On simultaneous linearization of diffeomorphisms of $\mathbb{T}^2$

Dominique MALICET

Abstract

In this paper, we prove that if $R_1, \dots, R_m$ are Anzai cocycles of $\mathbb{T}^2$ (i.e. on the form $R_i(x, y) = (x+\alpha_i, y+\beta_i(x))$, and if $f_1, \dots, f_m$ are C^∞-close to $R_1, \dots, R_m$ and satisfy some diophantine condition, then considering the Markov system $x_{n+1} = f_{i_n}(x_n)$ where (i_n) is a sequence of i.i.d. variables on $\{1, \dots, m\}$, for every stationary measure of this system at least one associated Lyapunov exponent is negative unless $f_1, \dots, f_m$ are simultaneously conjuted to Anzai cocycles with a smooth conjugation close to a linear action.

1 Introduction

Krikorian and Dolgopyat studied in [1] Markov system on S^d ($d \geq 2$) of the form $x_{n+1} = f_{i_n}(x_n)$, where (i_n) is an i.i.d sequence on $\{1, \dots, m\}$, and $f_1, \cdots, f_m$ are smooth diffeomorphisms C^∞-close to rotations $R_1, \dots, R_m$ whose action on S^d is minimal, and proved that for each stationary measure of this system, at least one associated Lyapunov exponent is non-positive unless $f_1, \dots, f_m$ are simultaneously conjugated to rotations, with a conjugation C^∞-close to Identity.

Our purpose is to obtain an analogous result on $\mathbb{T}^d$: if $f_1, \dots, f_m$ are diffeomorphisms of $\mathbb{T}^d$ close to rotations, does nullity of Lyapunov exponents imply simultaneous conjugation? We proved in a previous paper that the answer is yes when $d = 1$, under some diophantine condition on rotations numbers $\rho(f_1), \dots, \rho(f_m)$. If $d \geq 2$, it does not hold anymore. Indeed, if each function f_i is on the form

$$f_i(x_1, \dots, x_d) = (x_1 + \alpha_1, x_2 + \alpha_2(x_1), x_3 + \alpha_3(x_1, x_2), \dots, x_d + \alpha_d(x_1, \dots, x_{d-1})) \quad (1)$$

where α_k is a smooth map on $\mathbb{T}^{k-1}$, then it is not difficult to show that all the Lyapunov exponents of the system are null, though there is not in general simultaneous conjugation to rotations since in general, the f_i's do not commute. Thus, the natural extention of the result would be rather:
if $f_1, \cdots, f_m$ are diffeomorphisms of $\mathbb{T}^d$ C^∞-close to diffeomorphisms of the form (1), does nullity of Lyapunov exponents implies simultaneous conjugation of $f_1, \dots, f_m$ to diffeomorphisms of the form (1)? In this paper we will prove this

assertion, under some diophantine condition, in the case $d = 2$. Moreover, we will also prove that simultaneous conjugation is equivalent to some algebraic relation between $f_1, \dots, f_m$ (precisely: the group of diffeomorphisms generated by $f_1, \dots, f_m$ is solvable of order 2, that is, for every i, j, k, l, $f_i f_j f_i^{-1} f_j^{-1}$ and $f_k f_l f_k^{-1} f_l^{-1}$ commute. This can be viewed as an analog of Moser's theorem [2] which states that in a perturbative context, conjugation to translation is equivalent to the fact that the diffeomorphisms commute).

2 Notations and results

Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the d-dimensional torus. We will identify functions on $\mathbb{T}^d$ with their liftings on $\mathbb{R}^d$ by the map $x \mapsto x + \mathbb{Z}^d$. Thus, in this context, we will denote:

- $C^k(\mathbb{T}^d, \mathbb{R}^n)$ the space of the C^k functions on $\mathbb{R}^d$ into $\mathbb{R}^n$, 1-periodic in each variable, or only $C^k(\mathbb{T}^d)$ if $n = 1$. If $k < +\infty$, we endow $C^k(\mathbb{T}^d)$ with the norm $\|\varphi\|_k = \sup_{|\alpha| \leq k, x \in \mathbb{R}^d} |\partial^\alpha \varphi(x)|$, and we endow $C^\infty(\mathbb{T}^2)$ with the metric

$$d_\infty(\varphi, \psi) = \sum_{k \in \mathbb{N}} \frac{\min(1, \|\varphi - \psi\|_k)}{2^k}$$

- $Diff^k(\mathbb{T}^d)$ the space of the functions which induce a C^k diffeomorphism of $\mathbb{T}^d$, that is the bijective functions $f : \mathbb{R}^d \to \mathbb{R}^d$ of the form $f(x) = Ax + \phi(x)$ where $A \in Sl_d(\mathbb{Z})$, $\phi \in C^k(\mathbb{T}^d, \mathbb{R}^d)$. A is the *linear part* of f, and ϕ its *periodic part*.

- $Diff_0^k(\mathbb{T}^d)$ the connected component of Id in $Diff^k(\mathbb{T}^d)$, that is functions of $Diff^k(\mathbb{T}^d)$ whose linear part is Id.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a finite probability space. If E is one of the function spaces described above, we will denote by $\widetilde{E}$ the space of random variables $f : \omega \mapsto f^\omega$ from Ω into E. Such a random variable f will be called random function/diffeomorphism, and if x belongs to $\mathbb{T}^d$, $f(x)$ will denote the random variable $\omega \mapsto f^\omega(x)$. If $\|\cdot\|_\alpha$ is some functional norm and φ a random map, we will denote

$$|||\varphi|||_\alpha = \mathbb{E}\left[\|\varphi\|_\alpha^2\right]^{\frac{1}{2}}.$$

A probability measure μ is said *stationary* for $f \in \widetilde{Diff}^0(\mathbb{T}^d)$ if $\mathbb{E}[f_*\mu] = \mu$. A stationary measure is said *ergodic* if it is extremal in the convex set of stationary measures.

If $f = Id + \Phi \in \widetilde{Diff_0^0}(\mathbb{T}^d)$, we will call *random rotation vector* of f a random variable X on $\mathbb{R}^d$ of the form $X = \int_{\mathbb{T}^d} \phi d\mu$ where μ is a stationary measure of f.

If A, σ are positive reals, a random variable Θ in $\mathbb{R}^d$ will be said (A, σ)-diophantine if

$$\forall q \in \mathbb{Z}^d, \mathbb{E}\left[\inf_{p\in\mathbb{Z}} |q.\Theta - p|\right] \geq \frac{A}{|q|^\sigma}.$$

From now on we will essentially work in $\mathbb{T}^2$:

We will call Anzai cocycles the maps of $Diff^\infty(\mathbb{T}^2)$ of the form $R(x, y) = (x + \alpha, y + \beta(x))$. We will write $\hat{\beta}$ as a shortcut for $\hat{\beta}(0) = \int_{\mathbb{T}} \beta(x)dx$.

If f belongs to $\widetilde{Diff}^1(\mathbb{T}^2)$ and μ is a stationary ergodic measure of f, then letting (f_n) be a sequence of independant copies of f and $F_n = f_{n-1} \circ \cdots \circ f_0$, we set

$$\begin{cases} \gamma_1(f, \mu) = \lim\limits_{n\to+\infty} \dfrac{1}{n}\mathbb{E}\displaystyle\int_{\mathbb{T}^2} \ln \|dF_n(x)\| d\mu(x) \\ \gamma_2(f, \mu) = \lim\limits_{n\to+\infty} \dfrac{1}{n}\mathbb{E}\displaystyle\int_{\mathbb{T}^2} \ln \|dF_n(x)^{-1}\|^{-1} d\mu(x) \end{cases}$$

the *Lyapunov exponents* of f associated to μ.

Theorem 1. *Let $A, \sigma, M_0 > 0$. There exists $\varepsilon_0 > 0$ such that for every random Anzai cocycle R satisfying $d_\infty(R, Id) \leq M_0$, and for every random diffeomorphism f in $\widetilde{Diff}_0^\infty(\mathbb{T}^2)$ satisfying $d_\infty(f, R) \leq \varepsilon_0$ and admitting a (A, σ)-diophantine random rotation vector, the following assertions are equivalent:*

(I) $\gamma_2(f, \mu) \geq 0$ for some probability measure μ stationary for f.

(II) There exists a solvable subgroup $\mathcal{G}$ of $Diff_0^0(\mathbb{T}^2)$ such that $f \in \mathcal{G}$ almost surely.

(III) f is smoothly conjugated to a random Anzai cocycle, by a conjugation C^∞-close to a linear automorphism of $\mathbb{T}^2$.

Remark: Random rotation vectors are not in general invariant by conjugation. But it is when the stationary measure of f is almost surely invariant by f, which is the case when f is conjugated to a random Anzai cocycle. (Anzai cocycles are Lebesgue invariant).

(III) trivially implies (I) and (II) (and in fact implies more precisely that $\gamma_i(f, \mu) = 0$ $(i = 1, 2)$ and that f almost surely belongs to some solvable subgroup of $Diff_0(\mathbb{T}^2)$ with index 2) ,thus we will be interest with the converse implications. The way we will use assumption (II) relies on the following proposition:

Proposition 1. *Let X a compact space, and $\mathcal{G}$ a solvable group of homeomorphisms of X. Then there exists a $\mathcal{G}$-invariant probability measure μ (that is $g_*\mu = \mu$ for every g in $\mathcal{G}$).*

Proof. Let $\mathcal{G}$ a solvable group of homeomorphisms of X. Thus, there exists a chain of subgoup $\mathcal{G}_0 = \{Id\}, \mathcal{G}_1, \ldots, \mathcal{G}_{n-1}, \mathcal{G}_n = \mathcal{G}$ such that for $0 \leq i \leq n - 1$, $\mathcal{G}_i \lhd \mathcal{G}_{i+1}$ and $\mathcal{G}_{i+1}/\mathcal{G}_i$ is abelian. Let $\Pi(X) = C^0(X)'$ the set of probability measures on X endowed with the weak topology, and $\mathcal{M}_i$ the set of $\mathcal{G}_i$-invariant probability measures, which is convex and closed in $\Pi(X)$. We assume that $\mathcal{M}_i \neq \emptyset$ for some i, and then we define for $f \in \mathcal{G}_{i+1}$ the linear operator T_f on

$\mathcal{M}_i$ by $T_f(\mu) = f_*\mu$. Since $\mathcal{G}_{i+1} \lhd \mathcal{G}_i$, for every g in $\mathcal{G}_i$ and every $\mu \in \mathcal{M}_i$ we have $(f^{-1}gf)_*\mu = \mu$ and so $g_*(f_*\mu) = f_*\mu$, thus $T_f(\mu) = f_*\mu \in \mathcal{M}_i$. Moreover, if $f, g \in \mathcal{G}_{i+1}$, then $f^{-1}g^{-1}fg \in \mathcal{G}_i$, so for every μ in $\mathcal{M}_i$, $(f^{-1}g^{-1}fg)_*\mu = \mu$ and so $T_f \circ T_g(\mu) = T_g \circ T_f(\mu)$. Thus, $(T_f)_{f\in\mathcal{G}_{i+1}}$ is a commuting family of linear operators of $\mathcal{M}_i$ into itself. By Kakutani-Markov fixed point theorem, there exists a common fixed point of these operators in $\mathcal{M}_i$, that is there exists μ in $\mathcal{M}_i$ such that $f_*\mu = \mu$ for every μ in $\mathcal{G}_{i+1}$, so $\mathcal{M}_{i+1} \neq \emptyset$. By immediate induction, $\mathcal{M}_n \neq \emptyset$.

□

We recall here properties of C^k-calculus we will need in the sequel:

Proposition 2. *See [1]*
i) For every ϕ, ψ in $C^k(\mathbb{T}^2)$,

$$\|\varphi\psi\|_k \leq C(k)(\|\phi\|_k\|\psi\|_0 + \|\phi\|_0\|\psi\|_k)$$

ii) For every $f = P + \phi$ and $g = Q + \psi$ in $Diff^k(\mathbb{T}^2)$ such that $\|\psi\| \leq M_0$,

$$\|f \circ g - P \circ Q\| \leq C(k, M_0)(\|P\|.\|\psi\|_k + \|Q\|.\|\phi\|_k + \|\phi\|_k\|\psi\|_k)$$

iii) For every $f = P + \phi$ in $Diff^k(\mathbb{T}^2)$ such that $\|\phi\|_1 \leq \frac{1}{2\|P\|}$,

$$\|f^{-1} - P^{-1}\|_k \leq C(k)\|f - P\|_k$$

2.1 Scheme of the proof

We describe here the differents steps of the proof:

- **Step 1**: If R is a random Anzai map whose random rotation vector $(\alpha, \hat{\beta})$ satisfies a diophantine condition, and if we denote $T_0 : \varphi \mapsto \mathbb{E}[\varphi \circ R]$ the associated transition operator, for ψ in $C^\infty(\mathbb{T}^2)$ we will prove that the cohomological equation

$$\varphi - T_0\varphi = \psi - \int_{\mathbb{T}^2} \psi(z)dz$$

  admits a solution φ in $C^\infty(\mathbb{T}^2)$ whose norms of derivatives are controlled by derivatives of ψ.

- **Step 2**: Using estimates of step 1, we prove that a random diffeomorphism of $\mathbb{T}^2$ close to R has stationary measures close to Lebesgue measure, noticing that such a stationary measure will be small on functions of the form $\varphi - T_0\varphi$. Moreover, by a finer analysis we will obtain a more precise approximation of stationary measures by some density measure $(1 + h(z))dz$, where h is small.

Next, we conjugate the diffomorphism by a conjugation close to Identity in order to vanish the term $h(z)dz$. Thus, the new diffeomorphism has stationary measures closer to Lebesgue measure.

- **Step 3**: Using the formula

$$\gamma_1(f,\mu) + \gamma_2(f,\mu) = \mathbb{E}\int_{\mathbb{T}^2} \ln \det d_z f \, d\mu(z)$$

($d_z f$ is the differential of f at point z), when μ is close to Lebesgue measure, we verify that the condition $\gamma_2(f,\mu) \geq 0$ implies by Jensen inequality some uniform estimates from which we deduce that Lebesgue mesure is close to be invariant by realisations of f, and that in fact, $\gamma_1(f,\mu)$ and $\gamma_2(f,\mu)$ are close to 0.

- **Step 4**: We study the action of the differential of f on the space of vectorial lines of $\mathbb{R}^2$ denoted $\mathcal{P}(\mathbb{R}^2)$ (which can be identified to $\mathbb{T}$): we define a natural extension of f on $\mathbb{T}^2 \times \mathcal{P}(\mathbb{R}^2)$ by $\hat{f}(z,h) = (f(z), d_z f(h))$. Using step 2, we can assume, up to conjugating the system, that stationary measures of the cocycle have a projection on $\mathbb{T}^2$ close to Lebesgue measure. Moreover, using analog arguments we can also assume that stationary measures of $\hat{f}$ are close to product measures.

  When f is close to a translation, $d_z f$ is close to the identity, and by Taylor expansions the transition operator S of the cocycle has on $C^\infty(\mathbb{T})$ the form, up to negligibles terms:

$$S\varphi(h) \approx a(h)\varphi'(h) + b(h)\varphi''(h),$$

  with $b \geq 0$. Next we prove a dichotomy:

  -If b has a bound by below not too small, this implies some ellipticity on the operator S and then stationary measures can be approximated by some density $dz \otimes g(h)dh$, where g is smooth and has a controlled size.

  -If there exists h such that $b(h)$ is very small, then this direction h is close to be invariant by $d_z f$, uniformly in z.

  If the first case of this alternative occurs, then using that the Lyapunov exponent of the cocycle is close to 0 (consequence of the fact that Lyapunov exponents of f are close to 0), we prove that the cocycle is close to be linearizable, that is conjugated to one on the form $(z,h) \mapsto (f(z), e^{i\omega(z)}h)$ up to negligible terms. And we prove next that such a linearization is impossible in this context, and so that the first case cannot occur. Thus, the second case of the alternative necessarly occurs and there exists a direction h close to be invariant by the cocycle. It is then not difficult to deduce

that f is linearly conjugated to a diffeomorphism close to an Anzai cocycle.

When f is not close to a translation, using a conjugation of the cocycle by $(z,h) \mapsto (z, D(h))$ where D is some diagonal matrix, we can obtain a cocycle whose second term is close to the identity, and thus use previous arguments and prove the same result in this case, and more precisely in this case the diffeomorphism f is close to an Anzai cocycle without algebraic conjugation.

- **Step 5** At this point we have proved that a random diffeomorphism close to a diophantine Anzai cocyle whose Lyapunov exponents are positive is conjugated to a random diffeomorphism closer to an Anzai map, and in general, the algebraic part of the conjugation is the identity. Iterating this fact, by the use of classical KAM methods we prove that the random diffeomorphism is conjugated to an Anzai cocycle.

3 Cohomological equation

For all the sequel, we will assume σ and M_0 are fixed, and we will not explicit the dependence of constants which will appear in σ and M_0. We will neither explicit their depedence on A, but we will need the easily checkable fact that this dependance is polynomial (there are bounded by CA^{-m} for some $m > 0$ with C not depending on A) We will use the notation $O(Z)$ to represent a quantity whose absolute value is bounded by CZ for some constant C, and if E is some normed vectorial space, $O_E(Z)$ will represent an element of E whose norm is bounded by CZ.

Let $R : (x,y) \mapsto (x+\alpha, y+\beta(x))$ be a random Anzai cocycle, T_0 the associated transition operator defined by $T_0\varphi = \mathbb{E}[\varphi \circ R]$, $\hat{\beta} = \int_{\mathbb{T}} \beta(x)dx$ and $\Delta = |||R \circ \tilde{R} - \tilde{R} \circ R|||_{L^2(\mathbb{T}^2)}$ where $\tilde{R}$ is an independant copy of R. We will assume that $(\alpha, \hat{\beta})$ is (A,σ)-diophantine and that $d_\infty(R, Id) \leq M_0$.

If k_1 and k_2 are two integers and φ is a C^∞ function on $\mathbb{T}^2$, we will denote

$$\|\varphi\|_{k_1,k_2} = \sup_{i_1,i_2 \in \mathbb{N}, 0 \leq \frac{i_1}{k_1} + \frac{i_2}{k_2} \leq 1} \left\| \frac{\partial^{i_1+i_2}\varphi}{\partial x^{i_1} \partial y^{i_2}} \right\|_{L^2(\mathbb{T}^2)}$$

(with the convention $\frac{1}{0} = +\infty$, $\frac{0}{0} = 0$).

Proposition 3. *The operator $I - T_0$ is a bijection from the space*

$$E_0 = \left\{ \varphi \in C^\infty(\mathbb{T}^2) | \int_{\mathbb{T}^2} \varphi(x,y)dxdy = 0 \right\}$$

onto itself. Moreover:

i)For every integer k there exists an integer K such that for every ψ in E_0,

$$\|(I - T_0)^{-1}\psi\|_k \leq C\|\psi\|_K$$

for some $C = C(k)$.

ii)If $\Delta \neq 0$, there exists an integer a such that for every integers $k_1, k_2 \geq 1$ and every ψ in E_0,

$$\|(I - T_0)^{-1}\psi\|_{k_1,k_2} \leq \frac{C}{\Delta^{8k_1}}\|\psi\|_{k_1+a,k_2+ak_1},$$

for some constant $C = C(k_1, k_2)$.

iii)For every $\delta > 0$ and every integer p, there exists an integer k_0 such that for every ψ in E_0, there exists φ in E_0 such that for every k_1 and k_2 larger thant k_0,

$$\begin{cases} \|\varphi\|_{k_1,k_2} \leq C\|\psi\|_{(1+\delta)k_1,(1+\delta)k_2} \\ \|(I - T_0)\varphi - \psi\|_{k_1,k_2} \leq C\Delta^p\|\psi\|_{(1+\delta)k_1,(1+\delta)k_2} \end{cases}$$

for some constant $C = C(\delta, p, k_1, k_2)$.

Proof. We will use the decomposition of the space E_0 on two subspaces $E_0 = E_{00} \oplus F$ where

$$E_{00} = \left\{\varphi \in C^\infty(\mathbb{T}^2) | \forall x \in \mathbb{T}, \int_{\mathbb{T}} \varphi(x, y)dy = 0\right\}$$

and F is the subspace of functions of E_0 which depend only on the first variable. Notice that the spaces E_{00} and F are stable by T_0, and by the derivations ∂_1 and ∂_2. It is easy to check that $I - T_0$ is a bijection of F onto itself satisfying estimates i),ii) and iii) (it satisfies in fact stronger estimates), since for φ in F, $T_0\varphi(x) = \mathbb{E}[\varphi(x + \alpha)]$, so that the cohomological equation $\varphi - T_0\varphi = \psi$ can be solved using Fourier series expansion.. Thus, we will mainly study the restriction of $I - T_0$ on E_{00}.

Up to replacing R by URU^{-1} with $U(x, y) = (x, y + u(x))$, u being defined by the relation $u(x) - \mathbb{E}[u(x + \alpha)] = \mathbb{E}[\beta(x) - \hat{\beta}]$ (see Lemma 1 of previous chapter), we will from now on assume that

$$\forall x \in \mathbb{T}, \mathbb{E}[\beta(x) - \hat{\beta}] = 0.$$

If $\tilde{R}$ is an independant copy of R, then

$$(R \circ \tilde{R} - \tilde{R} \circ R)(x, y) = (0, (\beta(x + \tilde{\alpha}) + \tilde{\beta}(x)) - (\tilde{\beta}(x + \alpha) + \beta(x))).$$

Since $\mathbb{E}[\tilde{\beta}]$ = constant,

$$\mathbb{E}[R \circ \tilde{R} - \tilde{R} \circ R|\alpha, \beta] = (0, T_0\beta - \beta),$$

and in consequence

$$|||T_0\beta - \beta|||_{L^2(\mathbb{T})} \leq \Delta.$$

and for every integer k,

$$|||\beta - \hat{\beta}|||_k \leq C|||T_0\beta - \beta|||_{k+2\sigma} \leq C\sqrt{\Delta}$$

for some $C = C(k)$.

When $\Delta = 0$, this implies that R is almost surely a translation on $\mathbb{T}^2$: in this case we do not need to use the previous decomposition of E_0, Proposition 3 can be proved by Fourier arguments:

Lemma 1. *If R is almost surely a translation, then $(I - T_0)$ is a bijection from E_0 onto itself, then for every $\delta > 0$ there exists an integer k_0 such that for every integers k_1 and k_2 larger than k_0 and for every ψ in E_0,*

$$\|(I - T_0)^{-1}\psi\|_{k_1,k_2} \leq C\|\psi\|_{(1+\delta)k_1,(1+\delta)k_2}$$

for some $C = C(\delta, k_1, k_2)$.

Proof. The equation $\varphi - T_0\varphi = \psi$ is equivalent to

$$\forall p = (p_1, p_2) \in \mathbb{Z}^2, \hat{\varphi}(p)(1 - \mathbb{E}[e^{2i\pi(p_1\alpha + p_2\beta)}]) = \hat{\psi}(p)$$

Thus defining φ by

$$\varphi(x) = \sum_{p \in \mathbb{Z}^2 - \{0\}} \frac{\hat{\psi}(p)}{1 - \mathbb{E}[e^{2i\pi(p_1\alpha + p_2\beta)}]} e^{2i\pi(p_1 x + p_2 y)},$$

we have $\varphi - T_0\varphi = \psi$ and using that

$$|1 - \mathbb{E}[e^{2i\pi(p_1\alpha + p_2\beta)}]| \geq \frac{A^2}{|p_1|^{2\sigma} + |p_2|^{2\sigma}}$$

we deduce that ψ is C^∞ and that there exists an integer a such that

$$\|\varphi\|_{L^2(\mathbb{T}^2)} \leq C(\|\partial_1^a\psi\|_{L^2(\mathbb{T}^2)} + \|\partial_2^a\psi\|_{L^2(\mathbb{T}^2)}).$$

If k_1 and k_2 are larger than $\frac{a}{\delta}$ and if i_1 and i_2 are integers satisfying $\frac{i_1}{k_1} + \frac{i_2}{k_2} \leq 1$, then since $(I - T_0)\partial_1^{i_1}\partial_2^{i_2}\varphi = \partial_1^{i_1}\partial_2^{i_2}\psi$ we get by the previous inequality

$$\|\partial_1^{i_1}\partial_2^{i_2}\varphi\|_{L^2(\mathbb{T}^2)} \leq C(\|\partial_1^{i_1+a}\partial_2^{i_2}\psi\|_{L^2(\mathbb{T}^2)} + \|\partial_1^{i_1}\partial_2^{i_2+a}\psi\|_{L^2(\mathbb{T}^2)}).$$

with

$$\frac{i_1 + a}{(1+\delta)k_1} + \frac{i_2}{(1+\delta)k_2} = \frac{1}{1+\delta}\left[\left(\frac{i_1}{k_1} + \frac{i_2}{k_2}\right) + \frac{a}{k_1}\right] \leq 1$$

so that $\|\partial_1^{i_1+a}\partial_2^{i_2}\psi\|_{L^2(\mathbb{T}^2)} \leq \|\psi\|_{(1+\delta)k_1,(1+\delta)k_2}$, and $\|\partial_1^{i_1}\partial_2^{i_2+a}\psi\|_{L^2(\mathbb{T}^2)} \leq \|\psi\|_{(1+\delta)k_1,(1+\delta)k_2}$ in the same way. The result follows. □

In the sequel, we will assume that $\Delta \neq 0$. Now we will prove the point iii) of Proposition 3

Lemma 2. *For every integer p and every $\delta > 0$, there exists k_0 such that for every k_1, k_2 larger than k_0 and every ψ in E_0, there exists φ in $C^\infty(\mathbb{T}^2)$ such that*

$$\begin{cases} \|\varphi\|_{k_1,k_2} \leq C\|\psi\|_{(1+\delta)k_1,(1+\delta)k_2} \\ \|(I-T_0)\varphi - \psi\|_{k_1,k_2} \leq C\Delta^p \|\psi\|_{(1+\delta)k_1,(1+\delta)k_2}. \end{cases}$$

Proof. We set $\tilde{T}_0\varphi(x,y) = \mathbb{E}[\varphi(x+\alpha, y+\hat{\beta})]$. This operator $\tilde{T}_0$ is close to T_0 when Δ is small, in the sense that for every φ in E_0,

$$\|(T_0 - \tilde{T}_0)\varphi\|_{k_1,k_2} \leq C\|\beta - \hat{\beta}\|_{k_1}\|\varphi\|_{k_1,k_2+1} \leq C\sqrt{\Delta}\|\varphi\|_{k_1,k_2+1}$$

Moreover, we know by Lemma 1 that $I - \tilde{T}_0$ is invertible on E_0. For ψ in E_0, we define functions φ_p and ψ_p by induction by $\varphi_0 = 0$, $\psi_p = (I - T_0)\varphi_p$ and $\varphi_{p+1} = \varphi_p + (I - \tilde{T}_0)^{-1}(\psi - \psi_p)$. Now let p in $\mathbb{N}$, let us assume that for every $\delta > 0$ if k_1 and k_2 are large enough we have

$$\begin{cases} \|\varphi_p\|_{k_1,k_2} \leq C\|\psi\|_{(1+\delta)k_1,(1+\delta)k_2} \\ \|\psi_p - \psi\|_{k_1,k_2} \leq C\Delta^{\frac{p}{2}} \|\psi\|_{(1+\delta)k_1,(1+\delta)k_2}. \end{cases}$$

for some constant $C = C(\delta, k_1, k_2)$. Then

$$\|\varphi_{p+1} - \varphi_p\|_{k_1,k_2} = \|(I-\tilde{T}_0)^{-1}(\psi - \psi_p)\|_{k_1,k_2} \leq C\|\psi - \psi_p\|_{(1+\delta)k_1,(1+\delta)k_2} \leq C\Delta^{\frac{p}{2}}\|\psi\|_{(1+\delta)^2k_1,(1+\delta)^2k_2},$$

hence

$$\begin{cases} \|\varphi_{p+1}\|_{k_1,k_2} \leq \|\varphi_p\|_{k_1,k_2} + C\Delta^{\frac{p}{2}}\|\psi\|_{(1+\delta)^2k_1,(1+\delta)^2k_2} \leq C\|\psi\|_{(1+\delta)^2k_1,(1+\delta)^2k_2} \\ \|\psi_{p+1} - \psi\|_{k_1,k_2} = \|(T_0 - \tilde{T}_0)(\varphi_{p+1} - \varphi_p)\|_{k_1,k_2} \leq C\|\varphi_{p+1} - \varphi_p\|_{k_1,k_2+1} \leq C\Delta^{\frac{p+1}{2}}\|\psi\|_{(1+\delta)^2k_1,(1+\delta)^2(k_2+1)} \end{cases}.$$

We deduce by induction that $\varphi = \varphi_{2p}$ satisfies the claimed estimates. □

Lemma 3. *There exists an integer k_0 such that for every φ in E_{00},*

$$\|\varphi\|_{L^2(\mathbb{T}^2)} \leq \frac{C}{\Delta^8}\|\partial_2^{k_0}(\varphi - T_0\varphi)\|_{L^2(\mathbb{T}^2)}$$

Proof. Let $\varepsilon = \mathbb{E}[\|\partial_2^k(\varphi \circ R - \varphi)\|^2_{L^2(\mathbb{T}^2)}]^{\frac{1}{2}} = \mathbb{E}[\|\partial_2^k(\varphi - \varphi \circ R^{-1})\|^2_{L^2(\mathbb{T}^2)}]^{\frac{1}{2}}$ for some integer k we will chose later. We have

$$\begin{aligned} \varepsilon^2 &= 2\int_{\mathbb{T}^2} \partial_2^k\varphi(x,y)^2 dxdy - 2\int_{\mathbb{T}^2} \partial_2^k\varphi(x,y)T_0(\partial_2^k\varphi)(x,y)dxdy \\ &= \int_{\mathbb{T}^2} \varphi\partial_2^{2k}(\varphi - T_0\varphi)dxdy \\ &\leq 2\|\varphi\|_{L^2(\mathbb{T}^2)}\|\partial_2^{2k}(\varphi - T_0\varphi)\|_{L^2(\mathbb{T}^2)} \end{aligned}$$

Notice that if S is a composition of n any copies (not necessarly independant) of R or R^{-1}, then

$$\mathbb{E}[\|\partial_2^k(\varphi \circ S - \varphi)\|^2_{L^2(\mathbb{T}^2)}]^{\frac{1}{2}} \leq n\varepsilon.$$

Letting $\tilde{R} : (x, y) \mapsto (x + \tilde{\alpha}, y + \tilde{\beta})$ be an independant copy of R, we set

$$S_0 = R^{-1} \circ \tilde{R}^{-1} \circ R \circ \tilde{R},$$

which is on the form

$$S_0(x, y) = (x, y + b(x))$$

where $b(x) = -\beta(x + \alpha) - \tilde{\beta}(x + \alpha + \tilde{\alpha}) + \beta(x + \tilde{\alpha}) + \tilde{\beta}(x)$.
Next, letting $R_i : (x, y) \mapsto (x + \alpha_i, y + \beta_i(x))$ $(i = 1, \dots, n)$ be n independant copies of R for some n we will chose later, we set

$$S = (R_n \circ \cdots \circ R_1)^{-1} \circ S_0 \circ (R_n \circ \cdots \circ R_1),$$

which is on the form

$$S(x, y) = (x, y + b(x + s_n))$$

where $s_n = \alpha_1 + \cdots + \alpha_n$.
By the previous remark, we have

$$\mathbb{E}[\|\partial_2^k(\varphi \circ S - \varphi)\|^2_{L^2(\mathbb{T}^2)}]^{\frac{1}{2}} \leq (2n + 2)\varepsilon.$$

Writing

$$\varphi(x, y) = \sum_{p \in \mathbb{Z}} \varphi_p(x) e^{2ip\pi y}, \tag{2}$$

we have

$$\begin{aligned} \varphi \circ S(x, y) - \varphi(x, y) &= \varphi(x, y + b(x + s_n)) - \varphi(x, y) \\ &= \sum_{p \in \mathbb{Z}} \varphi_p(x)(e^{2i\pi pb(x+s_n)} - 1)e^{2i\pi py}. \end{aligned}$$

Seeing $\varphi_p(x)(e^{2ip\pi b(x+s_n)} - 1)$ as a Fourier coefficient L^2-norm of $y \mapsto \varphi \circ S(x, y) - \varphi(x, y)$ leads to

$$|\varphi_p(x)(e^{2ip\pi b(x+s_n)} - 1)| \leq \frac{C}{|p|^k} \|\varphi \circ S(x, \cdot) - \varphi(x, \cdot)\|_k$$

Denoting $u = \mathbb{E}[e^{2ip\pi b}]$, we have $\mathbb{E}[e^{2ip\pi b(x+s_n)}] = T_0^n u(x)$ (we still denote T_0 the restriction of T_0 on functions depending only of the first variable), and so

$$\|\varphi_p(T_0^n u - 1)\|_{L^2(\mathbb{T}^2)} \leq (2n + 2)\varepsilon,$$

hence

$$\left\| \varphi_p \left(\frac{1}{n} \sum_{j=0}^{n-1} T_0^j u - 1 \right) \right\|_{L^2(\mathbb{T})} \leq (2n + 2)\varepsilon. \tag{3}$$

Setting $m_p = \mathbb{E} \int_{\mathbb{T}} e^{2i\pi pb(t)} dt$, writing $u = v - T_0 v + m_p$ leads to

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} T_0^j u - m_p \right| \leq \frac{C\|u\|_{l_0}}{n} \leq \frac{C|p|^{l_0}}{n}$$

for some constant C and integer l_0, and so

$$\left|\frac{1}{n}\sum_{j=0}^{n-1} T_0^j u - 1\right| \geq |1 - m_p| - \frac{Cp^{l_0}}{n}.$$

Choosing $n = \dfrac{2Cp^{l_0}}{|1-m_p|}$, we obtain

$$\left|\frac{1}{n}\sum_{j=0}^{n-1} T^j u - 1\right| \geq \frac{|1-m_p|}{2}$$

and so (3) becomes

$$\|\varphi_p\|_{L^2(\mathbb{T})} \leq \frac{Cn\varepsilon}{|1-m_p|} \leq \frac{Cp^{l_0}\varepsilon}{|1-m_p|^2} \tag{4}$$

for some constant C.

Let us look for a lower bound for $|1 - m_p|$. We have

$$|1 - m_p| \geq \mathbb{E}\int_{\mathbb{T}} (1 - \cos(2\pi p b(t))dt \geq 2\pi^2 \mathbb{E}\int_{\mathbb{T}} \{pb(t)\}^2 dt,$$

where $\{t\}$ is the distance of t to $\mathbb{Z}$. Next, for a fixed event, using that $\hat{b}(0) = 0$, we have the alternative:

-either there exists t_0 such that $|pb(t_0)| = \frac{1}{2}$, and then for $|t - t_0| \leq \frac{1}{4|p|\|b'\|_0}$ we have

$$|pb(t) - pb(t_0)| \leq |p|\|b'\|_0 |t - t_0| \leq \frac{1}{4}$$

hence $\{pb(t)\} \geq \frac{1}{4}$, and hence

$$\int_{\mathbb{T}} \{pb(t)\}^2 dt \geq \int_{\left[t_0 - \frac{1}{4|p|\|b'\|_0}, t + \frac{1}{4|p|\|b'\|_0}\right]} \frac{dt}{16} = \frac{1}{64|p|\|b'\|_0},$$

-or $|pb(t)| < \frac{1}{2}$ for every t in $\mathbb{T}$ and

$$\int_{\mathbb{T}} \{pb(t)\}^2 dt = \int_{\mathbb{T}} (pb(t))^2 dt = p^2\Delta^2.$$

Thus,

$$\left(\mathbb{E}\left[\int_{\mathbb{T}} \{pb(t)\}^2 dt\right]\right)^{-1} \leq \mathbb{E}\left[\left(\int_{\mathbb{T}} \{pb(t)\}^2 dt\right)^{-1}\right] \leq \mathbb{E}\left[64|p|\|b'\|_0 + \frac{1}{p^2\Delta^2}\right] \leq C|p|\Delta^2$$

for some C, and

$$|1 - m_p| \geq \frac{C}{|p|\Delta^2}.$$

Finally, using these lower bounds of $|1 - m_p|$, (4) becomes

$$\|\varphi_p\|_{L^2(\mathbb{T})} \leq \frac{C\varepsilon}{|p|^{k-l_0-2}\Delta^4},$$

and, choosing $k = l_0 + 4$, we have by (2)

$$\|\varphi\|_{L_2(\mathbb{T}^2)} \leq \sum_{p\in\mathbb{Z}} \|\varphi_p\|_{L^2(\mathbb{T})} \leq \frac{C\varepsilon}{\Delta^4} \leq \frac{C}{\Delta^4}\|\varphi\|^{\frac{1}{2}}_{L^2(\mathbb{T}^2)}\|\partial_2^{2k}(\varphi - T_0\varphi)\|^{\frac{1}{2}}_{L^2(\mathbb{T}^2)},$$

and the result follows. □

Lemma 4. *There exists an integer a such that for every φ in E_0 and every integers k_1 and k_2 larger than* 1*,*

$$\|\varphi\|_{k_1,k_2} \leq \frac{C}{\Delta^{8k_1}}\|\varphi - T\varphi\|_{k_1,k_2+ak_1}$$

Proof. The relation $(I-T_0)\partial_1\varphi = \partial_1(\varphi-T_0\varphi)-\mathbb{E}[\beta'\partial_2\varphi\circ R]$ leads to the inequality

$$\|\partial_1^{k_1}(I - T_0)\partial_1\varphi\|_{L^2(\mathbb{T}^2)} \leq \|\partial_1^{k_1+1}(I - T_0)\varphi\|_{L^2(\mathbb{T}^2)} + C\|\partial_1^{k_1}\partial_2\varphi\|_{L^2(\mathbb{T}^2)}$$

Thus, for every positive integer j less than k_1,

$$\|\partial_1^{k_1-j}(I - T_0)\partial_1^{j}\varphi\|_{L^2(\mathbb{T}^2)} \leq \|\partial_1^{k_1-(j-1)}(I - T_0)\partial_1^{j-1}\varphi\|_{L^2(\mathbb{T}^2)} + C\|\partial_1^{k_1-1}\partial_2\varphi\|_{L^2(\mathbb{T}^2)},$$

and we deduce by succesive use of this inequality that

$$\|(I - T_0)\partial_1^{k_1}\varphi\|_{L^2(\mathbb{T}^2)} \leq \|\partial_1^{k_1}(I - T_0)\varphi\|_{L^2(\mathbb{T}^2)} + C\|\partial_1^{k_1-1}\partial_2\varphi\|_{L^2(\mathbb{T}^2)}$$

for some constant C. Next, Using Lemma 3 and the fact that ∂_2 and T_0 commute, we get

$$\|\partial_1^{k_1}\partial_2^{k_2}\varphi\|_{L^2(\mathbb{T}^2)} \leq \frac{C}{\Delta^8}\|(I-T_0)\partial_1^{k_1}\partial_2^{k_2+k_0}\varphi\|_{L^2(\mathbb{T}^2)} \leq \frac{C}{\Delta^8}(\|\partial_1^{k_1}\partial_2^{k_2+k_0}(I-T_0)\varphi\|_{L^2(\mathbb{T}^2)}+\|\partial_1^{k_1-1}\partial_2^{k_2+k_0+1}\varphi\|_{L^2(\mathbb{T}^2)})$$

and by iteration of this inequality,

$$\|\partial_1^{k_1}\partial_2^{k_2}\varphi\|_{L^2(\mathbb{T}^2)} \leq \frac{C}{\Delta^{8k_1}}\sum_{i=0}^{k_1}\|\partial_1^{k_1-i}\partial_2^{k_2+ai}(I - T_0)\varphi\|_{L^2(\mathbb{T}^2)}$$

with $a = k_0 + 1$. Now, If i_1, i_2 are integers satisfying $\frac{i_1}{k_1} + \frac{i_2}{k_2} \leq 1$, then

$$\|\partial_1^{i_1}\partial_2^{i_2}\varphi\|_{L^2(\mathbb{T}^2)} \leq \frac{C}{\Delta^{8k_1}}\sum_{i=0}^{i_1}\|\partial_1^{i_1-i}\partial_2^{i_2+ai}(I - T_0)\varphi\|_{L^2(\mathbb{T}^2)}$$

with

$$\frac{i_1 - i}{k_1} + \frac{i_2 + ai}{k_2 + ak_1} \leq \frac{i_1}{k_1} + \frac{i_2}{k_2} + \left(-\frac{1}{k_1} + \frac{a}{k_1 + a}\right)i \leq 1$$

hence

$$\|\partial_1^{i_1}\partial_2^{i_2}\varphi\|_{L^2(\mathbb{T}^2)} \leq \frac{C}{\Delta^{8k_1}}\|(I-T_0)\varphi\|_{k_1,k_2+ak_1}$$

and hence

$$\|\varphi\|_{k_1,k_2} \leq \frac{C}{\Delta^{8k_1}}\|(I-T_0)\varphi\|_{k_1,k_2+ak_1}.$$

□

Lemma 5. *For every ψ in E_{00}, there exists φ in E_{00} such that $\varphi - T\varphi = \psi$*

Proof. Let ψ in E_{00}. Using Lemma 3, we have for every n the inequality

$$\|S_n\psi\|_{L^2(\mathbb{T}^2)} \leq \frac{C}{\Delta^8}\|(I-T_0)S_n\psi\|_{0,k_0} = \frac{C}{\Delta^8}\|(I-T_0^n)\psi\|_{0,k_0} \leq \frac{C}{\Delta^8}\|\psi\|_{0,k_0},$$

hence $(S_n\psi)$ is bounded in $L^2(\mathbb{T}^2)$. Setting

$$\varphi_N = \frac{1}{N}\sum_{n=0}^{N-1} S_n\psi,$$

we have

$$(I-T_0)\varphi_N = \psi - \frac{1}{N}S_N\psi,$$

hence $(I-T_0)\varphi_N$ converges to ψ in $L^2(\mathbb{T}^2)$ as N tends to $+\infty$. Since $(S_n\psi)$ is bounded in $L^2(\mathbb{T}^2)$, so does (φ_N) and there exists a subsequence of (φ_N) weakly converging to some function φ in $L^2(\mathbb{T}^2)$, which satisfies $(I-T_0)\varphi = \psi$.

Moreover, since ∂_2 and T_0 commute, for every integer k $\partial_2^k\varphi$ belongs to $L^2(\mathbb{T}^2)$ and $(I-T_0)\partial_2^k\varphi = \partial_2^k\psi$.

For k_1 in $\mathbb{N}$, let us assume that $\partial_1^{k_1}\partial_2^{k_2}\varphi$ belongs to $L^2(\mathbb{T}^2)$ for every integer k_2. Then for every k_2, we easily verify by induction on k_1 that

$$\partial_1^{k_1}\partial_2^{k_2}\psi = (I-T_0)\partial_1^{k_1}\partial_2^{k_2}\varphi + r$$

where $\partial_1\partial_2^{k_2'}$ belongs to $L^2(\mathbb{T}^2)$ for every k_2'. Next, denoting

$$\Delta_h\varphi(x,y) = \frac{\varphi(x+h,y)-\varphi(x,y)}{h},$$

we have

$$\Delta_h\partial_1^{k_1}\partial_2^{k_2}\psi = (I-T_0)\Delta_h\partial_1^{k_1}\partial_2^{k_2}\varphi + \mathbb{E}[\partial_1^{k_1}\partial_2^{k_2}\varphi(x+\alpha, y+\beta(x+h)) - \partial_2^{k_2}\varphi(x+\alpha, y+\beta(x+h))] + \Delta_h r$$

and we deduce that $(I-T_0)\Delta_h\partial_1^{k_1}\partial_2^{k_2}\varphi$ converges in $L^2(\mathbb{T}^2)$ when h tends to 0, and next that for every k_2', $\|(I-T_0)(\Delta_h\partial_1^{k_1}\partial_2^{k_2}\varphi - \Delta_{h'}\partial_1^{k_1}\partial_2^{k_2}\varphi\|_{0,k_2'}$ converges to 0 when h and h' tend to 0, and next by Lemma 3 that $\Delta_{h'}\partial_1^{k_1}\partial_2^{k_2}\varphi$ converges in $L^2(\mathbb{T}^2)$, and so that $\partial_1^{k_1+1}\partial_2^{k_2}\varphi$ belongs to $L^2(\mathbb{T}^2)$.

□

Now, we can finish the proof of Proposition 3. At this point we have proved that $I - T_0$ is a bijection of E_0 satisfying the estimates ii) and iii). Let us now prove i). Let ψ in E_0, and $\varphi = (I - T_0)^{-1}\psi$. By Lemma 2, we can find $\tilde{\varphi}$ such that for k_1, k_2 large enough

$$\begin{cases} \|\tilde{\varphi}\|_{k_1,k_2} \leq C\|\psi\|_{2k_1,2k_2} \\ \|(I - T_0)(\tilde{\varphi} - \varphi)\|_{k_1,k_2} = \|(I - T_0)\tilde{\varphi} - \psi\|_{k_1,k_2} \leq C\Delta^p\|\psi\|_{2k_1,2k_2}. \end{cases}$$

Using Lemma 4, the second inequality implies that

$$\|\varphi - \tilde{\varphi}\|_{k_1,k_2} \leq \frac{C}{\Delta^{8k_1}}\|(I - T_0)(\varphi - \tilde{\varphi})\|_{k_1+a,k_2+ak_1} \leq \Delta^{p-8k_1}\|\psi\|_{\bar{k}_1,\bar{k}_2}$$

with $\bar{k}_1 = 2(k_1 + a)$, $\bar{k}_2 = 2(k_2 + ak_1)$. Choosing $p = 8k_1$, we deduce

$$\|\varphi\|_{k_1,k_2} \leq \|\tilde{\varphi}\|_{k_1,k_2} + \|\varphi - \tilde{\varphi}\|_{k_1,k_2} \leq C\|\psi\|_{\bar{k}_1,\bar{k}_2}.$$

□

4 Estimates on the stationary measure and the sum of Lyapunov exponents, first conjugation

We fix a diophantine random Anzai cocycle R as in the previous section, $f = R+\zeta$ in $\widetilde{Diff_0}^\infty(\mathbb{T}^2)$ a perturbation of R, and μ a stationary measure of f. For φ in $C^\infty(\mathbb{T}^2)$, we will denote

$$U\varphi = (I - T_0)^{-1}(\varphi - \hat{\varphi}(0)).$$

If μ and ν are probability measures on $\mathbb{T}^2$, we denote

$$d_k^*(\mu,\nu) = \sup\left\{\int_{\mathbb{T}^2}\varphi d\mu - \int_{\mathbb{T}^2}\varphi d\nu | \varphi \in C^k(\mathbb{T}^d), \|\varphi\|_k \leq 1\right\}.$$

Proposition 4. *There exists an integer K such that for every φ in $C^\infty(\mathbb{T}^2)$, we have*

$$\int_{\mathbb{T}^2}\varphi d\mu = \int_{\mathbb{T}^2}\varphi dz + \int_{\mathbb{T}^2}(\vec{\nabla}U\varphi).\bar{\zeta}dz + O(\varepsilon^2\|\varphi\|_K),$$

where $\bar{\zeta} = \mathbb{E}[\zeta \circ R^{-1}]$, $\varepsilon = |||\zeta|||_K$. In particular, if $\bar{\zeta} = constant + O(\varepsilon^2)$, then $d_K^*(\mu, dz) = O(\varepsilon^2)$.

Proof. For φ in $C^\infty(\mathbb{T}^2)$, setting $T\varphi = \mathbb{E}[\varphi \circ f]$ and $T_0\varphi = \mathbb{E}[\varphi \circ R]$, we have $|T\varphi - T_0\varphi| \leq \varepsilon\|\varphi\|_1$. Using the T-invariance of μ, we get

$$\left|\int_{\mathbb{T}^2}(\varphi - T_0\varphi)d\mu\right| \leq \varepsilon\|\varphi\|_1.$$

We now use Proposition 3. For ψ in $C^\infty(\mathbb{T}^2)$, applying the previous inequality to $\varphi = U\psi$ gives for some C and K

$$\left| \int_{\mathbb{T}^2} \psi d\mu - \int_{\mathbb{T}^2} \psi dz \right| \leq C\varepsilon \|U\psi\|_1 \leq C\varepsilon \|\psi\|_K.$$

Thus μ is equal to Lebesgue measure in first approximation. To get a more precise approximation, we now write $T\varphi - T_0\varphi = \mathbb{E}[(\vec{\nabla}\varphi \circ R).\zeta] + O(\varepsilon^2\|\varphi\|_2)$, which gives by integration and our previous estimate on μ:

$$\begin{aligned} \int_{\mathbb{T}^2} (\varphi - T_0\varphi) d\mu &= \int_{\mathbb{T}^2} \mathbb{E}[(\vec{\nabla}\varphi \circ R).\zeta] d\mu + O(\varepsilon^2\|\varphi\|_2) \\ &= \int_{\mathbb{T}^2} \mathbb{E}[(\vec{\nabla}\varphi \circ R).\zeta] dz + O(\varepsilon^2\|\varphi\|_K) \\ &= \int_{\mathbb{T}^2} \vec{\nabla}\varphi.\bar{\zeta} dz + O(\varepsilon^2\|\varphi\|_K). \end{aligned}$$

Applying as previously this formula to $\varphi = U\psi$ this gives for some K_1:

$$\int_{\mathbb{T}^2} \psi d\mu - \int_{\mathbb{T}^2} \psi dz = \int_{\mathbb{T}^2} (\vec{\nabla} U\psi).\bar{\zeta} dz + O(\varepsilon^2\|\psi\|_{K_1}).$$

□

Proposition 5. *i) There exists a diffeomorphism $G = Id + g$ such that, denoting $\tilde{f} = G^{-1}fG = R + \tilde{\zeta}$:*

- *For any k_1 ,k_2 larger than 1,*

$$\|g\|_{k_1,k_2} \leq \frac{C}{\Delta^{16k_1}} |||\zeta|||_{k_1+a,k_2+ak_1}.$$

  for some $C = C(k_1, k_2)$.

- *For any k, there exists K such that*

$$\|g\|_k \leq C|||\zeta|||_K$$

  for some $C = C(k)$.

- *For any integer k, there exists an integer K such that*

$$\left\| \mathbb{E}\left[\tilde{\zeta} \circ R^{-1} - \int_{\mathbb{T}^2} \tilde{\zeta}(z) dz \right] \right\|_k \leq C|||\zeta|||_K^2$$

ii) For any $\delta > 0$ and integer p, there exists a diffeomorphism $G = Id + g$ such that, denoting $\tilde{f} = G^{-1}fG = R + \tilde{\zeta}$:

- *For any k_1 ,k_2 large enough,*

$$\|g\|_{k_1,k_2} \leq C|||\zeta|||_{(1+\delta)k_1,(1+\delta)k_2}.$$

  for some $C = C(k_1, k_2)$.

- *For any integer k, there exists an integer K such that*

$$\left\| \mathbb{E}\left[\tilde{\zeta} \circ R^{-1} - \int_{\mathbb{T}^2} \tilde{\zeta}(z) dz \right] \right\|_k \leq C(\Delta^p |||\zeta|||_K + |||\zeta|||_K^2)$$

Proof. Let $G = Id + g$ in $Diff_0^\infty(\mathbb{T}^2)$ with $\|g\|_1 = O(|||\zeta|||_{k_0})$ for some integer k_0, and let $\tilde{f} = G^{-1} f G = R + \tilde{\zeta}$. Setting $\zeta = (\zeta_1, \zeta_2)$, $\tilde{\zeta} = (\tilde{\zeta}_1, \tilde{\zeta}_2)$ and $G = (G_1, G_2) = Id + (g_1, g_2)$, we have

$$\begin{cases} \tilde{\zeta}_1 = \zeta_1 + (g_1 - g_1 \circ R) + O_{L^2(\Omega)}(|||\zeta|||_{k_0}^2) \\ \tilde{\zeta}_2 = \zeta_2 + (\beta \circ G_1 - \beta) + (g_2 - g_2 \circ R) + O(|||\zeta|||_{k_0}^2), \end{cases}$$

hence, denoting $\tilde{T}_0 \varphi = \mathbb{E}[\varphi \circ R^{-1}]$, $\eta_1 = \mathbb{E}[\zeta_1 \circ R^{-1}]$ and $\eta_2 = \mathbb{E}[(\zeta_2 + \beta \circ G_1 - \beta) \circ R^{-1}]$, we have

$$\begin{cases} \tilde{\zeta}_1 = \eta_1 - (g_1 - \tilde{T}_0 g_1) + O(|||\zeta|||_{k_0}^2) \\ \tilde{\zeta}_2 = \eta_2 - (g_2 - \tilde{T}_0 g_2) + O(|||\zeta|||_{k_0}^2) \end{cases}.$$

Thus, using Proposition 3 with $\tilde{T}_0$ instead of T_0 and denoting $\tilde{U}$ the associated inverse operator, setting $g_1 = \tilde{U}\left(\tilde{T}_0 \zeta_1\right)$ and next $g_2 = \tilde{U}\left(\tilde{T}_0(\zeta_2 \circ R^{-1} + (\beta \circ G_1 - \beta))\right)$ we obtain the point i). If instead we define g_1 and g_2 using the point iii) of Proposition 3, we obtain the point ii). □

Proposition 6. *If $f = R + \tilde{\zeta}$ satisfies for some $\varepsilon > 0$ and some integer k_0 large enough the inequalities*

$$\begin{cases} |||\zeta|||_{k_0} \leq \varepsilon \\ \left\| \mathbb{E}\left[\zeta \circ R^{-1} - \int_{\mathbb{T}^2} \zeta(z) dz \right] \right\|_k \leq \varepsilon^2 \end{cases},$$

and if moreover $\gamma_1(f, \mu) + \gamma_2(f, \mu) \geq 0$, or $f_* \mu = \mu$ a.s., then we have,

$$\det(d_z f) = 1 + O_{L^2(\mathbb{T}^2)}(\varepsilon^{\frac{3}{2}}).$$

where $d_z f$ is the differential of f at the point z.

Proof. By Proposition 4, $d_k^*(\mu, dz) = O(\varepsilon^2)$. In consequence, we have

$$\begin{aligned} \gamma_1(f, \mu) + \gamma_2(f, \mu) &= \mathbb{E} \int_{\mathbb{T}^2} \ln \det(d_z f) d\mu(z) \\ &= \mathbb{E} \int_{\mathbb{T}^2} \ln \det(d_z f) dz + O(\varepsilon^3) \\ &= \mathbb{E} \int_{\mathbb{T}^2} (\det(d_z f) - 1) dz - \tfrac{1}{2} \mathbb{E} \int_{\mathbb{T}^2} (\det(d_z f) - 1)^2 dz + O(\varepsilon^3). \end{aligned}$$

Since f is a diffeomorphism of $\mathbb{T}^2$, $\int_{\mathbb{T}^2} \det(d_z f) dz = 1$ hence

$$\gamma_1(f, \mu) + \gamma_2(f, \mu) = -\frac{1}{2} \mathbb{E} \int_{\mathbb{T}^2} (\det(d_z f) - 1)^2 dz + O(\varepsilon^3).$$

This gives the expected estimate when $\gamma_1(f, \mu) + \gamma_2(f, \mu) = 0$.

To obtain this same estimate when $f_*\mu = \mu$ a.s., we write that for φ in $C^\infty(\mathbb{T}^2)$,

$$\int_{\mathbb{T}^2} \varphi d\mu = \int_{\mathbb{T}^2} \varphi(z)dz + O(\varepsilon^2\|\varphi\|_k)$$

and

$$\begin{aligned}\int_{\mathbb{T}^2} \varphi \circ f^{-1} d\mu &= \int_{\mathbb{T}^2} \varphi(f^{-1}(z))dz + O(\varepsilon^2\|\varphi \circ f^{-1}\|_k) \\ &= \int_{\mathbb{T}^2} \varphi(z)\det(d_z f)dz + O(\varepsilon^2\|\varphi\|_k)\end{aligned},$$

hence

$$\int_{\mathbb{T}^2} \varphi(z)(\det(d_z f))dz = \int_{\mathbb{T}^2} \varphi(z)dz + O(\varepsilon^2\|\varphi\|_k)$$

Setting $\varphi = \det(d_z f) - 1$ and taking the expectation of the last formula, we get

$$\mathbb{E}\int_{\mathbb{T}^2} (\det(d_z f) - 1)^2 dz = O(\varepsilon^3)$$

□

Thus, using the last two propositions, under assumption (I) or (II) of Theorem 1 we can conjugate f to some $\tilde{f}$ which is close to be Lebesgue invariant. In the next section, we will use more finely these assumptions in studying the action of the differential of $\tilde{f}$ on the projective space $\mathcal{P}(\mathbb{R}^2)$, and we will obtain that in fact $\tilde{f}$ is close to an Anzai cocycle, up to an algebraic conjugation.

5 Projective action

The aim of this section is to prove the following proposition:

Proposition 7. *Assume that $f = R + \zeta$ satisfies assumption (I) or (II) of Theorem 1, and that in addition, $\mathbb{E}[\beta(x-\alpha)] = \hat{\beta}$, and $\|\mathbb{E}[\zeta \circ R^{-1} - \hat{\zeta}(0)]\|_k \leq M_0\varepsilon^2$ for some M_0, k. Then, there exist $\delta_0 > 0$ such that for every $\kappa > 0$, if K is large enough and if $\varepsilon = |||f - R|||_K$ is small enough, there exists a matrix P in $Sl_2(\mathbb{Z})$ and an Anzai cocyle $\tilde{R} : (x, y) \to (x + \tilde{\alpha}, y + \tilde{\beta}(x))$ such that $|||P^{-1}fP - \tilde{R}|||_0 \leq C\varepsilon^{1+\delta_0}$ for some $C = C(M_0)$. Morever, $\|P\| \leq C\varepsilon^{-\kappa}$, and $P = Id$ if $|||\beta'|||_0 \geq 2\varepsilon$.*

Proof. By proposition 6, we already know that

$$\det(d_z f) = 1 + O(\varepsilon^{\frac{3}{2}}) \tag{5}$$

To prove the more precise fact that f is close, up to an algebraic conjugation, to a random Anzai cocycle, we will study the action of the random cocycle $(z, h) \to (f(z), d_z f(h))$ on $\mathbb{T}^2 \times \mathcal{P}(\mathbb{R}^2)$. Notice that in the case where f is close to a translation of $\mathbb{T}^2$, $d_z f$ is close to Id. And in the general case where f is close to an Anzai cocycle, we can conjugate the cocycle by $(z, h) \to (z, D(h)))$ where

$D = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$ to obtain for an appropriated choice of λ a cocycle whose second component is close to Id.

More precisely: Let $m = |||\beta'|||_0$, $\lambda \geq 1$ a real number we will choose later, $D = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}$ and $M_z = D(J_z f)D^{-1}$ where $J_z f$ is the Jacobian matrix of f at point z. Thus, $M_z = Id + N_z$ with

$$N_\cdot = \begin{pmatrix} \partial_1\zeta_1 & \frac{\partial_2\zeta_1}{\lambda} \\ \lambda(\partial_1\zeta_2 + \beta') & \partial_2\zeta_2 \end{pmatrix},$$

which satisfies $\|N_z\| = O(\frac{\|\zeta\|_1}{\lambda} + \lambda\|\beta'\|_0) = O(\varepsilon_1)$ where we have set

$$\varepsilon_1 = \frac{\varepsilon}{\lambda} + m\lambda.$$

Let

$$u_z = Id + \eta_z$$

the diffeomorphism of $\mathbb{T}$ induced by the projective action of M_z, that is such that $\frac{M_z(e^{i\pi\theta})}{|M_z(e^{i\pi\theta})|} = e^{i\pi u_z(\theta)}$. Writing that

$$e^{2i\pi u_z(\theta)} = \frac{M_z e^{i\pi\theta}}{\overline{M_z e^{i\pi\theta}}} = \frac{e^{i\pi\theta} + N_z e^{i\pi\theta}}{e^{-i\pi\theta} + N_z \bar{e}^{i\pi\theta}} = e^{2i\pi\theta}\left(1 + 2Im(N_z e^{i\pi\theta}.e^{-i\pi\theta}) + O(\|N_z\|^2)\right)$$

leads to the estimate

$$\begin{aligned} \eta_\cdot(\theta) &= \tfrac{1}{2\pi} Im(N_\cdot e^{i\pi\theta}.e^{-i\pi\theta}) + O(\|N_\cdot\|^2) \\ &= \lambda(\partial_1\zeta_2 + \beta')\cos^2(\pi\theta) + (\partial_2\zeta_2 - \partial_1\zeta_1)\cos(\pi\theta)\sin(\pi\theta) \\ &\quad - \tfrac{\partial_2\zeta_1}{\lambda}\sin^2(\pi\theta) + O(\|N_\cdot\|^2) \end{aligned} \tag{6}$$

Finally, let U the random cocycle $(z,\theta) \mapsto (f(z), u_z(\theta))$ on $\mathbb{T}^2 \times \mathbb{T}$, $\hat{\mu}$ a stationary measure of U, and μ, ν the projections of $\hat{\mu}$ on the first and second factor.

The probability μ is stationary for f, so we know by Proposition 4 that μ is close to Lebesgue measure. Adapting the proof, we will prove that in fact, π is close to a product measure:

Lemma 6. *Under assumptions of Proposition 7 and previous notations, there exists an integer k such that for every φ in $C^\infty(\mathbb{T}^2 \times \mathbb{T})$,*

$$\iint_{\mathbb{T}^2\times\mathbb{T}} \varphi(z,\theta)d\hat{\mu}(z,\theta) = \iint_{\mathbb{T}^2\times\mathbb{T}} \varphi(z,\theta)dzd\nu(\theta) + O(\varepsilon_1^2\|\varphi\|_k)$$

Proof. For φ in $C^\infty(\mathbb{T}^2 \times \mathbb{T})$,

$$\begin{aligned} \iint_{\mathbb{T}^2\times\mathbb{T}} \varphi(z,\theta)d\hat{\mu}(z,\theta) &= \mathbb{E}\iint_{\mathbb{T}^2\times\mathbb{T}} \varphi(f(z), u_z(\theta))d\hat{\mu}(z,\theta) \\ &= \mathbb{E}\iint_{\mathbb{T}^2\times\mathbb{T}} \varphi(R(z),\theta)d\hat{\mu}(z,\theta) + O(\varepsilon_1\|\varphi\|_1) \end{aligned}$$

and so, for $\psi \in C^\infty(\mathbb{T}^2 \times \mathbb{T})$, choosing $\varphi(\cdot,\theta) = U(\psi(\cdot,\theta))$ (where U is defined as previously by $U\varphi = (I - T_0)^{-1}(\varphi - \hat{\varphi}(0))$)

$$\int_{\mathbb{T}^2} \psi(z,\theta) d\hat{\mu}(z,\theta) = \iint_{\mathbb{T}^2\times\mathbb{T}} \psi(z,\theta) dz d\nu(\theta) + O(\varepsilon_1 \|\psi\|_k) \tag{7}$$

Next, for φ in $C^\infty(\mathbb{T}^2 \times \mathbb{T})$,

$$\mathbb{E}[\varphi(f(z), u_z(\theta))] = \mathbb{E}[\varphi(R(z),\theta)] + \mathbb{E}[\vec{\nabla}_z \varphi(R(z),\theta).\zeta(z)] + \mathbb{E}[\partial_\theta \varphi(R(z),\theta)\eta_z(\theta)] + O(\varepsilon_1^2 \|\varphi\|_2)$$

hence by (7)

$$\begin{array}{ll} \iint_{\mathbb{T}^2\times\mathbb{T}} (\varphi(z) - \mathbb{E}[\varphi(R(z),\theta)]) d\hat{\mu}(z,\theta) = & \iint_{\mathbb{T}^2\times\mathbb{T}} \vec{\nabla}_z \varphi(z,\theta) \mathbb{E}[\zeta(R^{-1}(z))] dz d\nu(\theta) \\ & + \iint_{\mathbb{T}^2\times\mathbb{T}} \partial_\theta \varphi(z,\theta) \mathbb{E}[\eta_{R^{-1}(z)}(\theta)] dz d\nu(\theta) \\ & + O(\varepsilon_1^2 \|\varphi\|_k). \end{array}$$

Since $\mathbb{E}[\zeta_i \circ R^{-1}] = c_i + O(\varepsilon^2)$ by assumption and $\varepsilon \geq \varepsilon_1$, the integral $\iint_{\mathbb{T}^2\times\mathbb{T}} \vec{\nabla}_z \varphi(z,\theta) \mathbb{E}[\zeta(R^{-1}(z))] dz d\nu(\theta)$ is bounded by above by $C\varepsilon_1^2 \|\varphi\|_1$ for some C. We estimate the integral $\iint_{\mathbb{T}^2\times\mathbb{T}} \partial_\theta \varphi(z,\theta) \mathbb{E}[\eta_{R^{-1}(z)}(\theta)] dz d\nu(\theta)$ thanks to (6). We have

$$\begin{array}{l} \partial_2 \left(\mathbb{E}[\zeta_i \circ R^{-1}] \right) = \mathbb{E}[\partial_2 \zeta_i \circ R^{-1}] \\ \partial_1 \left(\mathbb{E}[\zeta_i \circ R^{-1}] \right) = \mathbb{E}[\partial_1 \zeta_i \circ R^{-1}] + \mathbb{E}[\beta' \partial_2 \zeta_i \circ R^{-1}] = \mathbb{E}[\partial_1 \zeta_i \circ R^{-1}] + O(\varepsilon_1^2) \\ \mathbb{E}[\beta'(x-\alpha)] = 0, \end{array}$$

hence $\mathbb{E}[\eta_{R^{-1}(z)}(\theta)] = O(\varepsilon_1^2)$, and the integral $\iint_{\mathbb{T}^2\times\mathbb{T}} \partial_\theta \varphi(z,\theta) \mathbb{E}[\eta_{R^{-1}(z)}(\theta)] dz d\nu(\theta)$ is bounded by above by $C\varepsilon_1^2 \|\varphi\|_1$.

Consequently,

$$\iint_{\mathbb{T}^2\times\mathbb{T}} (\varphi(z) - \mathbb{E}[\varphi(R(z),\theta)]) d\hat{\mu}(z,\theta) = O(\varepsilon_1^2 \|\varphi\|_k),$$

and choosing $\varphi(\cdot,\theta) = U(\psi(\cdot,\theta))$ for ψ in $C^\infty(\mathbb{T}^2 \times \mathbb{T})$ gives the result.

□

We do not still have any information on the measure ν. This is the aim of the next lemma:

Lemma 7. *For every $\delta > 0$, we have the following alternative:*

- *either there exists θ_0 such that*

$$|||\eta(.,\theta_0)|||_{L^2(\mathbb{T}^2)} \leq \varepsilon_1^{1+\delta}$$

- *or there exists a smooth density $g(\theta)d\theta$ satisfying $\|g\|_k \leq C\varepsilon_1^{-\tilde{\delta}}$, $\left\|\ln g\right\|_k \leq C\varepsilon_1^{-\tilde{\delta}}$ and $d_k^*(\nu, g(\theta)d\theta) \leq C\varepsilon_1^{\frac{1}{3}-\tilde{\delta}}$ for some $C = C(k,\delta)$ and $\tilde{\delta} = \tilde{\delta}(k,\delta) = o_{\delta\to 0}(1)$.*

Proof. We set

$$\begin{cases} \bar{\eta}(\theta) = \displaystyle\int_{\mathbb{T}^2} \eta_z(\theta) dz \\ b(\theta) = \dfrac{1}{2}\mathbb{E}\displaystyle\int_{\mathbb{T}} \eta_z(\theta)^2 dz = \dfrac{1}{2}|||\eta(\cdot,\theta)|||^2_{L^2(\mathbb{T}^2)} \end{cases}, \tag{8}$$

We have the estimates $|||\eta_z|||_k = O(\varepsilon_1)$, $\|\bar{\eta}\|_k = O(\varepsilon_1^2)$ and $\|b\|_k = O(\varepsilon_1^2)$. Moreover we fix $\delta > 0$, and we will assume that

$$b \geq \varepsilon_1^{2+\delta}$$

on $\mathbb{T}$.(if it does not hold, we have the conclusion $|||\eta(\cdot,\theta)|||_{L^2(\mathbb{T}^2)} \leq \sqrt{2}\varepsilon_1^{1+\frac{\delta}{2}}$). With this assumption, we want to approximate ν by a smooth density whose derivatives are controlled.

If φ is a C^∞ function on $\mathbb{T}$, then

$$\begin{aligned} \int_{\mathbb{T}} \varphi(\theta) d\nu(\theta) &= \mathbb{E}\iint_{\mathbb{T}^2\times\mathbb{T}} \varphi(u_z(\theta)) d\hat{\mu}(z,\theta) \\ &= \mathbb{E}\iint_{\mathbb{T}^2\times\mathbb{T}} \left(\varphi(\theta) + \varphi'(\theta)\eta_z(\theta) + \tfrac{1}{2}\varphi''(\theta)\eta_z(\theta)^2\right) d\hat{\mu}(z,\theta) + O(\varepsilon_1^3\|\varphi\|_2) \\ &= \int_{\mathbb{T}} \varphi(\theta) d\nu(\theta) + \int_{\mathbb{T}} \left(\bar{\eta}(\theta)\varphi'(\theta) + b(\theta)\varphi''(\theta)\right) d\nu(\theta) + O(\varepsilon_1^3\|\varphi\|_k), \end{aligned}$$

hence

$$\int_{\mathbb{T}} \left(\bar{\eta}(\theta)\varphi'(\theta) + b(\theta)\varphi''(\theta)\right) d\nu(\theta) = O(\varepsilon_1^3\|\varphi\|_k). \tag{9}$$

Thus, ν is small on the space $\{\bar{\eta}\varphi' + b\varphi'', \varphi \in C^\infty(\mathbb{T})\}$, so we will naturally try to approximate ν by a density measure $g(\theta)d\theta$ null on this space, that is such that $(\bar{\eta} - b')g + bg'$ is constant.

Sublemma 1. *Let a, b in $C^\infty(\mathbb{T}^2)$ such that $\|a\|_k, \|b\|_k \leq \varepsilon_1^2$ and $\inf b \geq \varepsilon_1^{2+\delta}$ for some reals numbers $\varepsilon, \delta > 0$ and integer k. Then there exists an unique smooth periodic function g such that $\int_{\mathbb{T}} g(t)dt = 1$ and $ag + bg'$ is constant. Moreover, g is positive, $\|g\|_k \leq C\varepsilon_1^{-\tilde{\delta}}$ and $\|\ln g\|_k \leq C\varepsilon_1^{-\tilde{\delta}}$ for some $C = C(k,\delta)$ and $\tilde{\delta} = \tilde{\delta}(k,\delta) = o_{\delta\to 0}(1)$.*

Proof. For c_0 in $\mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ is solution of the equation $ag + bg' = c_0$ if and only if there exists d_0 in $\mathbb{R}$ such that

$$g(\theta) = d_0 e^{-Q(\theta)} + c_0 R(\theta),$$

where $Q(\theta) = -\int_0^\theta \frac{a(t)}{b(t)}dt$ and $R(\theta) = \int_0^\theta \frac{e^{Q(t)-Q(\theta)}}{b(t)}dt$. By Cauchy-Lipschitz theorem, g is 1-periodic if and only if $g(1) = g(0)$ since $\theta \mapsto g(1+\theta)$ is solution of the same equation, and so if and only if $d_0(e^{-Q(1)} - 1) + c_0 R(1) = 0$. Since $R(1)$ is clearly non null, we deduce that the space of 1-periodic functions g such that $ag + bg'$ is constant has dimension 1.

If g in $C^\infty(\mathbb{T})$ is such that $ag + bg'$ is constant equal to c_0, then:
-If $c_0 = 0$, g cannot vanish by Cauchy-Lipschitz theorem, unless it is identically

null.
-If $c_0 \neq 0$, at each point θ where g vanishes we must have $b(\theta)g'(\theta) = c_0$, so $g'(\theta)$ has the same strict sign as c_0. But if θ_1 and θ_2 are two consecutive zeros of g, $g'(\theta_1)$ and $g'(\theta_2)$ cannot have same sign. Consequently, g cannot vanish on $\mathbb{R}$. Thus, unless g is identically null, g does not vanish on $\mathbb{T}$, and in particular, $\int_{\mathbb{T}} g(t)dt \neq 0$.
In any case, there exists a unique function g such that $ag + bg'$ is constant and $\int_{\mathbb{T}} g(t)dt = 1$. Moreover, since g does not vanish, it is necessarily positive.

Let us assume now that $ag+bg' = c_0$ for some constant c_0 and that $\int_{\mathbb{T}} g(t)dt = 1$. Let θ_0 be a point where g is minimal. Then $g'(\theta_0) = 0$ so $|c_0| = |a(\theta_0)g(\theta_0)| \leq \varepsilon_1^2 \inf(g)$, and so

$$\left|\frac{g'}{g}\right| \leq \left|\frac{c_0}{bg}\right| + \left|\frac{a}{b}\right| \leq \frac{2}{\varepsilon_1^\delta}.$$

For $k \geq 2$, derivating $k-1$ times the relation $ag + bg' = c_0$ gives

$$bg^{(k)} + \sum_{j=0}^{k-1} q_j g^{(j)} = 0,$$

where q_j is some linear combination of derivatives of a and b (in particular, $q_j = O(\varepsilon_1^2)$). We deduce that

$$\left|\frac{g^{(k)}}{g}\right| \leq \frac{1}{|b|} \sum_{j=0}^{k-1} |q_j| \left|\frac{g^{(j)}}{g}\right| \leq \frac{C}{\varepsilon_1^\delta} \sup_{0 \leq j \leq k-1} \left|\frac{g^{(j)}}{g}\right|$$

for some $C = C(k)$, and so by induction,

$$\left|\frac{g^{(k)}}{g}\right| \leq \frac{C}{\varepsilon_1^{k\delta}} \tag{10}$$

for some $C = C(k)$.

For every θ,

$$\left|g(\theta) - \int_{\mathbb{T}} g(t)dt\right| \leq \int_{\mathbb{T}} |g'(t)|dt \leq \frac{2}{\varepsilon_1^\delta} \int_{\mathbb{T}} g(t)dt \leq \frac{2}{\varepsilon_1^{2\delta}}$$

so $|g| \leq \frac{C}{\varepsilon_1^\delta}$, and by (10), $|g^{(k)}| \leq \frac{C}{\varepsilon_1^{(k+1)\delta}}$.

Finally, one can easily verify by induction that

$$(\ln g)^{(k)} = \sum_{0 \leq i_1 + \cdots + i_k \leq k} a_{i_1,\ldots,i_k} \left(\frac{g'}{g}\right)^{i_1} \cdots \left(\frac{g^{(k)}}{g}\right)^{i_k}$$

where the a_i's depend only on k. It follows by (10) that $\|\ln g\|_k \leq C\varepsilon_1^{-k^2\delta}$ for some $C = C(k)$.

□

We will now prove Lemma 7. Let g be given by Sublemma 1 with b defined in (8) and $a = b' - \bar{\eta}$, so that

$$ag + bg' = \varepsilon_1^2 c_0 \tag{11}$$

for some constant c_0. We will distinguish the cases where c_0 is small and large.

Let us assume that $|c_0| \geq \varepsilon_1^{1/3}$, in which case we will prove that ν is close to $g(\theta)d\theta$.
Notice that $g(\theta)d\theta$ is null on the functions of the form $\bar{\eta}\varphi' + b\varphi''$, so if ψ in $C^\infty(\mathbb{T}^2)$ satisfies $\int_{\mathbb{T}} \psi(t)g(t)dt = 0$, there exists φ such that $\varepsilon_1^2\psi = \bar{\eta}\varphi' + b\varphi''$, and we can assume $\int_{\mathbb{T}} \varphi(t)dt = 0$ up to adding a constant to φ. In order to use (9) and conclude that $\int_{\mathbb{T}} \varphi d\nu$ is small, we need to estimate $\|\varphi\|_k$. Using that $(b' - \bar{\eta})g + bg' = \varepsilon_1^2 c_0$ or equivalently $\bar{\eta}g = (bg)' - \varepsilon_1^2 c_0$, we write

$$\varepsilon_1^2\psi g = \bar{\eta}g\varphi' + bg\varphi'' = ((bg)' - \varepsilon_1^2 c_0)\varphi' + bg\varphi'' = (bg\varphi')' - \varepsilon_1^2 c_0\varphi'.$$

If $|bg\varphi'|$ is maximum at θ_0, $(bg\varphi')'(\theta_0) = 0$, hence

$$|c_0\varphi'(\theta_0)| = |\psi(\theta_0)g(\theta_0)| \leq \frac{C}{\varepsilon_1^\delta}\|\psi\|_0,$$

and $|c_0(bg\varphi')(\theta_0)| \leq \varepsilon_1^{2-\delta}\|\psi\|_0$. Thus,

$$\|c_0 bg\varphi'\|_0 \leq C\varepsilon_1^{2-\delta}\|\psi\|_0.$$

But we know that $b \geq \varepsilon_1^{2+\delta}$, we have assumed that $|c_0| \geq \varepsilon_1^{\frac{1}{3}}$, and if g is minimal at point θ_1, we have by (11)

$$|\varepsilon_1^2 c_0| = |a(\theta_1)g(\theta_1)| \leq C\varepsilon_1^2|g(\theta_1)|,$$

so that $g \geq \frac{1}{C}|c_0| \geq \frac{1}{C}\varepsilon_1^{\frac{1}{3}}$. These lower bounds leads to

$$\|\varphi'\|_0 \leq \varepsilon_1^{-\frac{2}{3}-\delta}\|\psi\|_0.$$

Using this estimate, successive derivations of the relation $\varepsilon_1^2\psi = \bar{\eta}\varphi' + b\varphi''$ gives $\|\varphi\|_k \leq C\varepsilon_1^{-\frac{2}{3}-\delta}\|\psi\|_k$ for some C, and thus hence by (9),

$$\int_{\mathbb{T}} \psi d\nu = \frac{1}{\varepsilon_1^2}\int_{\mathbb{T}} (\bar{\eta}\varphi' + b\varphi'')d\nu = O(\varepsilon_1\|\varphi\|_k) = O(\varepsilon_1^{\frac{1}{3}-\delta}\|\psi\|_k),$$

hence $d_k^*(d\nu(\theta), g(\theta)d\theta) = O(\varepsilon_1^{\frac{1}{3}-\delta})$, which proves Lemma 7 in the case $|c_0| \geq \varepsilon_1^{\frac{1}{3}}$.

Now we can assume $|c_0| \leq \varepsilon_1^{\frac{1}{3}}$. For φ in $C^\infty(\mathbb{T})$, applying (9) to $\varphi_1(\theta) = \int_0^\theta (\varphi(t) - \varphi(0))dt$ gives

$$\int_{\mathbb{T}} (\bar{\eta}\varphi + b\varphi')d\nu = \varepsilon_1^2 \tilde{c}_0 \int_{\mathbb{T}} \varphi(\theta)d\theta + O(\varepsilon_1^3 \|\varphi\|_k)$$

where $\varepsilon_1^2 \tilde{c}_0 = \int_{\mathbb{T}} \bar{\eta}(\theta)d\nu(\theta)$, and we also have

$$\int_{\mathbb{T}} (\bar{\eta}\varphi + b\varphi')g(\theta)d\theta = -\varepsilon_1^2 c_0 \int_{\mathbb{T}} \varphi(\theta)d\theta.$$

Thus, using Sublemma 1 we choose φ such that $\bar{\eta}\varphi' + b\varphi$ is constant with $\int_{\mathbb{T}} \varphi(\theta)d\theta = 1$ and $\|\varphi\|_k \leq C\varepsilon_1^{-\tilde{\delta}}$. Thus we get $\tilde{c_0} = -c_0 + O(\varepsilon_1^{1-\tilde{\delta}}) = O(\varepsilon_1^{\frac{1}{3}})$, and in consequence, for every φ in $C^\infty(\mathbb{T})$,

$$\int_{\mathbb{T}} (\bar{\eta}\varphi + b\varphi')d\nu = O(\varepsilon_1^{\frac{7}{3}} \|\varphi\|_k). \tag{12}$$

We set $A(\theta) = \int_0^\theta \frac{\bar{\eta}(t)}{b(t)}dt$, which satisfies $|A'| \leq C\varepsilon_1^{-\delta}$ for some C since $\bar{\eta} = O(\varepsilon_1^2)$ and $b \geq \varepsilon_1^{2+\delta}$.

Sublemma 2. *Let $I = [\theta_0, \theta_1]$ an interval such that A does not have local minima on $]\theta_0 + \varepsilon_1^\delta, \theta_1 - \varepsilon_1^\delta[$. Then there exists a real λ such that for every φ in $C^\infty(\mathbb{T})$ supported in I,*

$$\int_{\mathbb{T}} \varphi d\nu = \lambda \int_{\mathbb{T}} \varphi(\theta) \frac{e^{A(\theta)}}{b(\theta)} d\theta + O(\varepsilon_1^{\frac{1}{3}-\tilde{\delta}} \|\varphi\|_k)$$

.

Proof. Let ψ in $C^\infty(\mathbb{T})$ whose support is included in I, and satisfying $\int_I \psi(t)\frac{e^{A(t)}}{b(t)}dt = 0$. Then the function $\varphi(\theta) = \varepsilon_1^2 e^{-A(\theta)} \int_0^\theta \frac{\psi(t)}{b(t)} e^{A(t)}dt$ is 1-periodic, satisfies $\bar{\eta}\varphi + b\varphi' = \varepsilon_1^2\psi$, has support in I, and:
-If $\theta \in [\theta_0, \theta_0 + \varepsilon_1^\delta]$, then for t in $[\theta_0, \theta]$, $|A(t) - A(\theta)| \leq C$ and so $|\varphi(\theta)| \leq \frac{e^C}{\varepsilon_1^\delta}\|\psi\|_0$.
-If $\theta \in [\theta_1 - \varepsilon_1^\delta, \theta_1]$, then using that $\varphi(\theta) = -e^{-A(\theta)} \int_\theta^{\theta_1} \frac{\psi(t)}{b(t)} e^{A(t)}dt$, we get in the same way $|\varphi(\theta)| \leq \frac{e^C}{\varepsilon_1^\delta}\|\psi\|_0$.
-If $\theta \in [\theta_0 + \varepsilon_1^\delta, \theta_1 - \varepsilon_1^{2\delta}]$, then either A is increasing on $[\theta_0 + \varepsilon_1^\delta, \theta]$ and

$$|\varphi(\theta)| \leq e^{-A(\theta)} \left(e^{A(\theta_0+\varepsilon_1^\delta)} |\varphi(\theta_0 + \varepsilon_1^\delta)| + \varepsilon_1^2 \int_{\theta_0+\varepsilon_1^\delta}^\theta \frac{|\psi(t)|}{b(t)} e^{A(t)}dt \right) \leq \frac{e^C + 1}{\varepsilon_1^\delta}\|\psi\|_0,$$

or A is decreasing on $[\theta, \theta_1 - \varepsilon_1^{2\delta}]$ and in the same way,

$$|\varphi(\theta)| \leq e^{-A(\theta)} \left(e^{A(\theta_1-\varepsilon_1^{2\delta})} |\varphi(\theta_1 - \varepsilon_1^\delta)| + \varepsilon_1^2 \int_\theta^{\theta_1-\varepsilon_1^{2\delta}} \frac{|\psi(t)|}{b(t)} e^{A(t)}dt \right) \leq \frac{e^C + 1}{\varepsilon_1^\delta}\|\psi\|_0.$$

Thus we conclude that $\|\varphi\|_0 \leq \frac{C}{\varepsilon_1^\delta}\|\psi\|_0$. Then, derivating the equality $\bar{\eta}\varphi + b\varphi' = \psi$, we have by induction $\|\varphi\|_k \leq C\|\psi\|_k$ for some C, and so by (12) $\int_{\mathbb{T}} \psi d\nu = O(\varepsilon_1^{\frac{1}{3}-\delta}\|\psi\|_k)$. □

Now if I and J are two successive intervals where previous lemma applies, with an intersection of lenght superior to ε_1^δ, we can approximate ν by a density of the form $h_1(\theta) = c_1 e^{A(\theta)} b(\theta)^{-1}$ on I and $h_2(\theta) = c_2 e^{A(\theta)} b(\theta)^{-1}$ on J, with for exemple $c_1 \leq c_2$. Let us cut $I \cup J$ in three intervals I_1, I_2, I_3 of lenght $\geq \frac{1}{3}\varepsilon^\delta$ with $I_1 \subset I$, $I_2 \subset J$, $I_3 \subset I \cap J$. Let χ a smooth function such that $0 \leq \chi \leq 1$, $\chi = 1$ on I_1, $\chi = 0$ on I_2, and $\|\chi\|_k \leq C\varepsilon_1^{k\delta}$, and let h defined by $\ln h = \chi \ln h_1 + (1-\chi)\ln h_2$. We clearly have $\|\ln h\|_k \leq C\varepsilon_1^{-\tilde{\delta}}$ for some C,$\tilde{\delta}$. We will verify that $h(\theta)d\theta$ is an approximation of ν on $I_1 \cup I_2$.

If φ is a C^∞ function supported on $I_1 \cup I_2$, we write $\varphi = \varphi_1 + \varphi_2 + \varphi_3$ with φ_j suported on I_j, $\|\varphi_j\|_k \leq C\varepsilon_1^{-k\delta}\|\varphi\|_k$. Then for $j = 1, 2$, φ_j is supported on I_j so $\int_{\mathbb{T}} \varphi_j(\theta)h(\theta)d\theta = \int_{\mathbb{T}} \varphi_j(\theta)h_j(\theta)d\theta$ is close to $\int_{\mathbb{T}} \varphi_j(\theta)d\nu(\theta)$ (up to a remainder $O(\varepsilon_1^{\frac{1}{3}-\delta}\|\varphi_j\|_k)$, and for $j = 3$, $\int_{\mathbb{T}} \varphi_3(\theta)h_1(\theta)d\theta$ is close to $\int_{\mathbb{T}} \varphi_3(\theta)d\nu(\theta)$ since φ_3 is supported on I_1 and I_2, and using that $h_1 \leq h \leq h_2$, we get that $\int_{\mathbb{T}} \varphi_3(\theta)h(\theta)d\theta$ is also close to $\int_{\mathbb{T}} \varphi_3(\theta)d\nu(\theta)$, up to a remainder $O(\varepsilon_1^{\frac{1}{3}-\delta}\|\varphi_3\|_k)$. Consequently, $\int_{\mathbb{T}} \varphi(\theta)h(\theta)d\theta$ is close to $\int_{\mathbb{T}} \varphi(\theta)d\nu(\theta)$ up to a remainder $O(\varepsilon_1^{\frac{1}{3}-\delta}(\|\varphi_1\|_k + \|\varphi_2\|_k + \|\varphi_3\|_k)) = O(\varepsilon_1^{\frac{1}{3}-(k+1)\delta}\|\varphi\|_k)$.

We can write $\bar{\eta} = \bar{\eta}_1 + r$ where $\bar{\eta}_1$ is a trigonometrical polynomial of degree 2 and r a negligible term of order $O(\varepsilon_1^3)$, hence up to replacing $\bar{\eta}$ by its approximation $\bar{\eta}_1$, we can assume that a has at most 4 zeros on $\mathbb{T}$, and so that A has at most 2 local minima. Thus we can write $\mathbb{T}$ as a union of at most two intervals where previous sublemma applies, and "connecting" the densities obtained using our previous argument, we obtain a smooth density on $\mathbb{T}$ satisfying the claimed property, and this completes the proof of Lemma 7 □

Lemma 8. *If f satisfies the assumption (I) or (II) of Theorem 1, we have for some $C = C(k,\delta)$ and $\tilde{\delta} = \tilde{\delta}(k,\delta)$ the alternative:*

- *either there exists θ_0 in $\mathbb{T}$ such that*

$$|||\eta(.,\theta_0)|||_{L^2(\mathbb{T}^2)} \leq C\varepsilon_1^{1+\delta},$$

- *or there exists a diffeomorphism G in $Dif f_0(\mathbb{T})$ satisfying $\|G - Id\|_k \leq C\varepsilon^{-\tilde{\delta}}$ and a family $(r_z = Id + \omega(z))_{z\in\mathbb{T}^2}$ of random translations on $\mathbb{T}$ such that:*

$$|||Gu_.G^{-1} - r_.)|||_{L^2(\mathbb{T}^2)} \leq C\varepsilon_1^{\frac{7}{6}-\tilde{\delta}}.$$

Proof. We can assume that the first alternative does not hold, so that we can apply Lemma 7 to approximate ν by some smooth density g. We set $G(\theta) = \int_0^\theta g(t)dt$.

We first treat the case where $\gamma_2(f,\mu) \geq 0$. For any real numbers θ_1 and θ_2, and any z in $\mathbb{T}^2$, the aera of the parallelogram generated by the vectors $M_z(e^{i\pi\theta_1})$ and $M_z(e^{i\pi\theta_2})$ is equal to $|\det(M_z)| \cdot |\sin(\pi(\theta_1 - \theta_2))|$, and also to $|M_z(e^{i\pi\theta_1})| \cdot |M_z(e^{i\pi\theta_2})| \cdot |\sin(u_z(\theta_1) - u_z(\theta_2))|$. Letting $\theta_2 - \theta_1$ tend to 0 and using (5) leads to the equality

$$|M_z(e^{i\pi\theta})|^2.|u_z'(\theta)| = |\det(M_z)| = 1 + O(\varepsilon_1^{\frac{3}{2}}).$$

Now, $\gamma_1(f,\mu) \geq \gamma_2(f,\mu) \geq 0$ and

$$\gamma_1(f,\mu) + \gamma_2(f,\mu) = \mathbb{E}\int_{\mathbb{T}^2} \ln\det d_z f d\mu(z) = O(\varepsilon_1^2)$$

hence we have $\gamma_i(f,\mu) = O(\varepsilon_1^{\frac{3}{2}})$. This implies that

$$\mathbb{E}\iint_{\mathbb{T}^2\times\mathbb{T}} \ln|M_z(e^{i\pi\theta})| d\hat{\mu}(z,\theta) = O(\varepsilon_1^{\frac{3}{2}}),$$

and hence

$$\mathbb{E}\iint_{\mathbb{T}^2\times\mathbb{T}} \ln u_z'(\theta) d\hat{\mu}(z,\theta) = O(\varepsilon_1^{\frac{3}{2}}).$$

Thus, the Lyapunov exponent of the cocycle U associated to the stationary measure $\hat{\mu}$ is close to 0. The idea is now tu use that since $\hat{\mu}$ is close to $g(\theta)dzd\theta$, the conjugation of U by $(z,\theta) \mapsto (z, G(\theta))$ should be almost $dzd\theta$-stationary, which would allow to easily estimate the associated Lyapunov exponent. But we will not exactly proceed like this because we do not have a good control on derivatives of G^{-1}. We will rather copy the proof of the invariance of Lyapunov exponent by conjugation, which leads to the following computation: using that $\hat{\mu}$ is stationary for U, that $d_k(\hat{\mu}, g(\theta)d\theta dz) = O(\varepsilon^{\frac{1}{3}-\delta})$ and the estimate

$$\|\mathbb{E}[\ln u_\cdot' + \ln g \circ u_\cdot - \ln g]\|_{C^k(\mathbb{T}^2\times\mathbb{T})} = \|\bar{\eta}_\cdot' + (\ln g)'\bar{\eta}_\cdot\|_{C^k(\mathbb{T}^2\times\mathbb{T})} + O(\varepsilon_1^{2-\delta}) = O(\varepsilon_1^{2-\delta}),$$

we have

$$\begin{aligned}
\mathbb{E}\iint_{\mathbb{T}^2\times\mathbb{T}} \ln u_z'(\theta)d\hat{\mu}(z,\theta) &= \mathbb{E}\iint_{\mathbb{T}^2\times\mathbb{T}} \left(\ln u_z'(\theta) + \ln g(u_z(\theta)) - \ln g(\theta)\right) d\hat{\mu}(z,\theta) \\
&= \mathbb{E}\iint_{\mathbb{T}^2\times\mathbb{T}} \left(\ln u_z'(\theta) + \ln g(u_z(\theta)) - \ln g(\theta)\right) g(\theta)d\theta dz \\
&\quad + O\left(\varepsilon_1^{\frac{7}{3}-\tilde{\delta}}\right) \\
&= \mathbb{E}\iint_{\mathbb{T}^2\times\mathbb{T}} \ln\left(\frac{(G\circ u_z)'(\theta)}{G'(\theta)}\right) g(\theta)d\theta dz + O\left(\varepsilon_1^{\frac{7}{3}-\tilde{\delta}}\right) \\
&= \mathbb{E}\iint_{\mathbb{T}^2\times\mathbb{T}} \left(\frac{(G\circ u_z - G)'(\theta)}{G'(\theta)}\right) g(\theta)d\theta dz \\
&\quad -\frac{1}{2}\mathbb{E}\iint_{\mathbb{T}^2\times\mathbb{T}} \left(\frac{(G\circ u_z - G)'(\theta)}{G'(\theta)}\right)^2 g(\theta)d\theta dz + O\left(\varepsilon_1^{\frac{7}{3}-\tilde{\delta}}\right) \\
&= -\frac{1}{2}\mathbb{E}\iint_{\mathbb{T}^2\times\mathbb{T}} (G\circ u_z - G)'(\theta)^2 \frac{d\theta}{g(\theta)}dz + O\left(\varepsilon_1^{\frac{7}{3}-\tilde{\delta}}\right).
\end{aligned}$$

Thus, we have

$$\mathbb{E}\iint_{\mathbb{T}^2\times\mathbb{T}} (G\circ u_z - G)'(\theta)^2 d\theta dz \leq C\varepsilon_1^{\frac{7}{3}-\tilde{\delta}}$$

for some C. Setting $\omega(z) = G(u_z(0))$,

$$\mathbb{E}\int_{\mathbb{T}^2} \|G\circ u_z - G - \omega(z)\|_0^2 dz \leq C\varepsilon_1^{\frac{7}{3}-\tilde{\delta}}.$$

If f satisfies assertion (II), and belongs almost surely to some solvable group G, then one can easily verify that the cocycle $(z,h)\mapsto(f(z),d_zf(h))$ also almost surely belongs to a solvable group, precisely to $\hat{G} = \{(z,h)\mapsto(g(z),d_zg(h)), g\in G\cap Diff^1(\mathbb{T}^2)\}$, and then that this is also the case for U. Hence, by Proposition 1, there exists a probability $\hat{\mu}$ almost surely invariant by U, which can be approximated by a density $g(\theta)d\theta dz$. Thus, for φ in $C^\infty(\mathbb{T}^2\times\mathbb{T})$, the integral of $\varphi\circ U^{-1}-\varphi$ along $\hat{\mu}$ is null, and differs from its integral along $g(\theta)d\theta dz$ by a rest $O(\varepsilon_1^{2+\frac{1}{3}-\tilde{\delta}}(\|\varphi\circ U^{-1}\|_k + \|\varphi\|_k)) = O(\varepsilon_1^{\frac{7}{3}-\tilde{\delta}}\|\varphi\|_k)$. By a change of variable,

$$\int_{\mathbb{T}\times\mathbb{T}^2} \varphi\circ U^{-1}(z,\theta)g(\theta)d\theta dz = \int_{\mathbb{T}^2\times\mathbb{T}} \varphi(z,\theta)g(u_z(\theta))u_z'(\theta)\det(d_zf)dzd\theta$$

and by Proposition 5, $\det d_zf = 1 + O(\varepsilon^{\frac{3}{2}})$, hence

$$\int_{\mathbb{T}^2\times\mathbb{T}} \varphi(z,\theta)(G\circ u_z - G)'dzd\theta = O(\varepsilon_1^{\frac{7}{3}-\tilde{\delta}}\|\varphi\|_k)$$

Choosing $\varphi(z,\theta) = (G\circ u_z - G)'(\theta)$, we deduce as previously that

$$\int_{\mathbb{T}^2} \|G\circ u_z - G - \omega(z)\|_0^2 dz = O(\varepsilon_1^{\frac{7}{3}-\tilde{\delta}}).$$

□

Lemma 9. *Under assumptions of Proposition 7 there exists a vector p in $\mathbb{Z}^2$, a map $k : \mathbb{T}^2 \to \mathbb{R}$ and some positive constants C and δ_0 such that*

$$|||d_z f(p) - k(z)p|||_{L^2(\mathbb{T}^2)} \leq C\varepsilon^{1+\delta_0}$$

with $|p| \leq \varepsilon^{-\kappa}$, and $p = (0,1)$ if $m = |||\beta'|||_0 \geq 2\varepsilon$.

Proof. Now, we will fix the value λ in the definition of M_z and ε_1 of the begining of the section by:

$$\begin{cases} \lambda = \sqrt{\frac{\varepsilon}{m}} \text{ if } m > 2\varepsilon, \text{ in which case } \varepsilon_1 = 2\sqrt{m\varepsilon}. \\ \lambda = 1 \text{ if } m \leq 2\varepsilon, \text{ in which case } \varepsilon_1 = \varepsilon + m \leq 3\varepsilon. \end{cases}$$

We will first prove that this is necessarly the first alternative of Lemma 8 which will occur. Let $\delta > 0$ and let us assume that there exists G and ω as in Lemma 8 such that

$$Gu_{\cdot}G^{-1}(\theta) = \theta + \omega(\cdot) + O_{L^2(\mathbb{T}^2)}\left(\varepsilon_1^{\frac{7}{6}-\delta}\right),$$

or equivalently

$$g(\theta)\eta_z(\theta) = \omega(z) + O_{L^2(\mathbb{T}^2)}\left(\varepsilon_1^{\frac{7}{6}-\delta}\right). \tag{13}$$

If $\inf g \leq \varepsilon_1^{\delta}$, then choosing θ in (13) such that g is minimal gives $\|\omega(\cdot)\|_{L^2(\mathbb{T}^2)} \leq C\varepsilon_1^{1+\delta}$ for some C, and next choosing θ_0 such that $g(\theta_0) = 1$ gives $\|\eta_{\cdot}(\theta_0)\|_{L^2(\mathbb{T}^2)} \leq C\varepsilon_1^{\frac{7}{6}-\delta}$.

We now assume that $\inf g \geq \varepsilon_1^{\delta}$. If W is a complex measure on $\mathbb{T}^2$, we set $N_W = \int_{\mathbb{T}^2} N_z dW(z)$ and $\eta_W = \int_{\mathbb{T}^2} \eta_z dW(z)$. Assume that for some θ_0, $N_W(e^{i\pi\theta_0}) = 0$. Then $\eta_W(\theta_0) = 2Im(N_W(e^{i\pi\theta_0})e^{-i\pi\theta_0}) + O(\varepsilon_1^2) = O(\varepsilon_1^2)$, which implies, by the estimates (13) and the inequality $\varepsilon_1^{\delta} \leq g \leq C\varepsilon_1^{-\tilde{\delta}}$, that $\|\eta_W\|_0 = O(\varepsilon_1^{\frac{7}{6}-\tilde{\delta}})$ and so that $N_W = O(\varepsilon_1^{1+\tilde{\delta}})$. Using this remark:

If $m > 2\varepsilon$, we choose $dW = \delta_x \times dy$, so that $N_W = \begin{pmatrix} \times & 0 \\ \times & 0 \end{pmatrix}$ vanishes at the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. By the above remark, $N_W = O(\varepsilon_1^{\frac{7}{6}-\delta})$, which implies, setting $V(x) = \beta'(x) + \int_{\mathbb{T}} \partial_1\zeta_2(x,y)dy$, that $\lambda V(x) = O(\varepsilon_1^{\frac{7}{6}-\delta})$, and using that $|||V|||_0 \geq m - \varepsilon \geq \frac{m}{2}$ we get $\varepsilon_1 = \lambda m = O(\varepsilon_1^{\frac{7}{6}-\tilde{\delta}})$, which is absurd if ε_1 and $\tilde{\delta}$ are small enough.

If $m \leq 2\varepsilon$, then $N_z = J\Phi(z)$ where $\Phi = f - Id$. For $p = (p_1, p_2)$ in $\mathbb{Z}^2$, we choose $dW = e^{2i\pi p.z}dz$, so that $N_W = (p_j\hat{\Phi}_i(p))_{1\leq i,j\leq 2}$ is null at vector $\begin{pmatrix} p_2 \\ -p_1 \end{pmatrix}$, hence $N_W = O(\varepsilon_1^{\frac{7}{6}-\tilde{\delta}})$. And since this estimate is valid for any choice of p, we

deduce that $N_z = O(\varepsilon_1^{\frac{7}{6}-\tilde{\delta}})$ and so that.

Thus, if δ_0 is small enough, there necessarily exists θ_0 such that $|||\eta_\cdot(\theta_0)|||_{L^2(\mathbb{T}^2)} \leq \varepsilon_1^{1+\delta_0}$. This implies that

$$N_\cdot e^{i\pi\theta_0} = k(z)e^{i\pi\theta_0} + O_{L^2(\mathbb{T}^2)}(\varepsilon_1^{1+\delta_0})$$

where $k(z) = Re(N_z(e^{i\pi\theta_0})e^{-i\pi\theta_0})$.

If $m > 2\varepsilon$, for $W = \delta_x \times dy$, N_W and $V(x)$ defined as previously and $k_W = \int_{\mathbb{T}^2} k(z)dW(z)$, we have $N_W e^{i\pi\theta_0} = k_W e^{i\pi\theta_0} + O(\varepsilon_1^{1+\delta_0})$ with $N_W = \begin{pmatrix} 0 & 0 \\ \lambda V(x) & 0 \end{pmatrix} + O(\varepsilon_1^2)$, hence

$$\begin{cases} \lambda V(x)\cos(\pi\theta_0) = k_W \sin(\pi\theta_0) + O(\varepsilon_1^{1+\delta_0}) \\ k_W \cos(\pi\theta_0) = O(\varepsilon_1^{1+\delta_0}) \end{cases}.$$

Combining these two equalities gives $\lambda V(x)\cos^2(\pi\theta_0) = O(\varepsilon_1^{1+\delta_0})$ with $|||V|||_0 \geq \frac{m}{2}$, hence $\cos^2(\pi\theta_0) = O(\varepsilon_1^{1+\delta_0})$. Thus, $e^{i\pi\theta_0} = \pm i + O(\varepsilon_1^{\delta_0})$, hence, denoting e_2 the second canonical vector of $\mathbb{R}^2$, we successively obtain $N_z e_2 = k(z)e_2 + O(\varepsilon_1^{1+\delta_0})$, $M_z e_2 = (1 + k(z))e_2 + O(\varepsilon_1^{1+\delta_0})$ and $d_z f(e_2) = (1 + k(z))e_2 + O(\lambda\varepsilon_1^{1+\delta_0})$, with $\lambda\varepsilon_1^{1+\delta_0} = m^{\frac{\delta_0}{2}}\varepsilon^{1+\frac{\delta_0}{2}} = O(\varepsilon^{1+\frac{\delta_0}{2}})$.

If $m \leq 2\varepsilon$, we set $dW = e^{2i\pi p.z}dz$, where $p = (p_1, p_2) \in \mathbb{Z}^2$. On one hand, we have $N_W e^{i\pi\theta_0} = k_W e^{i\pi\theta_0} + O(\varepsilon^{1+\delta})$, and on another hand, N_W vanishes at vector $\begin{pmatrix} p_2 \\ -p_1 \end{pmatrix}$, hence writing N_W in the basis $\begin{pmatrix} p_2 \\ -p_1 \end{pmatrix}, \begin{pmatrix} \cos(\pi\theta_0) \\ \sin(\pi\theta_0) \end{pmatrix}$ gives the relation

$$\mathrm{Tr}(N_W) = k_W + O(\varepsilon^{1+\delta_0}).$$

But $\mathrm{Tr}(N_z) = \det(M_z) - 1 + O(\varepsilon^2) = O(\varepsilon^{1+\delta_0})$ so $k_W = O(\varepsilon^{1+\delta_0})$ and so $N_W(e^{i\pi\theta_0}) = O(\varepsilon^{1+\delta_0})$, that is

$$(p_1\cos(\pi\theta_0) + p_2\sin(\pi\theta_0))\hat{\zeta}_i(p_1, p_2) = O(\varepsilon^{1+\delta_0})$$

for $i = 1, 2$. Now we fix $\kappa_1 < \frac{\delta}{6}$, and we distinguish two cases:

1) If $|\hat{\zeta}_i(p)| \leq \varepsilon^{1+\frac{\delta_0}{2}}$ $(i = 1, 2)$ for every $p \in \mathbb{Z}^2 - \{(0, 0)\}$ such that $|p| \leq \varepsilon^{-\kappa}$, then, for k large enough,

$$|||\zeta_i - \hat{\zeta}_i(0)|||_0 \leq \sum_{p\in\mathbb{Z}^2-\{(0,0)\}} |\hat{\zeta}_i(p)| \leq \sum_{|p|\leq\varepsilon^{-\kappa}} \varepsilon^{1+\frac{\delta_0}{2}} + \sum_{|p|\geq\varepsilon^{-\kappa}} \frac{|||\zeta_i|||_k}{|p|^k} = O(\varepsilon^{1+\frac{\delta_0}{3}}).$$

If $|\hat{\zeta}_i(p)| \geq \varepsilon^{1+\frac{\delta_0}{2}}$ for some i in $\{1, 2\}$ and some $p = (p_1, p_2)$ in $\mathbb{Z}^2 - \{(0, 0)\}$ such that $|p| \leq \varepsilon^{-\kappa}$, then $|p_1\cos(\pi\theta_0) + p_2\sin(\pi\theta_0)| \leq \varepsilon^{\frac{\delta_0}{2}}$. Thus, $e^{i\pi\theta_0} = \pm\frac{\tilde{p}}{|\tilde{p}|} + O(\varepsilon^{\frac{\delta_0}{2}})$, where $\tilde{p} = (-p_2, p_1)$. Consequently, $N_z(\tilde{p}) = O(\varepsilon^{1+\frac{\delta_0}{3}})$.

□

We can complete the proof of Proposition 7. Let $p = (p_1, p_2)$ and $k(z)$ given by Lemma 9. Up to dividing p_1 and p_2 by some common divisor, we can assume $\gcd(p_1, p_2) = 1$. According to Bezout Lemma, there exists P in $Sl_2(\mathbb{Z})$ such that $Pe_2 = \tilde{p}$ and $\|P\| \leq |p| \leq \varepsilon^{-\delta_1}$,and we can choose $P = Id$ if $p = (0, 1)$, hence if $m \geq 2\varepsilon$. The diffeomorphism $\tilde{f} = P^{-1}fP = (\tilde{f_1}, \tilde{f_2})$ satisfies

$$d_z\tilde{f}(e_2) = e_2 + P^{-1}N_zPe_2 = e_2 + O(\|P^{-1}\|\varepsilon^{1+\delta_0}) = e_2 + O(\varepsilon^{1+\frac{\delta_0}{2}}),$$

that is

$$\partial_2\tilde{f_1} = O(\varepsilon^{1+\frac{\delta_0}{2}}). \tag{14}$$

Thus, $\tilde{f_1}$ depends only on x, up to a remainder $O(\varepsilon^{1+\delta_0})$. Next, we have by (5)

$$\det(d_z\tilde{f}) = \det(d_zf) = 1 + O(\varepsilon^{\frac{3}{2}}),$$

which leads to

$$\partial_1\tilde{f_1} + \partial_2\tilde{f_2} = 2 + O(\varepsilon^{\frac{3}{2}}).$$

Using that $\int_{\mathbb{T}} \partial_2\tilde{f_2}(x, y)dy = 1$ and that $\int_{\mathbb{T}} \partial_1\tilde{f_1}(x, y)dy = \partial_1\tilde{f_1}(x, \cdot) + O(\varepsilon^{1+\frac{\delta_0}{2}})$, we obtain

$$\begin{cases} \partial_1\tilde{f_1} = 1 + O(\varepsilon^{1+\frac{\delta_0}{2}}) \\ \partial_2\tilde{f_2} = 1 + O(\varepsilon^{1+\frac{\delta_0}{2}}). \end{cases} \tag{15}$$

Estimates (14) and (15) imply that $\tilde{f} = \tilde{R} + O(\varepsilon^{1+\frac{\delta_0}{2}})$ for some Anzai cocycle $\tilde{R}$. □

6 KAM iteration

Let $f = R + \zeta$ a random diffeomorphism of $\mathbb{T}^2$ having a finite number of realizations, where $R : (x, y) \mapsto (x + \alpha, y + \beta(x))$ is a diophantine random Anzai cocycle satisfying $\mathbb{E}[\beta(x - \alpha)] = 0$. We will denote $\varepsilon_{k_1,k_2} = |||\zeta|||_{k_1,k_2}$ and $\Delta = |||R \circ \tilde{R} - \tilde{R} \circ R|||$. As a corollary of the two previous sections (more precisely of Proposition 5 and 7), we have the following conjugation result:

Lemma 10. *There exists a and k_0 in $\mathbb{N}$ and $\delta_0, \varepsilon > 0$ such that if f satisfies assumption (I) or (II) of Theorem 1, and if $\varepsilon_{k_0,k_0} \leq \varepsilon$ then:*

i)There exists a diffeomorphism $G = P + g$ where P is the algebraic part of G, and an Anzai cocycle R_1, such that

$$|||GfG^{-1} - R_1|||_0 \leq C\varepsilon_{k_0,k_0}^{1+\delta_0}$$

and for every k_1, k_2

$$\|g\|_{k_1,k_2} \leq \frac{C}{\Delta^{16k_1}}\varepsilon_{k_1+a,k_2+ak_1+a}.$$

for some $C = C(k_1, k_2)$.

ii)For every integer p, there exists $G = P + g$ where P is an algebraic automorphism, and an Anzai cocycle R_1, such that

$$\|GfG^{-1} - R_1\|_0 \leq C(\varepsilon_{k_0,k_0} + \sqrt{\Delta^p \varepsilon_{k_0,k_0}})^{1+\delta_0}$$

and for every $\delta > 0$ and every integers k_1, k_2 large enough,

$$\|g\|_{k_1,k_2} \leq C\varepsilon_{(1+\delta)k_1,(1+\delta)k_2}.$$

Moreover, $\|P\| \leq C\varepsilon_{k_0,k_0}^{-\kappa}$ and $P = Id$ if $\|\|\beta'\|\|_0 \geq 2\|\zeta\|_{k_0}$.

In order to iterate this conjugation procedure, we will need some estimates on large derivatives of the conjugated diffeomorphism. In this view, we will replace the conjugation G by some approximation by a trigonometric polynomial.

If φ belongs to $C^\infty(\mathbb{T}^2)$ and λ_1, λ_2 are real numbers, we denote

$$P_{\lambda_1,\lambda_2}\varphi(x,y) = \sum_{|p_1|\leq\lambda_1,|p_2|\leq\lambda_2} \hat{\varphi}(p)e^{2i\pi(p_1x+p_2y)}.$$

The operator P_{λ_1,λ_2} satisfies the estimates:

Lemma 11. *For every $\delta > 0$ and every integers k_1 ,k_2, and $\bar{k}_1$, $\bar{k}_2$ large enough, there exists a constant C such that for every φ in $C^\infty(\mathbb{T}^2)$:*

$$\begin{cases} \|P_{\lambda_1,\lambda_2}\varphi\|_{\bar{k}_1,\bar{k}_2} \leq C(\lambda_1^{\bar{k}_1-k_1} + \lambda_2^{\bar{k}_2-k_2})\|\varphi\|_{k_1,k_2} \\ \|\varphi - P_{\lambda_1,\lambda_2}\varphi\|_{k_1,k_2} \leq C\left(\dfrac{1}{\lambda_1^{(\bar{k_1}-k_1)-\delta\bar{k}_1}} + \dfrac{1}{\lambda_2^{(\bar{k_2}-k_2)-\delta\bar{k}_2}}\right)\|\varphi\|_{\bar{k_1},\bar{k_2}} \end{cases}$$

Proof. If $\frac{i_1}{\bar{k}_1} + \frac{i_2}{\bar{k}_2} \leq 1$, then

$$\begin{aligned} \|\partial_1^{i_1}\partial_2^{i_2}P_{\lambda_1,\lambda_2}\varphi\|^2_{L^2(\mathbb{T}^2)} &\leq C \sum_{|p_1|\leq\lambda_1,|p_2|\leq\lambda_2} |p_1|^{2i_1}|p_2|^{2i_2}|\hat{\varphi}(p_1,p_2)|^2 \\ &\leq C \sum_{|p_1|\leq\lambda_1,|p_2|\leq\lambda_2} (|p_1|^{2\bar{k}_1} + |p_2|^{2\bar{k}_2})\hat{\varphi}(p_1,p_2)|^2 \\ &\leq C(\lambda_1^{2(\bar{k}_1-k_1)}\|\partial_1^{k_1}\varphi\|^2_{L^2(\mathbb{T}^2)} + \lambda_2^{2(\bar{k_2}-k_2)}\|\partial_2^{k_2}\varphi\|^2_{L^2(\mathbb{T}^2)}) \\ &\leq C(\lambda_1^{2(\bar{k}_1-k_1)} + \lambda_2^{2(\bar{k}_2-k_2)})\|\varphi\|^2_{k_1,k_2} \end{aligned}$$

and if $\frac{i_1}{k_1} + \frac{i_2}{k_2} \leq 1$,

$$\begin{aligned}
\|\partial_1^{i_1}\partial_2^{i_2}(\varphi - P_{\lambda_1,\lambda_2}\varphi)\|_{L^2(\mathbb{T}^2)} &\leq C\left(\sum_{|p_1|>\lambda_1, p_2\in\mathbb{Z}} + \sum_{p_1\in\mathbb{Z}, |p_2|>\lambda_2}\right)|p_1|^{i_1}|p_2|^{i_2}|\hat{\varphi}(p_1,p_2)| \\
&\leq C\left(\sum_{|p_1|>\lambda_1, p_2\in\mathbb{Z}} + \sum_{p_1\in\mathbb{Z}, |p_2|>\lambda_2}\right)\left(\frac{|p_1|^{k_1}+|p_2|^{k_2}}{|p_1|^{\bar{k}_1}+|p_2|^{\bar{k}_2}}\right)\|\varphi\|_{\bar{k}_1,\bar{k}_2} \\
&\leq C\left(\sum_{|p_1|>\lambda_1, p_2\in\mathbb{Z}} \frac{|p_1|^{k_1}}{|p_1|^{(1-\delta)\bar{k}_1}|p_2|^{\delta\bar{k}_2}} + \sum_{p_1\in\mathbb{Z}, |p_2|>\lambda_1} \frac{|p_2|^{k_2}}{|p_1|^{\delta\bar{k}_1}|p_2|^{(1-\delta)\bar{k}_2}}\right)\|\varphi\|_{\bar{k}_1,\bar{k}_2} \\
&\leq C\left(\frac{1}{\lambda_1^{(\bar{k}_1-k_1)-\delta\bar{k}_1}} + \frac{1}{\lambda_2^{(\bar{k}_2-k_2)-\delta\bar{k}_2}}\right)\|\varphi\|_{\bar{k}_1,\bar{k}_2}
\end{aligned}$$

□

Lemma 12. *Let δ a positive real number. We set $\varepsilon_k = \varepsilon_{\delta k,k}$.*

There exists a and k_0 in $\mathbb{N}$ and $\delta_0, \varepsilon > 0$ such that if f satisfies assumption (I) or (II) of Theorem 1, and if $\varepsilon_{k_0} \leq \varepsilon$ then:

i) For every $\lambda > 1$, there exist a diffeomorphism $G_0 = P + g_0$, an Anzai cocycle $R_1 : (x,y) \mapsto (x+\alpha_1, y+\beta_1(x))$ and a family of constants $(C_k)_{k\in\mathbb{N}}$ depending on δ and λ such that, denoting $f_1 = G_0 f G_0^{-1} = R_1 + \zeta_1$ and $\varepsilon_k^{(1)} = |||\zeta_1|||_{\delta k,k}$, we have for every integer $k, \bar{k}$:

$$\varepsilon_k^{(1)} \leq \frac{C_{\bar{k}}}{\Delta^{16\bar{k}}}\left(\lambda^{k+a}\varepsilon_{k_0}^{1+\delta_0} + \frac{\varepsilon_{\bar{k}}}{\lambda^{(\bar{k}-k)-\delta\bar{k}-a}}\right)..$$

ii)For every integer p and every $\lambda > 1$, there exists a diffeomorphism $G_0 = P + g_0$, an Anzai cocycle $R_1 : (x,y) \mapsto (x+\alpha_1, y+\beta_1(x))$ and a family of constants $(C_k)_{k\in\mathbb{N}}$ depending on δ, λ and p such that, denoting $f_1 = G_0 f G_0^{-1} = R_1 + \zeta_1$ and $\varepsilon_k^{(1)} = |||\zeta_1|||_{\delta k,k}$, we have

$$\varepsilon_k^{(1)} \leq C_k\left(\lambda^{k+a}(\varepsilon_{k_0} + \sqrt{\Delta^p\varepsilon_{k_0}})^{1+\delta_0} + \frac{\varepsilon_{\bar{k}}}{\lambda^{(\bar{k}-k)-\delta\bar{k}-a}}\right)$$

Moreover in the two cases, we have $\mathbb{E}[\beta_1'(x-\alpha_1)] = 0$, $\|P\| \leq C\varepsilon_{k_0}^{-\kappa}$ and $P = Id$ if $|||\beta'|||_0 \geq 2\varepsilon_{k_0}$ or $|||\tilde{\beta}'|||_0 \leq 2\varepsilon_{k_0}^{(1)}$.

Proof. Let $G = P + g$ and R_1 given by point i) of Lemma 10, $G_0 = P + g_{\lambda_1,\lambda_2}$ and $f_\lambda = G_0 f G_0^{-1} = R_1 + \zeta_1$ where $g_{\lambda_1,\lambda_2} = P_{\lambda_1,\lambda_2} g$.

We know that

$$\|g\|_{k_1-a,k_2-ak_1} \leq \frac{C}{\Delta^{16k_1}}\varepsilon_{k_1,k_2},$$

hence

$$\|g_{\lambda_1,\lambda_2}\|_{k_1,k_2} \leq \frac{C}{\Delta^{16k_1}}(\lambda_1^a + \lambda_2^{ak_1})\varepsilon_{k_1,k_2}$$

and

$$\|g - g_{\lambda_1,\lambda_2}\|_{k_1,k_2} \leq \frac{C}{\Delta^{16\bar{k}_1}}\left(\frac{1}{\lambda_1^{(\bar{k}_1-k_1)-\delta\bar{k}_1}} + \frac{1}{\lambda_2^{(\bar{k}_2-k_2)-\delta\bar{k}_2}}\right)\varepsilon_{\bar{k}_1,\bar{k}_2}.$$

Next, we have for some constant C the following estimates:

$$\begin{aligned}
|||Gf G^{-1} - R_1|||_0 &\leq C\varepsilon_{k_0,k_0}^{1+\delta_0} \\
|||G_0 f G_0^{-1} - Gf G^{-1}|||_0 &\leq |||G_0 f G_0^{-1} - Gf G_0^{-1}|||_0 + \|Gf G_0^{-1} - Gf G^{-1}\|_0 \\
&\leq \|G_0 - G\|_0 + \|Gf - Gf G^{-1}G_0\|_0 \\
&\leq (1 + |||(Gf G^{-1})'|||_0)\|G - G_0\|_0 \\
&\leq C\left(\frac{1}{\lambda_1^{\bar{k}_1-\delta\bar{k}_1}} + \frac{1}{\lambda_2^{\bar{k}_2-\delta\bar{k}_2}}\right)\varepsilon_{\bar{k}_1,\bar{k}_2}
\end{aligned}$$

Combining the two last inequalities, we obtain:

$$\varepsilon_{0,0}^{(1)} \leq |||G_0 f G_0^{-1} - R_1|||_0 \leq C\left[\varepsilon_{k_0,k_0}^{1+\delta_0} + \left(\frac{1}{\lambda_1^{\bar{k}_1-\delta\bar{k}_1}} + \frac{1}{\lambda_2^{\bar{k}_2-\delta\bar{k}_2}}\right)\varepsilon_{\bar{k}_1,\bar{k}_2}\right].$$

Moreover, we also have

$$\varepsilon_{\bar{k}_1,\bar{k}_2}^{(1)} \leq C(\varepsilon_{\bar{k}_1,\bar{k}_2} + \|g\|_{\bar{k}_1,\bar{k}_2}) \leq \frac{C}{\Delta^{16\bar{k}_1}}(\lambda_1^a + \lambda_2^{a\bar{k}_1})\varepsilon_{\bar{k}_1,\bar{k}_2}$$

and consequently, using the decomposision $\zeta = P_{\lambda_1,\lambda_2}\zeta + (\zeta - P_{\lambda_1,\lambda_2}\zeta)$, we deduce

$$\begin{aligned}
\varepsilon_{k_1,k_2}^{(1)} &\leq C\left[(\lambda_1^{k_1+a} + \lambda_2^{k_2+a})\varepsilon_0^{(1)} + \left(\frac{1}{\lambda_1^{(\bar{k}_1-k_1)-\delta\bar{k}_1}} + \frac{1}{\lambda_2^{(\bar{k}_2-k_2)-\delta\bar{k}_2}}\right)\varepsilon_{\bar{k}_1,\bar{k}_2}^{(1)}\right] \\
&\leq \frac{C}{\Delta^{16\bar{k}_1}}\left[(\lambda_1^{k_1+a} + \lambda_2^{k_2+a})\varepsilon_{k_0,k_0}^{1+\delta_0} + (\lambda_1^{k_1+a} + \lambda_2^{k_2+a})\left(\frac{1}{\lambda_1^{\bar{k}_1-\delta\bar{k}_1}} + \frac{1}{\lambda_2^{\bar{k}_2-\delta\bar{k}_2}}\right)\varepsilon_{\bar{k}_1,\bar{k}_2}\right. \\
&\quad \left. + (\lambda_1^a + \lambda_2^{ak_1})\left(\frac{1}{\lambda_1^{(\bar{k}_1-k_1)-\delta\bar{k}_1}} + \frac{1}{\lambda_2^{(\bar{k}_2-k_2)-\delta\bar{k}_2}}\right)\varepsilon_{\bar{k}_1,\bar{k}_2}\right]
\end{aligned}$$

Now we set $\lambda_1 = \lambda$, $\lambda_2 = \lambda^\delta$, $k_1 = k$, $k_2 = \delta^{-1}k$, $\bar{k}_1 = \bar{k}$, $\bar{k}_2 = \delta^{-1}\bar{k}$. The previous inequality becomes

$$\varepsilon_k^{(1)} \leq \frac{C}{\Delta^{16k}}\left(\lambda^{k+a}\varepsilon_{k_0}^{1+\delta_0} + \frac{1}{\lambda^{(\bar{k}-k)-(a+1)\delta\bar{k}-a}}\varepsilon_{\bar{k}}\right).$$

The proof of ii) is analog.

Finally, we can assume that $\mathbb{E}[\beta_1'(x-\alpha)] = 0$ up to conjugate by some diffeomorphism $U : (x, y) \mapsto (x, y + u(x))$, and if $|||\beta_1'|||_0 \leq 2\varepsilon_{k_0}$, then f_1 is close to a translation and hence $P^{-1}fP$ is stillclose to a translation so that we can assume $P = Id$.

□

The convergence of the procedure will be assured by the following combinatorial Lemma

Lemma 13. *Let a and k_0 in $\mathbb{N}$ and $\delta_0 > 0$. There exists k_1 and s in $\mathbb{N}$ and $\delta > 0$ such that, setting $Q = 1 + \frac{\delta_0}{2}$ and $\lambda_n = e^{Q^n}$, if $(\varepsilon_k^{(n)})_{k,n\in\mathbb{N}}$ and $(C_k)_{k\in\mathbb{N}}$ are families of positive real numbers satisfying for every integers n, k, $\bar{k}$ with $\bar{k} \geq k \geq k_0$*

$$\varepsilon_k^{(n+1)} \leq C_{\bar{k}} \left(\lambda_n^{k+a} (\varepsilon_{k_0}^{(n)})^{1+\delta_0} + \frac{\varepsilon_{\bar{k}}^{(n)}}{\lambda_n^{(\bar{k}-k)-\delta\bar{k}-a}} \right) \tag{16}$$

and $\varepsilon_{k_1}^{(0)} \leq C_{k_1}^{-s}$, then for every k,p in $\mathbb{N}$, $\varepsilon_k^{(n)} = O(\lambda_n^{-p})$ as n goes to $+\infty$.

Proof. Up to replacing ε_k^n by $\varepsilon_{k+k_0}^{(n)}$, we can assume $k_0 = 0$. We fix $\delta > 0$ small enough. Let $\bar{k}$ and p be large enough and satisying the following conditions:

$$\begin{cases} p(1+\delta_0) > \bar{k}(1+\delta) + 2a \\ (1 - \delta\dfrac{Q}{Q-1})\bar{k} - a\dfrac{Q}{Q-1} \geq (1+\delta)p \\ p(1+\delta_0) - a \geq (1+\delta)p \end{cases} \tag{17}$$

(when $\delta = 0$, these inequalities become $p(1+\delta_0) > \bar{k} + \text{constant}$, $\bar{k} \geq p + \text{constant}$ and $p \geq$ constant, hence are compatible, and still are when δ is small enough)
We fix an integer N and we set $M = \sup_{n\leq N} \varepsilon_0^{(n)} \lambda_n^p$.
Using inequality (16) with $k = \bar{k}$ we have

$$\varepsilon_{\bar{k}}^{(n+1)} \leq C_{\bar{k}} \left(\frac{M^{1+\delta_0}}{\lambda_n^{p(1+\delta_0)-\bar{k}-a}} + \frac{\varepsilon_{\bar{k}}^{(n)}}{\lambda_n^{-\delta\bar{k}-a}} \right) \leq C_{\bar{k}} (M^{1+\delta_0} + \varepsilon_{\bar{k}}^{(n)}) \lambda_n^{\delta\bar{k}+a},$$

and we deduce

$$M^{1+\delta_0} + \varepsilon_k^{(n)} \leq 2C_{\bar{k}} \lambda_{n-1}^{\delta\bar{k}+a} (M^{1+\delta_0} + \varepsilon_k^{(n-1)}) \leq \cdots \leq (2C_{\bar{k}})^n (M^{1+\delta_0} + \varepsilon_{\bar{k}}^{(0)}) (\lambda_0 \cdots \lambda_{n-1})^{\delta\bar{k}+a} \leq (2C_{\bar{k}})^n \lambda_n^{\frac{\delta\bar{k}+a}{Q-1}}$$

and hence

$$\varepsilon_{\bar{k}}^{(n)} \leq C_{\bar{k}}^n (M^{1+\delta_0} + \varepsilon_{\bar{k}}^{(0)}) \lambda_n^{\frac{\delta\bar{k}+a}{Q-1}} \tag{18}$$

(up to replacing $C_{\bar{k}}$ by $3C_{\bar{k}}$). Next, inequality (16) with $k = 0$ gives

$$\begin{aligned} \varepsilon_0^{(n+1)} &\leq C_{\bar{k}} \left(\lambda_n^a (\varepsilon_0^{(n)})^{1+\delta_0} + \frac{\varepsilon_{\bar{k}}^{(n)}}{\lambda_n^{(1-\delta)\bar{k}-a}} \right) \\ &\leq C_{\bar{k}} \left(\frac{M^{1+\delta_0}}{\lambda_n^{p(1+\delta_0)-a}} + \frac{C_k^n (M^{1+\delta_0} + \varepsilon_{\bar{k}}^{(0)})}{\lambda_n^{(1-\delta\frac{Q}{Q-1})\bar{k} - a\frac{Q}{Q-1}}} \right) \\ &\leq \frac{\widetilde{M}}{\lambda_{n+1}^{(1+\delta)p}} \end{aligned}$$

where $\widetilde{M} = C_{\bar{k}}^{n+1}(2M^{1+\delta_0} + \varepsilon_{\bar{k}}^{(0)})$. A first consequence is that the sequence $M_n = \sup_{k\leq n} \varepsilon_0^{(k)}\lambda_k^p$ satisfies the inequality

$$M_{n+1} \leq C_{\bar{k}}^{n+1}(M_n^{1+\delta_0} + \varepsilon_{\bar{k}}^{(0)}),$$

which implies that (M_n) is bounded if

$$\varepsilon_{\bar{k}}^{(0)} \leq C_{\bar{k}}^{-\frac{1}{\delta_0}}.$$

A second consequence is that once we have proved that $\varepsilon_0^{(n)} = O(\lambda_n^{-p})$ for some p, we have $\varepsilon_0^{(n)} = O(\lambda_n^{-(1+\delta)p})$, and by induction we deduce that $\varepsilon_0^{(n)} = O(\lambda_n^{-p})$ for every $p > 0$.

Finally, we deduce by (18) that for every $\bar{k}$,

$$\varepsilon_{\bar{k}}^{(n)} = O(\lambda_n^{\frac{\delta\bar{k}}{Q-1}+a})$$

and next by inequality (16) that for every $p, k, \bar{k}$,

$$\varepsilon_k^{(n+1)} = O\left(\frac{1}{\lambda_n^{p(1+\delta_0)-k-a}} + \frac{1}{\lambda_n^{(1-\delta\frac{Q}{Q-1})\bar{k}-k-a\frac{Q}{Q-1}}}\right)$$

which gives the result choosing $\bar{k}$ and p large enough. □

By successive uses of the Lemma 12, we will now define a sequence of random diffeomorphisms f_n conjugated to f and converging to a random Anzai cocycle, defined by $f_0 = f$ and the induction relation $f_{n+1} = G_n f_n G_n^{-1}$ where G_n is constructed using point i) or ii) of Lemma 12.

Thus, we assume now that $f = R + \zeta$ satisfies assumptions of theorem 1. If $(\alpha_0, \hat{\beta_0})$ is a (A, σ)-diophantine rotation vector of f, we can assume that $\int_{\mathbb{T}^2}(R - Id)dz = (\alpha_0, \hat{\beta_0})$. Indeed, we have $\int_{\mathbb{T}^2}(R - Id)d\mu = (\alpha_0, \hat{\beta_0})$ for some stationary measure μ for f. The first coordinate of $R - Id$ is constant, hence close to α_0. In consequence μ is small on maps on the form $\mathbb{E}[\varphi(x + \alpha_0)] - \varphi(x)$, and using that α_0 is diophantine, we deduce that μ is close to Lebesgue on maps depending only on x, and so $\int_{\mathbb{T}^2}(R - Id)d\mu$ is close to $\int_{\mathbb{T}^2}(R - Id)dz$.

Now we fix $\delta > 0$, we set $\lambda_n = e^{(1+\delta_0/2)^n}$ as in Lemma 13, we define $f_0 = f$, $R_0 = R$, $\zeta_0 = \zeta$, and once we have constructed $f_n = R_n + \zeta_n$, if Lemma 12 applies we define $f_{n+1} = G_n f_n G_n^{-1} = R_{n+1} + \zeta_{n+1}$ where $G_n = P_n + g_n$ is a conjugation constructed using point i) or ii) of Lemma 12 with $\lambda = \lambda_n$ (contruction ii) will be used with some fixed integer p we will chose later). We set $H_n = G_{n-1} \circ \cdots \circ G_0$, $\varepsilon_k^{(n)} = |||\zeta_n|||_{\delta k,k}$ and $\Delta_n = |||R_n \circ \tilde{R}_n - \tilde{R}_n \circ R_n|||_{L^2(\mathbb{T}^2)}$, where $\tilde{R}_n$ is an independant copy of R_n. More precisely, we will use construction ii) of Lemma 12 as long as

Δ_n is small behind ζ_n, and construction ii) once Δ_n is large behind ζ_n.

Using that $(\alpha_0, \hat{\beta}_0)$ is a diophantine random rotation vector of f, we can assume that R_n is diophantine as long as f_n is defined: indeed, by Proposition 4, the stationary measure $\mu_n = (H_n)_*\mu$ for f_n is close to the Lebesgue measure (up to a remainder $O\left(\left(\varepsilon_{k_0}^{(n-1)}\right)^2\right)$), hence

$$(\alpha_0, \hat{\beta}_0) = \int_{\mathbb{T}^2} (H_n \circ f_n - H_n) d\mu_n \approx \int_{\mathbb{T}^2} (H_n \circ R_n - H_n) dz = P(\alpha_n, \hat{\beta}_n)$$

and hence we can assume R_n is diophantine up to adding it a constant.

There will exist n_0 such that the diffeomorphism G_n will be defined using construction ii) for $n < n_0$ (with some fixed p we will chose later) and construction i) for $n \geq n_0$.

As long as f_n is defined, $\varepsilon_k^{(n)}$ satisfies for $n \leq n_0$

$$\varepsilon_k^{(n+1)} \leq C_{\bar{k}} \left(\lambda_n^{k+a} \left(\varepsilon_{k_0}^{(n)} + \sqrt{\Delta_n^p \varepsilon_{k_0}^{(n)}} \right)^{1+\delta_0} + \frac{\varepsilon_{\bar{k}}^{(n)}}{\lambda_n^{(\bar{k}-k)-ap}} \right)$$

and for $n \geq n_0$

$$\varepsilon_k^{(n+1)} \leq \frac{C_{\bar{k}}}{\Delta_n^{16\bar{k}}} \left(\lambda_n^{k+a} (\varepsilon_{k_0}^{(n)})^{1+\delta_0} + \frac{\varepsilon_{\bar{k}}^{(n)}}{\lambda_n^{(\bar{k}-k)-\delta\bar{k}-a}} \right).$$

If we control derivatives of H_n, then the numbers Δ_n are comparable. If we assume that $\Delta_n \leq 2\Delta_m$ for every n, m, then for $n \geq n_0$ we have

$$\varepsilon_k^{(n+1)} \leq \frac{C_{\bar{k}}}{\Delta_{n_0}^{16\bar{k}}} \left(\lambda_n^{k+a} (\varepsilon_{k_0}^{(n)})^{1+\delta_0} + \frac{\varepsilon_{\bar{k}}^{(n)}}{\lambda_n^{(\bar{k}-k)-\delta\bar{k}-a}} \right),$$

and so by Lemma 13, there exists integers s and k_1 such that if $\varepsilon_{k_1}^{(m)} \leq \left(\frac{\Delta_m}{C_{k_1}}\right)^s$, then for every $n \geq m$ and every $k, q > 0$

$$\varepsilon_k^{(n)} \leq \frac{C}{\lambda_n^q}$$

for some constant $C = C(k, q)$.

Thus, we use the following scheme: As long as $\Delta_n^s \leq C_{k_1}^s \varepsilon_{k_1}^{(n)}$ we use construction ii) with $p = s$ to define f_{n+1}. If $\Delta_{n_0}^s \geq C_{k_1}^s \varepsilon_{k_1}^{(n_0)}$ for some n_0, then we use construction i) to define f_k for $k > n$.

Hence, for $n < n_0$ we have

$$\varepsilon_k^{(n+1)} \leq C_k C_{k_1} \left(\lambda_n^{k+a} (\varepsilon_{k_1}^{(n)})^{1+\delta_0} + \frac{\varepsilon_{\bar{k}}^{(n)}}{\lambda_n^{(\bar{k}-k)-\delta\bar{k}-a}} \right)$$

Thus, there exists k_2 in $\mathbb{N}$ such that if $\varepsilon_{k_2}^{(0)}$ is small enough then we have the estimate $\varepsilon_k^{(n)} = O(\lambda_n^{-q})$ for any k, q as long as f_n is defined and $\|H_n'\|_0$ is controlled. Let us prove that this estimate of $\varepsilon_k^{(n)}$ implies an estimate on H_n.

Noticing that there is at most one integer n_0 such that $P_{n_0} = P \neq Id$, we have $H_n = P_n + h_n$ with

$$h_n = \sum_{j=0}^{n-1} P_j (g_j \circ H_{j-1}).$$

For every l, we have for some C and $\bar{k}$

$$\begin{aligned} \|h_n\|_k &\leq \sum_{j=0}^{n-1} \|P_j (g_j \circ H_j)\|_k \\ &\leq C \sum_{j=0}^{n-1} \|P_j\| \|g_j\|_k (1 + \|h_j\|_k) \\ &\leq C \varepsilon_{\bar{k}}^{(0)} \sup_{j<n} (1 + \|h_j\|_k). \end{aligned}$$

Using an induction we deduce that $\|h_n\|_k = O(\varepsilon_{\bar{k}}^{(0)})$, and next that

$$\sum_{j=0}^{n-1} \|P_j (g_j \circ H_m)\|_l = O(\varepsilon_{\bar{k}}^{(0)}).$$

In consequence, we deduce by induction that f_n, G_n and H_n are defined for every n, and that h_n normally converges in $C^l(\mathbb{T}^2)$ to some limit h satisfying $\|h\|_k = O(\varepsilon_{\bar{k}}^{(0)})$, and $H = P + h$ is thus C^∞, close to P and invertible if f is close enough to R, and HfH^{-1} is almost surely a limit of Anzai cocycles, hence is almost surely an Anzai cocycle.

References

[1] D. Dolgopyat and R. Krikorian. On simultaneous linearization of diffeomorphisms of the sphere. *Duke Mathematical Journal*, 136(3):475–506, 2007.

[2] J. Moser. On commuting circle mappings and simultaneous Diophantine approximations. *Mathematische Zeitschrift*, 205(1):105–121, 1990.