

Homework 3: Lie derivatives and interior products

Given a particular vector field on a manifold, we can define two more operations. The *Lie derivative* is a generalization of the directional derivative of a function to a derivative of a tensor field “along” the given vector field. The *interior product*, or contraction, is an operation that maps p -forms to $(p - 1)$ -forms by “contracting” with the given vector field.

Flows and Lie derivatives (Nakahara, section 5.3)

Let V be a vector field in a manifold M . A curve $x(t)$ whose tangent vector is $V|_{x(t)}$, *i.e.*, $dx^\mu(t)/dt = V^\mu(x(t))$, is called the *integral curve* of V .

Let $\sigma(t, x_0)$ be an integral curve of V which passes through $x_0 \in M$ at $t = 0$, *i.e.*, $d\sigma^\mu(t, x_0)/dt = V^\mu(\sigma(t, x_0))$ and $\sigma^\mu(0, x_0) = x_0^\mu$. The map $\sigma : \mathbb{R} \times M \rightarrow M$ is called the *flow generated by V* . Here t can be thought of as “time”: start at the point x_0 and then “flow” with V .

At fixed t , $\sigma(t, x_0)$ is a diffeomorphism $\sigma_t : M \rightarrow M$. For infinitesimal $t = \epsilon$, $\sigma_\epsilon : x^\mu \mapsto x^\mu + \epsilon V^\mu(x)$. The vector field V is the *generator* of the transformation σ_ϵ .

Consider a second vector field U . We would like to know the rate of change of the vector field U along the flow generated by the vector field V . We should therefore compare $U|_x$ with $U|_{x'=\sigma_\epsilon(x)}$, or more precisely with the push-forward of this quantity by $\sigma_{-\epsilon}$.

Def. The *Lie derivative* of a vector field U along the flow σ of a vector field V is

$$\mathcal{L}_V U \equiv \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [(\sigma_{-\epsilon})_* U|_{\sigma_\epsilon(x)} - U|_x]$$

In local coordinates $U|_x = U^\mu(x) \frac{\partial}{\partial x^\mu}$, and we get (check!)

$$\mathcal{L}_V U = (V^\mu \partial_\mu U^\nu - U^\mu \partial_\mu V^\nu) \frac{\partial}{\partial x^\nu}.$$

This can also be expressed as

$$\mathcal{L}_V U = [V, U], \tag{1}$$

where the RHS is the *Lie bracket*, defined by

$$[V, U]f \equiv V[U[f]] - U[V[f]]. \tag{2}$$

The Lie derivative of a dual vector field (one form) ω along a vector field V is similarly defined (using the pull-back) as

$$\mathcal{L}_V \omega \equiv \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [(\sigma_\epsilon)^* \omega|_{\sigma_\epsilon(x)} - \omega|_x],$$

and in local coordinates

$$\mathcal{L}_V \omega = (V^\mu \partial_\mu \omega_\nu + \omega_\mu \partial_\nu V^\mu) dx^\nu.$$

The Lie derivative of a function f along V is the usual directional derivative

$$\mathcal{L}_V f = V[f] = V^\mu(x) \frac{\partial f}{\partial x^\mu}.$$

For a general tensor field the Lie derivative satisfies

$$\begin{aligned}\mathcal{L}_V(t_1 + t_2) &= \mathcal{L}_V(t_1) + \mathcal{L}_V(t_2) \\ \mathcal{L}_V(t_1 \otimes t_2) &= (\mathcal{L}_V t_1) \otimes t_2 + t_1 \otimes (\mathcal{L}_V t_2).\end{aligned}$$

Problem 1: Properties of the Lie bracket

Show that the Lie bracket satisfies the following properties.

- a. Jacobi identity: $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$.
- b. X, Y are vector fields in M , ω is a p -form and g is a function in M . Show that

$$\begin{aligned}\mathcal{L}_{gX} Y &= g[X, Y] - Y[g]X \\ \mathcal{L}_X(gY) &= g[X, Y] + X[g]Y \\ [\mathcal{L}_X, \mathcal{L}_Y]\omega &= \mathcal{L}_{[X, Y]}\omega.\end{aligned}$$

The interior product (Nakahara, section 5.4.3)

The *interior product*, or *contraction*, maps p -forms to $(p - 1)$ -forms

$$i_V : \Lambda^p(M) \rightarrow \Lambda^{p-1}(M),$$

where $V \in T_p M$, and is defined for $\omega \in \Lambda^p(M)$ by

$$(i_V \omega)(V_1, \dots, V_{p-1}) \equiv \omega(V, V_1, \dots, V_{p-1}).$$

In terms of local coordinates $V = V^\mu \frac{\partial}{\partial x^\mu}$, $\omega = \omega_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$, and

$$\begin{aligned}i_V \omega &= \frac{p!}{(p-1)!} V^\mu \omega_{\mu \mu_2 \dots \mu_p} dx^{\mu_2} \wedge \dots \wedge dx^{\mu_p} \\ &= \sum_{s=1}^p V^{\mu_s} \omega_{\mu_1 \dots \mu_s \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_{s-1}} \wedge dx^{\mu_{s+1}} \wedge \dots \wedge dx^{\mu_p}.\end{aligned}$$

One can relate d , i_V and \mathcal{L}_V acting on differential forms:

$$\mathcal{L}_V = di_V + i_V d. \tag{3}$$

Problem 2: Properties of the interior product

Let X be vector field on M , $\omega \in \Lambda^r(M)$ and $\eta \in \Lambda^s(M)$.

- a. Show that i_X is an antiderivation: $i_X(\omega \wedge \eta) = i_X \omega \wedge \eta + (-1)^r \omega \wedge i_X \eta$.
- b. Show that i_X is nilpotent, $i_X^2 = 0$, and use this to show $\mathcal{L}_X i_X \omega = i_X \mathcal{L}_X \omega$.
- c. Show that

$$(\mathcal{L}_X \omega)(V_1, \dots, V_r) = X[\omega(V_1, \dots, V_r)] - \sum_{i=1}^r (-1)^{i+1} \omega([X, V_i], V_1, \dots, V_{i-1}, V_{i+1}, \dots, V_r)$$

- d. Let X, Y be vector fields on M . Show that $[\mathcal{L}_X, i_Y]\omega = i_{[X, Y]}\omega$.