

Lecture 1

Our main goal is to get acquainted with the language and tools of differential geometry and topology that are useful in theoretical physics.

This includes quantum field theory, general relativity, and condensed matter physics.

I assume everyone is familiar with these concepts:

Maps, equivalence relations^{, quotient space (example?)}, vector spaces, tensors, topological space, compactness, connectedness, homeomorphism (?).

If not, you should read Ch. 2 of Nakahara's book.

Course outline

1. Homotopy groups - applications to solitons.
2. Diff. manifolds, forms, homology and cohomology
3. Riemannian manifolds
4. Fiber bundles (gauge theory, Berry's phase)
- * 5. Characteristic classes

Introduction

How can we classify topological spaces?

How do we decide if two topological spaces are "equivalent"?

Homeomorphisms define an equivalence relation

$$X_1 \cong X_2$$

if \exists a homeomorphism $f: X_1 \rightarrow X_2$.

But the complete classification is unknown in general.

Take $f: (0,1) \rightarrow \{0\}$ with $f(x) = 0$

and $g: \{0\} \rightarrow (0,1)$ with $g(0) = \frac{1}{2}$

Then $fg = I$ but $gf \neq I$

• Euler number

X is a 2d surface $\subset \mathbb{R}^3$ homeo. to
a polyhedron K .

$$\chi(X) = V - E + F; \quad V = \# \text{ of vertices of } K$$

$$E = \# \text{ of edges " "}$$

$$F = \# \text{ of faces of } K$$

Theorem: (Poincaré - Alexander)

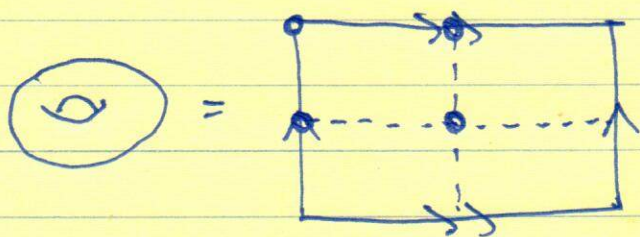
$\chi(X)$ is indep. of the specific K , as long as
it is homeo. to X .

Examples:


$$(1) \chi(S^2) = \chi(\text{tetrahedron}) = 4 - 6 + 4 = 2$$

$$= \chi(\text{cube}) = 8 - 12 + 6 = 2$$

(2) Torus - describe as a rectangle with identified edges



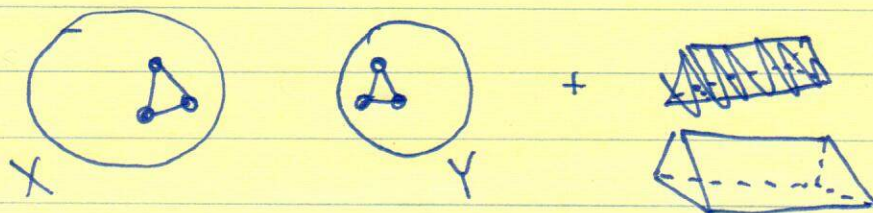
$$\chi(T^2) = 4 - 8 + 4 = 0$$

(3) Genus 2-surface 

We can also construct a ~~poly~~ polyhedron for this by identifying edges of a polygon, but it's easier to use the property of a connected sum of surfaces $X \# Y$:

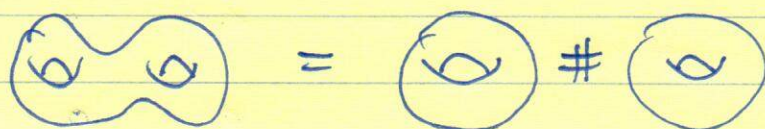
$$\chi(X \# Y) = \chi(X) + \chi(Y) - 2$$

Proof:



$$\begin{aligned}\chi(X \# Y) &= [\chi(X) - (3 - 3 + 1)] + [\chi(Y) - (3 - 3 + 1)] + [6 - 9 + 3] \\ &= \chi(X) + \chi(Y) - 2\end{aligned}$$

For the example:



$$\text{Then } \chi(\text{genus } 2) = \chi(T^2 \# T^2) = 0 + 0 - 2 = -2$$

$$\text{More generally } \chi(\text{genus } g) = 2 - 2g$$

1. Homotopy Groups (Nakahara)

The basic idea is to assign a group structure to continuous maps (homotopies) between topological spaces.

If we have two continuous maps

$$f: X \rightarrow Y, \quad g: X \rightarrow Y$$

we say that $f \sim g$ (homotopically equivalent)

if $f(X)$ can be continuously deformed to $g(X)$ in Y .

We take X to be standard top. spaces like S^n .

In this case the set of equiv. classes is called $\pi_n(Y)$.

We will see that these are actually groups.

A. The fundamental group $\pi_1(X)$

Basic ingredients are paths and loops (closed paths).

A path is a continuous map

$$\alpha: I \rightarrow X; \quad I = [0, 1]$$

$$\alpha(0) = x_0, \quad \alpha(1) = x_1.$$

If $x_0 = x_1$, α is a loop with base point x_0 .

The space of paths has the ingredients of a group:

(1) Identity element = constant path

$$C_x: I \rightarrow X \text{ with } C_x(s) = x \text{ for all } s \in I$$

(2) Multiplication: $\alpha, \beta: I \rightarrow X$ with $\alpha(1) = \beta(0)$

$$\alpha * \beta(s) \equiv \begin{cases} \alpha(2s) & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases}$$

(3) Inverse: $\alpha^{-1}(s) \equiv \alpha(1-s)$

But it's not a group, since:

$$\alpha * \alpha^{-1}(s) = \begin{cases} \alpha(2s) & 0 \leq s \leq \frac{1}{2} \\ \alpha^{-1}(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases} = \begin{cases} \alpha(2s) & \\ \alpha(2-2s) & \end{cases} \neq C_x(s)$$

Homotopy classes

$\alpha, \beta : I \rightarrow X$ loops at x_0

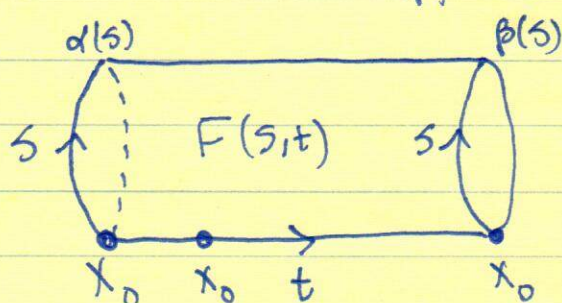
We say $\alpha \sim \beta$ if \exists continuous map

$F : I \times I \rightarrow X$, such that

$F(s, 0) = \alpha(s)$, $F(s, 1) = \beta(s)$ for all $s \in I$, and

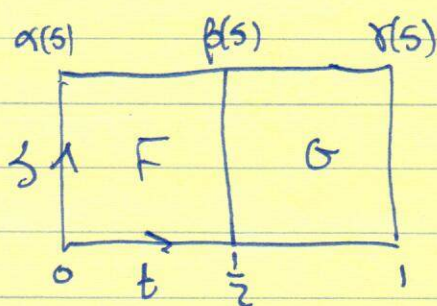
$F(0, t) = F(1, t) = x_0$ for all $t \in I$.

F is called a homotopy between α and β .



This defines an equivalence relation between α and β :

- (1) Reflexivity: $\alpha \sim \alpha$ using $F(s, t) = \alpha(s)$
- (2) Symmetry: If $\alpha \sim \beta$ using $F(s, t)$ then $\beta \sim \alpha$ using $F(s, 1-t)$
- (3) Transitivity: If $\alpha \sim \beta$ using $F(s, t)$ and $\beta \sim \gamma$ using $G(s, t)$ then $\alpha \sim \gamma$ using $H(s, t) \equiv \begin{cases} F(s, 2t) & 0 \leq t \leq \frac{1}{2} \\ G(s, 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$



Claim: ~~The set of~~ The set of homotopy classes $[\alpha]$ of loops at $x_0 \in X$ is a group, called the fund. group (or first homotopy group) of X at x_0 , $\pi_1(X, x_0)$.

$$[\alpha] * [\beta] = [\alpha * \beta]$$

\uparrow group product \uparrow product of paths

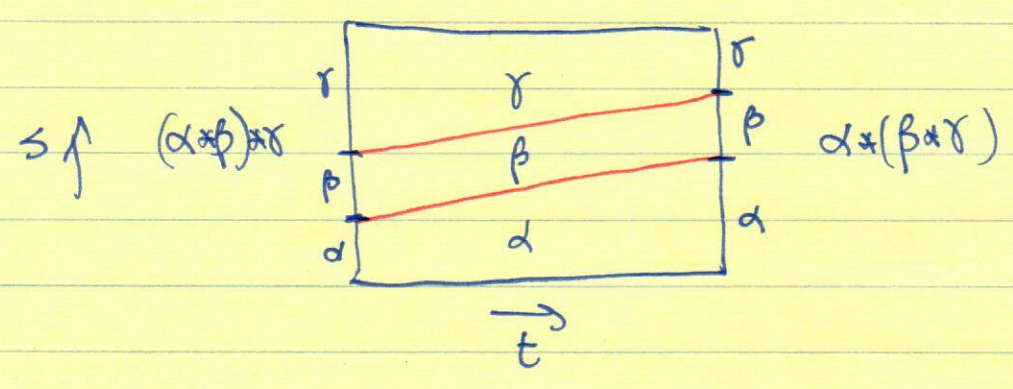
~~To~~ To prove this we need to show:

- (1) $([\alpha] * [\beta]) * [\gamma] = [\alpha] * ([\beta] * [\gamma])$ associativity
- (2) $[\alpha] * [c_x] = [c_x] * [\alpha] = [\alpha]$ property of identity element
- (3) $[\alpha] * [\alpha^{-1}] = [c_x]$ property of inverse element

Namely, we need to show \exists homotopies between the corresponding paths.

I'll do (1), you'll do (2) and (3).

A homotopy $F(s,t)$ between $(\alpha * \beta) * \gamma$ and $\alpha * (\beta * \gamma)$:



$$F(s,t) = \begin{cases} \alpha\left(\frac{4s}{1+t}\right) & 0 \leq s \leq \frac{1+t}{4} \\ \beta(4s-t-1) & \frac{1+t}{4} \leq s \leq \frac{2+t}{4} \\ \gamma\left(\frac{4s-t-2}{2-t}\right) & \frac{2+t}{4} \leq s \leq 1 \end{cases}$$

General properties

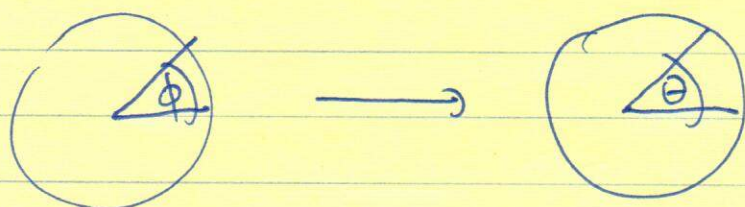
- (1) $\pi_1(X, x_0)$ is a topological invariant.
- (2) If X is (arcwise) connected, $\pi_1(X, x_0)$ is independent of the base point x_0 .
- (3) If X is contractible (cont. deformable to a point) the $\pi_1(X) \cong \{e\}$. X is "simply connected".
- (4) If X, Y are connected then

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \oplus \pi_1(Y, y_0)$$

Note: $\pi_1(X, x_0)$ is not necessarily Abelian (commutative).

Examples:

(1) S^1 : $\pi_1(S^1)$ is the group of equivalence classes of mappings $f: S^1 \rightarrow S^1$



$$\Theta = f(\Phi)$$

Take the base points $\phi=0$ and $\theta=0$.

Without loss of generality we can set $f(0)=0$.

Periodicity requires $f(2\pi) = 2\pi k$, $k \in \mathbb{Z}$.

Two maps f_1, f_2 with the same k are homotopic, with homotopy given by:

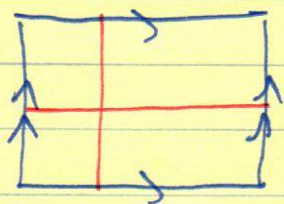
$$F(\phi, t) = t f_1(\phi) + (1-t) f_2(\phi) \quad (F(2\pi, t) = 2\pi k)$$

Therefore $\boxed{\pi_1(S^1) \cong \mathbb{Z}}$

$$k = \text{"winding number"} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta}{d\phi} d\phi$$

$$(2) \pi_1(T^2) = \pi_1(S^1 \times S^1) \cong \pi_1(S^1) \oplus \pi_1(S^1) = \mathbb{Z} \oplus \mathbb{Z}$$

The fund. group of the torus is labelled by two winding numbers:



For an n -diml. torus $\pi_1(T^n) = \overbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \dots \oplus \mathbb{Z}}^n$

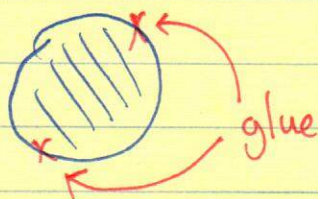
(3) Real projective plane $\mathbb{R}P^2$ (Define as the set of lines thru $(0,0,0)$ in \mathbb{R}^3)

Defined by identifying anti-podal points on S^2 :

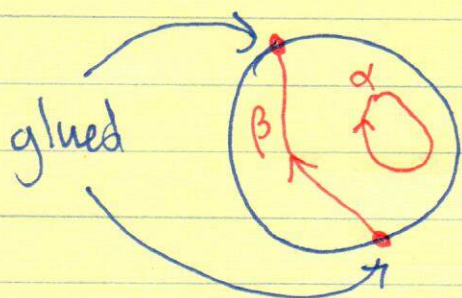
$$\mathbb{R}P^2 \cong S^2 / \mathbb{Z}_2$$



This is topologically equivalent to a 2d disc, where we glue opposite points on its boundary. (In HW 1 you'll construct a polyhedron for this).



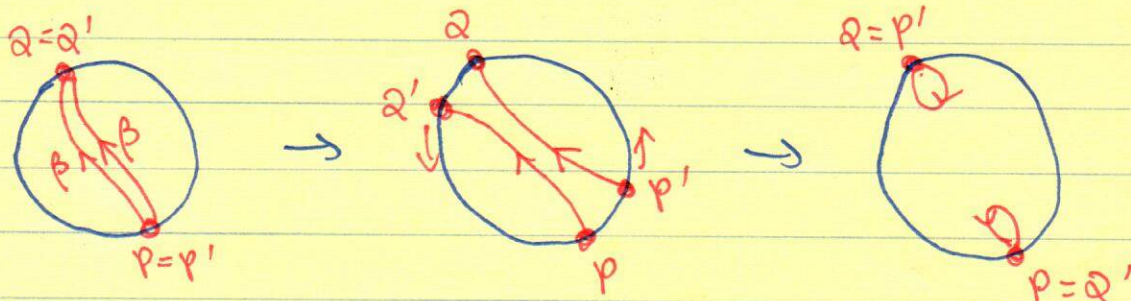
On the disc, all loops are homotopically trivial,
but in $\mathbb{R}P^2$ there's another class:



α is trivial in ~~$\mathbb{R}P^2$~~ $\pi_1(\mathbb{R}P^2)$

β is non-trivial in ~~$\mathbb{R}P^2$~~ $\pi_1(\mathbb{R}P^2)$

However the product $\beta \times \beta$ is trivial in $\pi_1(\mathbb{R}P^2)$,
as can be seen by performing a continuous deformation:



Therefore $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$.