $d\omega_p = 0$ is called a closed form
- $\delta\omega_p = 0$ " " " co-closed form
- $\Delta\omega_p = 0$ " " " harmonic form
- $\omega_p = d\alpha_{p-1}$ " " " an exact form
- $\omega_p = \delta\alpha_{p+1}$ " " " a co-exact form

Two theorems:

(1) ω_p is harmonic $\Leftrightarrow \omega_p$ is closed and co-closed

(2) If M is a compact manifold without boundary, any p -form can be uniquely decomposed as

$$\omega_p = d\alpha_{p-1} + \delta\beta_{p+1} + \gamma_p, \text{ with } \Delta\gamma_p = 0.$$

(Hodge's theorem)