

## Lecture 2

### B. Higher homotopy groups

Classify cont. maps from  $S^n$  to  $X$ .

We need to generalize the idea of a loop.

Let  $I^n = \{(s_1, \dots, s_n) \mid 0 \leq s_i \leq 1\}$  "unit n-cube"

$\partial I^n = \{(s_1, \dots, s_n) \in I^n \mid \text{one of } s_i = 0 \text{ or } 1\}$

Then an "n-loop"  $\alpha: I^n \rightarrow X$ , such that  $\partial I^n$  maps to a point  $x_0 \in X$ .

Two n-loops are homotopic,  $\alpha \sim \beta$ , if  $\exists$  a continuous map

$F: I^n \times I \rightarrow X$ , such that

$$F(s_1, \dots, s_n; 0) = \alpha(s_1, \dots, s_n)$$

$$F(s_1, \dots, s_n; 1) = \beta(s_1, \dots, s_n)$$

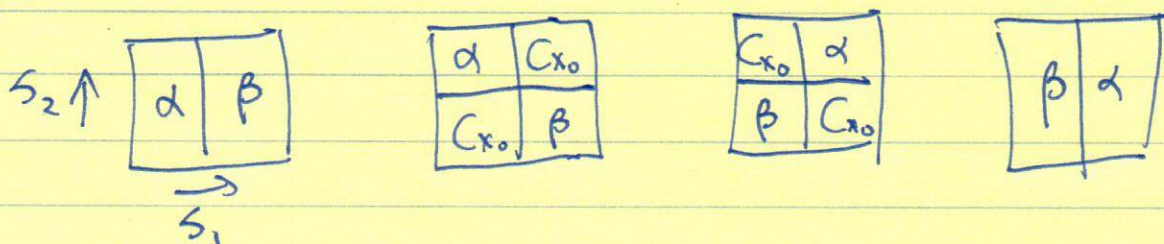
$$F(s_1, \dots, s_n; t) = x_0 \text{ for } (s_1, \dots, s_n) \in \partial I^n \text{ and } t \in I$$

The product and inverse are defined in a similar way to 1-loops (using one of the  $S_i$ ).

The set of homotopy classes  $[\alpha]$  of  $n$ -loops at  $x_0 \in X$  defines the  $n^{\text{th}}$  homotopy group of  $X$  at  $x_0$ ,  $\Pi_n(X, x_0)$ .

### General properties

- The same 4 properties of  $\Pi_1(X, x_0)$  (page 12)
- Unlike  $\Pi_1(X, x_0)$ ,  $\Pi_n(X, x_0)$  with  $n \geq 2$  is always Abelian, namely  $[\alpha] * [\beta] = [\beta] * [\alpha]$ .



~~You can construct the homotopy for this too.~~

\*  $\Pi_0(X, x_0)$  classifies maps from a point to  $X$ .

$\alpha, \beta: \{0\} \rightarrow X$  with  $\alpha(0) = x$ ,  $\beta(0) = y$

$\alpha \sim \beta \Leftrightarrow x, y$  are in the same connected component of  $X$ .

$\Pi_0$  denotes the number of connected components. Not a group in general.

## Examples

(1) n-sphere  $S^n$ :  $\Pi_n(S^n) \cong \mathbb{Z}$

$$f: S^n \rightarrow S^n$$

$$\phi \equiv (\phi_1, \dots, \phi_n) \in \text{domain } S^n, \quad \theta \equiv (\theta_1, \dots, \theta_n) \in \text{range } S^n$$

$$\theta = f(\phi)$$

For each mapping we can assign an integer  $h$ , computed in analogy with the  $n=1$  case.

$$\text{Basically } h = \frac{\text{Volume of range } S^n}{\text{Volume of domain } S^n}$$

$$= \# \text{ of times the range } S^n \text{ covers the domain } S^n$$

~~In HW you should study this more quantitatively.~~

## • Universal covering spaces

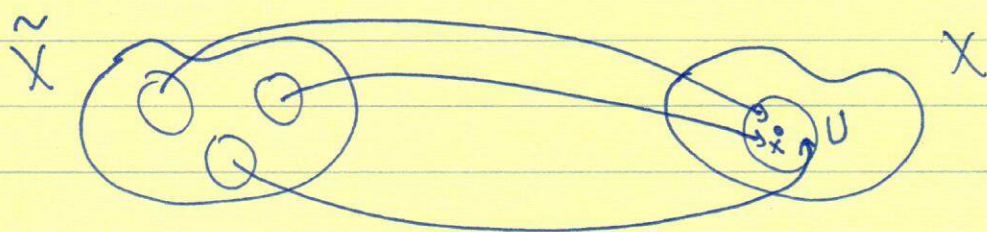
This is a useful concept for computing higher homotopy groups.

$X, \tilde{X}$  topological spaces

$\tilde{X}$  is the covering space of  $X$  if  $\exists$  a continuous map  $p: \tilde{X} \rightarrow X$ , such that

(1)  $p$  is surjective (onto)

(2) For each  $x \in X$ ,  $\exists$  a connected open set  $U \subset X$ , such that  $p^{-1}(U)$  is a disjoint union of open sets in  $\tilde{X}$ , each of which is mapped homeomorphically onto  $U$  by  $p$ .



In addition if  $\tilde{X}$  is simply connected,  $\pi_1(\tilde{X}) = 0$ , it's called the universal covering space of  $X$ .

Examples:

(1)  $\mathbb{R}$  is the universal covering space of  $S^1$

$$p: \mathbb{R} \rightarrow S^1, \quad p(x) = e^{2\pi i x}, \quad \pi_1(\mathbb{R}) = \{e\}$$

(2)  $S^n$  is the universal covering space of  $\mathbb{R}P^n$  ( $n \geq 2$ )

$$\pi_1(S^n) = \{e\} \quad \text{for } n \geq 2$$

Theorem: If  $\tilde{X}$  is the covering space of  $X$ , then

$$\pi_n(\tilde{X}, \tilde{x}_0) \cong \pi_n(X, x_0) \quad \text{for } n \geq 2$$

$$(x_0 = p(\tilde{x}_0))$$

We can use this to compute more examples:

- $\pi_n(S^1) \cong \pi_n(\mathbb{R}) \cong \{e\} \quad \text{for } n \geq 2$

since  $\mathbb{R}$  is contractible (but note that  $S^1$  isn't)

- $\pi_n(\mathbb{R}P^n) \cong \pi_n(S^n) \cong \mathbb{Z} \quad \text{for } n \geq 2$

## • Covering groups and Spin groups

Lie groups are topological spaces

Examples:

(1)  $\underline{U(1)} \cong S^1$ , any element is given by  $e^{i\theta}$ ,  $0 \leq \theta < 2\pi$

Thus  $\mathbb{R}$  is the universal covering group (under addition) of  $U(1)$ .  $\Rightarrow \pi_1(U(1)) \cong \pi_1(S^1) = \mathbb{Z}$ .

(2)  $\underline{SO(3)} \cong \mathbb{R}P^3$

$g \in SO(3)$  is specified by a rotation axis in  $\mathbb{R}^3$ ,  $\hat{n}$ , and a rotation angle  $0 \leq \theta \leq \pi$ .

Let  $\vec{\Omega} = \theta \hat{n}$  denote this element.

$\Rightarrow \vec{\Omega}$  takes values in the 3-disc  $D^3$  of radius  $\pi$ .

But ~~and~~  $\vec{\Omega} = \pm \pi \hat{n}$  are the same rotation, so opposite points on ~~the~~  $\partial D^3$  are identified  $\Rightarrow \mathbb{R}P^3$ .

$$\pi_n(SO(3)) = \pi_n(\mathbb{R}P^3)$$

In particular  $\pi_1(SO(3)) \cong \mathbb{Z}_2$

$$\pi_3(SO(3)) \cong \mathbb{Z}$$

$$(3) \underline{SU(2)} \cong S^3$$

$g \in SU(2)$  can be expressed as  $g = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$

where  $a, b \in \mathbb{C}$  and  $|a|^2 + |b|^2 = 1$

Let  $a = u + iv$ ,  $b = x + iy \Rightarrow u^2 + v^2 + x^2 + y^2 = 1$ ,  $S^3 \checkmark$

So we find that  $SU(2)$  is the universal covering group of  $SO(3)$ .

$$\pi_1(SU(2)) \cong \{e\}$$

$$\pi_n(SU(2)) \cong \pi_n(SO(3)) \text{ for } n \geq 2$$

~~Spin(3) is a spin group~~

This is an example of a spin group

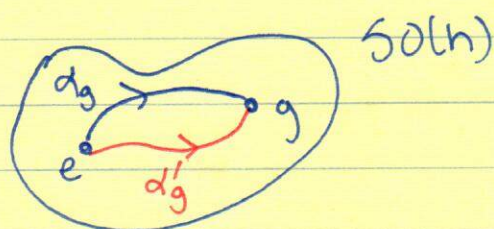
$$\text{Spin}(3) \cong SU(2)$$

## The spin group

The universal covering group of  $SO(n)$  is called  $Spin(n)$ ;  $\pi_1(Spin(n)) \cong \{e\}$

(In HW1 you'll show that  $\pi_1(SO(n)) \cong \mathbb{Z}_2$ , generalizing the  $SO(3)$  case).

Consider a path  $\alpha_g$  in  $SO(n)$  from the identity element  $e$  to the element  $g$ :



Since  $\pi_1(SO(n)) \cong \mathbb{Z}_2$  there are 2 classes of paths.

If  $\alpha_g, \alpha'_g$  belong to two different classes, then

$\alpha_g * (\alpha'_g)^{-1}$  is a loop that belongs to the non-trivial class of  $\pi_1(SO(n))$ .



$\text{Spin}(n)$  is defined as the set of classes of paths in  $\text{SO}(n)$  from  $e$  to  $g$ , for all  $g \in \text{SO}(n)$ .

This defines a group, which is simply connected, and for each element  $g \in \text{SO}(n) \exists$  two elements  $[\alpha_g], [\alpha'_g] \in \text{Spin}(n)$ .

- $\text{Spin}(n)$  has a  $\mathbb{Z}_2$  subgroup, generated by the non-trivial class of paths from  $e$  to  $e$ , namely loops,  $[\alpha_e]$ .

Since  $[\alpha'_g] = [\alpha_g] * [\alpha_e]$ ,  $\text{Spin}(n)/\mathbb{Z}_2 \cong \text{SO}(n)$ .

- Examples:

$$\text{Spin}(3) \cong \text{SU}(2)$$

$$\text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2)$$

$$\text{Spin}(5) \cong \text{USp}(4)$$

$$\text{Spin}(6) \cong \text{SU}(4)$$

- Coset spaces

$G =$  continuous group,  $H \subset G$  subgroup

$G/H =$  coset space (set of equivalence classes under  $H$ )

Examples:

(1)  $SO(3)/SO(2) \cong S^2$

$$g \in SO(3) \Rightarrow g = \vartheta \hat{n}$$

$$h \in SO(2) \Rightarrow h = e^{i\theta}$$

$$\Rightarrow SO(3)/SO(2) \cong S^2 \text{ spanned by } \hat{n}$$

(2) More generally  $SO(n)/SO(n-1) \cong S^{n-1}$

(3)  $SU(n)/SU(n-1) \cong S^{2n-1}$

- Homotopy groups of spheres

We showed that  $\pi_n(S^n) \cong \mathbb{Z}$

But what is  $\pi_n(S^m)$  more generally?

$n < m$ : This case is easy,  $\pi_n(S^m) \cong \{e\}$

We can always shrink a lower-diml. sphere in a higher-diml. sphere to a point.

$n > m$ : This is a non-trivial problem.

The result depends on  $n, m$ . No simple rule.

For  $m=1$ ,  $\pi_n(S^1) \cong \{e\}$  for  $n \geq 2$ .

Other cases are harder, more sophisticated tools needed.

(You'll compute  $\pi_3(S^2)$  in HW 1)

~~Weg. The exact homotopy sequence HW 1.~~

- The exact homotopy sequence

A more sophisticated tool that allows us to compute homotopy groups in some cases.

Let  $G$  be a continuous topological group, and

- $H$  a subgroup of  $G$ ,  $H \subset G$ .

There exists a sequence of homomorphisms:

$$\dots \pi_n(H) \xrightarrow{M_1} \pi_n(G) \xrightarrow{M_2} \pi_n(G/H) \xrightarrow{M_3} \pi_{n-1}(H) \rightarrow \dots$$

that is exact, namely such that:

$$\text{im}(M_i) \cong \text{ker}(M_{i+1})$$

The construction of the maps  $M_1, M_2, M_3$  and the proof of exactness is given in the book by Nakahara. I won't repeat them.

Our interest in this sequence is the following two nice properties:

(1) If any exact sequence contains a string:

$$\dots \rightarrow \{e\} \xrightarrow{f_1} A \xrightarrow{f_2} \{e\} \rightarrow \dots$$

then  $A \cong \{e\}$ .

Proof:  $\text{im}(f_1) \cong \{e\}$ , ~~therefore~~  $A \cong \ker(f_2) \cong \text{im}(f_1)$ , ~~therefore~~

(2) If any exact sequence contains a string:

$$\dots \rightarrow \{e\} \xrightarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} \{e\} \rightarrow \dots$$

then  $A \cong B$ , i.e.  $f_2$  is an isomorphism.

Proof:  $\text{im}(f_1) \cong \{e\} \cong \ker f_2 \Rightarrow f_2$  injective (1-to-1)

$\text{im}(f_2) \cong \ker f_3 \cong B \Rightarrow f_2$  surjective (onto)

These properties allow us to compute more cases.

Examples:

(1)  $\underline{\pi_2(S^2)}$ : (One of the cases of  $\pi_n(S^m)$  with  $n > m$ ).

$$\text{Take } G = SU(2) \cong S^3 \text{ and } H = U(1) \cong S^1$$

$$\text{Then } G/H = SU(2)/U(1) \cong S^2$$

Look at a part of the sequence:

$$\begin{array}{ccccccc} \dots \rightarrow \pi_3(U(1)) & \rightarrow & \pi_3(SU(2)) & \rightarrow & \pi_3(SU(2)/U(1)) & \rightarrow & \pi_2(U(1)) \rightarrow \dots \\ & & \cong \mathbb{Z} & & \cong \mathbb{Z} & & \cong \mathbb{Z} \end{array}$$

$$\text{So } \pi_3(S^2) \cong \pi_3(S^3) \cong \mathbb{Z}.$$

(2)  $\underline{\pi_1(SO(n))}$ : (Generalization of  $\pi_1(SO(3)) \cong \mathbb{Z}_2$ )

$$\text{Take } G = SO(n), H = SO(n-1) \Rightarrow G/H \cong S^{n-1}$$

$$\dots \rightarrow \pi_2(SO(n)/SO(n-1)) \rightarrow \pi_1(SO(n-1)) \rightarrow \pi_1(SO(n)) \rightarrow \pi_1(SO(n)/SO(n-1)) \rightarrow \dots$$

$$\begin{array}{l} \pi_2(S^{n-1}) = \{e\} \\ \text{for } n \geq 4 \end{array}$$

$$\begin{array}{l} \pi_1(S^{n-1}) = \{e\} \\ \text{for } n \geq 3 \end{array}$$

$$\text{So } \pi_1(SO(n)) \cong \pi_1(SO(n-1)) \text{ for } n \geq 4$$

$$\Rightarrow \pi_1(SO(n)) \cong \pi_1(SO(3)) \cong \mathbb{Z}_2 \text{ for } n \geq 3.$$

(3)  $\pi_1(SU(n))$ : (Generalization of  $\pi_1(SU(2)) \cong \{e\}$ )

Take  $G = SU(n)$ ,  $H = SU(n-1) \Rightarrow G/H \cong S^{2n-1}$

Look at the same part of the sequence as in the previous case.

$$\Rightarrow \pi_1(SU(n)) \cong \pi_1(SU(n-1)) \quad \text{for } n \geq 2$$

$$\vdots$$

$$\cong \pi_1(SU(2)) \cong \{e\}$$