

Homework 1: Fundamental groups of Polyhedra

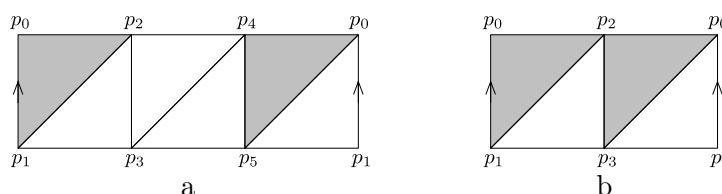
In this exercise you will learn a more systematic way of computing the fundamental group of topological spaces using triangulations (Nakahara, Sec. 4.4).

Triangulations

A triangulation of a topological space X is a polyhedron which is homeomorphic to X , and which is constructed using triangles and their analogs in other dimensions (“simplexes”). Thus, a 0-simplex is a point $\langle p_0 \rangle$, a 1-simplex is a line segment $\langle p_0 p_1 \rangle$, a 2-simplex is a filled triangle $\langle p_0 p_1 p_2 \rangle$, a 3-simplex is a filled tetrahedron $\langle p_0 p_1 p_2 p_3 \rangle$, etc. We define a q -face to be a q -sub-simplex of an r -simplex, for example $\langle p_1 p_2 p_3 \rangle$ is a 2-face of the 3-simplex $\langle p_0 p_1 p_2 p_3 \rangle$. A “simplicial complex” K is a finite set of simplexes (in \mathbb{R}^m) which are “nicely fitted together”, namely

1. Any face of any simplex of K belongs to K .
2. For any pair of simplexes of K , their intersection is either empty or a face of both.

The set K defines a polyhedron $|K| \subset \mathbb{R}^m$ as the union of all the simplexes of K . The dimension of $|K|$ is the highest simplex dimension in K . For example, Fig. a below shows a triangulation of the cylinder. Fig. b, on the other hand, is not a legal triangulation, since $\langle p_0 p_1 p_2 \rangle \cap \langle p_2 p_3 p_0 \rangle = \langle p_0 \rangle \cup \langle p_2 \rangle$, which is neither empty nor a simplex.



Free groups

A subset $X = \{x_j\}$ of a group G is called a *free set of generators* of G if any $g \in G - \{e\}$ is uniquely expressed as $g = x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ ($i_k \in \mathbb{Z} - \{0\}$). We assume that no adjacent x_j are equal (if they are, you must reduce to a form where they’re not). If G has a free set of generators it is a *free group*. Conversely, given a set X , we can construct a free group G whose free set of generators is X .

A general group G is specified by the generators and *constraints* on the generators, which take the form $r = x_{k_1}^{i_1} x_{k_2}^{i_2} \cdots x_{k_n}^{i_n} = 1$. These are called *relations*. The combination $(x_1, \dots, x_p : r_1, \dots, r_q)$ is called the *presentation* of G . For example, the group \mathbb{Z} is generated by a single element x with no relations, so $\mathbb{Z} = (x, \emptyset)$. The cyclic group of order n , \mathbb{Z}_n , is generated by a single element x with the relation $x^n = 1$, so $\mathbb{Z}_n = (x : x^n)$. Another example

is $\mathbb{Z} \oplus \mathbb{Z} = \{x^n y^m | n, m \in \mathbb{Z}\}$. This is a free abelian group generated by $X = \{x, y\}$, so $xy = yx$, or equivalently $xyx^{-1}y^{-1} = 1$. Thus the presentation is $\mathbb{Z} \oplus \mathbb{Z} = (x, y : xyx^{-1}y^{-1})$

The edge group of a polyhedron

An element of the fundamental group of a topological space X can be represented by loops in X . Therefore we expect similar loops in the triangulation $|K|$, which are made up of 1-simplexes, to generate a group isomorphic to the fundamental group of X . This is called the *edge group* of the polyhedron $|K|$. We will first define the edge group, and then introduce a convenient way of computing it for any polyhedron.

An *edge path* is a sequence $v_0 v_1 \cdots v_k$ of vertices of a polyhedron $|K|$, where $v_i v_{i+1}$ is either a 0-simplex (if $v_i = v_{i+1}$) or a 1-simplex. An *edge loop* is an edge path with $v_0 = v_k (= v)$. We classify these loops into equivalence classes according to an equivalence relation. Two edge loops α and β are equivalent if one is obtained from the other by repeating the following operations a finite number of times:

1. If the vertices u, v, w span a 2-simplex in K , the edge path uvw may be replaced by uw and vice-versa.
2. As a special case, if $u = w$ in (1), the edge path uvu may be replaced by a 0-simplex u and vice-versa.

Denote the equivalence class of edge loops at v , to which $vv_1 \cdots v_{k-1}v$ belongs by $\{vv_1 \cdots v_{k-1}v\}$. The set of equivalence classes forms a group under the product operation $\{vu_1 \cdots u_{k-1}v\} * \{vv_1 \cdots v_{l-1}v\} = \{vu_1 \cdots u_{k-1}vv_1 \cdots v_{l-1}v\}$. The unit element is $\{v\}$, and the inverse of $\{vv_1 \cdots v_{k-1}v\}$ is $\{vv_{k-1} \cdots v_1v\}$. This group is called the *edge group* of K at v , denoted by $E(K; v)$. **Theorem:** $E(K; v) \cong \pi_1(|K|; v)$.

Computing the edge group

To compute $E(K; v)$, and therefore $\pi_1(X)$, we need to find the generators and relations. Here's the recipe:

1. Find a triangulation $f : |K| \rightarrow X$.
2. Find a subcomplex L of K that is arcwise connected, simply connected and contains all the vertices of K . (Since L is simply connected, the edge loops in $|L|$ do not contribute to $E(K; v)$. Therefore we can effectively ignore the simplexes in L in our calculations.)
3. Assign a generator g_{ij} to each 1-simplex $\langle v_i v_j \rangle$ of $K - L$, for which $i < j$. (The inverse is given by $g_{ij}^{-1} = g_{ji}$.)
4. Impose a relation $g_{ij} g_{jk} = g_{ik}$ if there is a 2-simplex $\langle v_i v_j v_k \rangle$ such that $i < j < k$. If two of the vertices v_i, v_j and v_k form a 1-simplex of L , the corresponding generator should be set equal to 1.
5. The group generated by $\{g_{ij}\}$ with the above relations is the edge group $E(K; v)$.

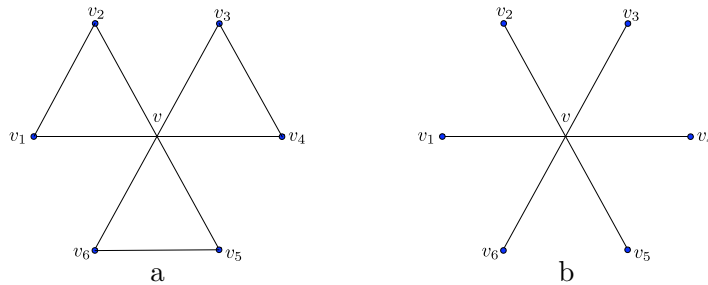
If K is arcwise connected, an L satisfying the conditions of 2 can always be constructed as the *maximal tree* of K . A *tree* is a 1-dimensional simplicial complex that is arcwise and simply connected. It is a *maximal tree* of K if it is not a subset of other trees of K .

Example 1

$K = \{v_1, v_2, v_3, \langle v_1v_2 \rangle, \langle v_1v_3 \rangle, \langle v_2v_3 \rangle\}$ is a simplicial complex of a circle S^1 . A maximal tree is given by $L = \{v_1, v_2, v_3, \langle v_1v_2 \rangle, \langle v_1v_3 \rangle\}$. There is only one generator g_{23} , and there are no relations since there are no 2-simplexes in K . Hence $\pi_1(S^1) \cong (g_{23} : \emptyset) \cong \mathbb{Z}$.

Example 2

An n -bouquet is defined by the one-point union of n circles. For example, Fig. (a) below is a triangulation of a 3-bouquet. Fig. (b) is the maximal tree. There are three generators g_{12} , g_{34} , and g_{56} , and no relations. The fundamental group is therefore $\pi_1(3\text{-bouquet}) \cong (x, y, z : \emptyset)$. Note that this is a free, but not Abelian-free group. Consider two loops at v encircling different holes: $\alpha = vv_2v_1v$ and $\beta = vv_4v_3v$. The product $\alpha * \beta * \alpha^{-1}$ cannot be continuously deformed into β , therefore $[\alpha] * [\beta] * [\alpha]^{-1} \neq [\beta]$, or equivalently $[\alpha] * [\beta] \neq [\beta] * [\alpha]$. In general $\pi_1(n\text{-bouquet}) \cong (x_1, \dots, x_n : \emptyset)$.



Exercise

Go over the other examples in Nakahara, including the torus T^2 and the real projective plane $\mathbb{R}P^2$, and then compute the fundamental groups of the Möbius strip and the Klein bottle.