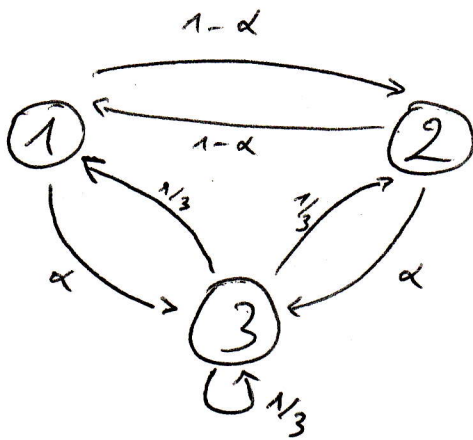


Exercice

a)



All the states are linked and the set of states is finite. Thus, the chain is irreducible, positive and recurrent.

b)

$$P = (\alpha - 1)I = \begin{pmatrix} 1-\alpha & 1-\alpha & \alpha \\ 1-\alpha & 1-\alpha & \alpha \\ \frac{1}{3} & \frac{1}{3} & \frac{4}{3}-\alpha \end{pmatrix}$$

$\underbrace{\qquad\qquad\qquad}_{C_1}$       $\underbrace{\qquad\qquad\qquad}_{C_2}$

" $C_i$ " stands for "column  $i$ "

$$C_1 = C_2 \Rightarrow P - (\alpha - 1)I \text{ is not invertible}$$

so that  $\det(P - (\alpha - 1)I) = 0$  and  $\underline{\alpha - 1}$  is an eigenvalue of  $P$

Since  $P$  is stochastic,  $\underline{1}$  is also an eigenvalue of  $P$

We then call  $\lambda_1, \lambda_2$  and  $\lambda_3$  the three eigenvalues of  $P$  (potentially non-distinct). The trace of  $P$  satisfies:

$$\text{Tr}(P) = \lambda_1 + \lambda_2 + \lambda_3$$

$$\Rightarrow \frac{1}{3} = 1 + \alpha - 1 + \lambda_3 \Rightarrow \lambda_3 = \frac{1}{3} - \alpha$$

so that  $\underline{\frac{1}{3} - \alpha}$  is an eigenvalue of  $P$

Ex

$$\alpha \neq \frac{2}{3} \Rightarrow \alpha - 1 \neq \frac{1}{3} - \alpha \Rightarrow 3 \text{ different eigenvalues}$$

c)

$$P \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ is an eigenvector for } 1$$

$$P \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha - 1 \\ 1 - \alpha \\ 0 \end{pmatrix} = (\alpha - 1) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \text{ is an eigenvector for } (\alpha - 1)$$

$$P \begin{pmatrix} 3\alpha \\ 3\alpha \\ -2 \end{pmatrix} = \begin{pmatrix} \alpha - 3\alpha^2 \\ \alpha - 3\alpha^2 \\ 2\alpha - \frac{2}{3} \end{pmatrix} = \left(\frac{1}{3} - \alpha\right) \begin{pmatrix} 3\alpha \\ 3\alpha \\ -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 3\alpha \\ 3\alpha \\ -2 \end{pmatrix} \text{ is an eigenvector for } \left(\frac{1}{3} - \alpha\right)$$

d)  $\mathcal{B}$  is composed of eigenvectors related to different eigenvalues so that  $\mathcal{B}$  is linearly independent. In addition  $\text{Vect}(\mathcal{B})$ , the vector space span by  $\mathcal{B}$  has dimension 3. Hence  $\text{Vect}(\mathcal{B}) = \mathbb{R}^3$ .

If we define  $A = \text{Mat}(\mathcal{B}, e)$ , we get:

$$A = \begin{pmatrix} 1 & 1 & 3\alpha \\ 1 & -1 & 3\alpha \\ 1 & 0 & -2 \end{pmatrix}$$

Easy calculations provide its inverse:

$$A^{-1} = \frac{1}{2(2+3\alpha)} \begin{pmatrix} 2 & 2 & 6\alpha \\ 2+3\alpha & -(2+3\alpha) & 0 \\ 1 & 1 & -2 \end{pmatrix}$$

e) Let  $D_\alpha$  be the diagonal matrix:  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha - 1 & 0 \\ 0 & 0 & \frac{1}{3} - \alpha \end{pmatrix}$

Considering the studies of the eigenvalues/vectors of  $P$ :

$$\boxed{P = A \cdot D_\alpha \cdot A^{-1}}$$

Ex e) (t.b.c)  $P^n = A D_\alpha^n A^{-1}$

So that  $P^n = A \begin{pmatrix} 1 & 0 & 0 \\ 0 & (\alpha-1)^n & 0 \\ 0 & 0 & (\frac{1}{3}-\alpha)^n \end{pmatrix} A^{-1}$

f). By continuity,  $\lim_{n \rightarrow +\infty} P^n = A \cdot (\lim_{n \rightarrow +\infty} D_\alpha^n) A^{-1}$

Since  $0 < \alpha < 1 \Rightarrow |\alpha-1| < 1$  and  $|\frac{1}{3}-\alpha| < 1$

and  $\lim_{n \rightarrow +\infty} P^n = A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} A^{-1}$

then:

$$\lim_{n \rightarrow +\infty} P^n = \begin{pmatrix} 1 & 1 & 3\alpha \\ 1 & -1 & 3\alpha \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} \frac{1}{2+3\alpha} & \frac{1}{2+3\alpha} & \frac{3\alpha}{2+3\alpha} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and  $\lim_{n \rightarrow +\infty} P^n = \begin{pmatrix} \frac{1}{2+3\alpha} & \frac{1}{2+3\alpha} & \frac{3\alpha}{2+3\alpha} \\ \frac{1}{2+3\alpha} & \frac{1}{2+3\alpha} & \frac{3\alpha}{2+3\alpha} \\ \frac{1}{2+3\alpha} & \frac{1}{2+3\alpha} & \frac{3\alpha}{2+3\alpha} \end{pmatrix}$

$\lim_{n \rightarrow +\infty} P^n (1, i) = \begin{cases} \frac{1}{2+3\alpha} & \text{if } i=1 \\ \frac{1}{2+3\alpha} & \text{if } i=2 \\ \frac{3\alpha}{2+3\alpha} & \text{if } i=3. \end{cases}$

$$\pi = \left( \frac{1}{2+3\alpha} ; \frac{1}{2+3\alpha} ; \frac{3\alpha}{2+3\alpha} \right)$$

We then check the invariance of  $\pi$ : ( $\pi \cdot P = \pi$ )

$$\rightarrow \pi \cdot P(1) = \sum_{i=1}^3 \pi(i) P(i,1) = \frac{1}{2+3\alpha} \times 0 + (\alpha-1) \times \frac{1}{2+3\alpha} + \frac{3\alpha}{2+3\alpha} \times \frac{1}{3} = \frac{1}{2+3\alpha} = \pi(1)$$

$$\rightarrow \pi \cdot P(2) = \sum_{i=1}^3 \pi(i) P(i,2) = \frac{1}{2+3\alpha} \times (\alpha-1) + \frac{1}{2+3\alpha} \times 0 + \frac{3\alpha}{2+3\alpha} \times \frac{1}{3} = \frac{1}{2+3\alpha} = \pi(2)$$

$$\rightarrow \pi \cdot P(3) = \sum_{i=1}^3 \pi(i) P(i,3) = \frac{1}{2+3\alpha} \times \alpha + \frac{1}{2+3\alpha} \times \alpha + \frac{3\alpha}{2+3\alpha} \times \frac{1}{3} = \frac{3\alpha}{2+3\alpha} = \pi(3)$$

# Problem Ruin of the two gamblers.

## Part 1

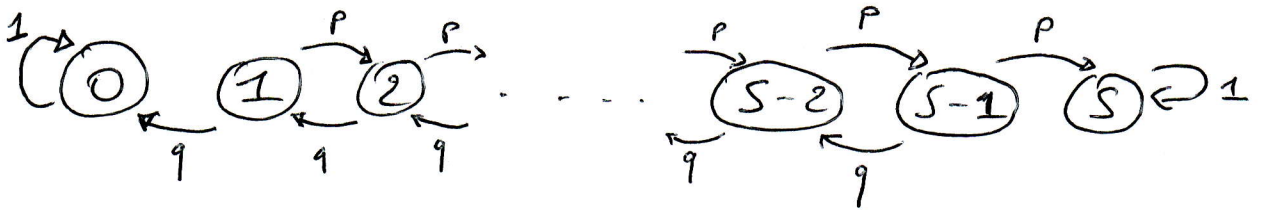
1). the value of  $X_{n+1}$  depends only on  $X_n$ , so that

$$\mathbb{P}(X_{n+1} = x_{n+1} | X_n = x_n) = \mathbb{P}(X_{n+1} = x_{n+1} | X_0 = x_0, \dots, X_n = x_n)$$

then this process  $(X_n)$  is a Markov chain.

since

$$\begin{aligned} \mathbb{P}(X_{n+1} = i+1 | X_n = i) &= p & 0 \leq i \leq N-1 \\ \mathbb{P}(X_{n+1} = i | X_n = i+1) &= q & 1 \leq i \leq N-1 \\ \mathbb{P}(X_{n+1} = S | X_n = S) &= 1 \\ \mathbb{P}(X_{n+1} = 0 | X_n = 0) &= 1 \end{aligned}$$



The transition matrix is given by:

$$P = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ q & 0 & p & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots \\ 0 & \dots & q & 0 & p \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix}$$

$\{0\}$  or  $\textcircled{0}$  means that A is broke  
 $\{S\}$  or  $\textcircled{S}$  ————— B is broke.

Since  $P(S,S) = P(0,0) = 1$ ,  $\{0\}$  and  $\{S\}$  are absorbing.

(Problem)

Part 1

$$T_S = \inf \{ n \geq 1, X_n = S \} \quad \text{and} \quad T_0 = \inf \{ n \geq 1, X_n = 0 \}$$

2) Considering the definitions of the two hitting-times,

$$\underline{\varphi_i = \mathbb{P}(T_S < T_0 \mid X_0 = i)}$$

•  $\varphi_0 = \mathbb{P}(T_S < T_0 \mid X_0 = 0)$ . But if  $X_0 = 0$ ,  $X_n = 0 \quad \forall n \geq 0$   
 $\Rightarrow \{T_S \leq T_0 \mid X_0 = 0\} = \emptyset$  game is stopped.

and  $\underline{\varphi_0 = 0}$

•  $\varphi_S = \mathbb{P}(T_S < T_0 \mid X_0 = S)$ . But if  $X_0 = S$ , then A has won the game.

and  $X_n = S \quad \forall n \geq 1 \Rightarrow \{T_S \leq T_0 \mid X_0 = S\} = \Omega$ .

then  $\underline{\varphi_S = 1}$

Then, if  $S = 1$ , the game is over before the beginning!

3) 3.1) •  $\{T_S < T_0\} \cap \{X_1 = 0\} = \emptyset$

$$\Rightarrow \underline{\mathbb{P}(T_S < T_0; X_1 = 0 \mid X_0 = i) = 0}$$

• The only possibility to reach  $\{S\}$  is to come from  $\{S-1\}$ , in all other case ( $0 \leq i < S-1$ ), the probability is 0.

$$\Rightarrow \underline{\mathbb{P}(T_S < T_0; X_1 = S \mid X_0 = i) = \begin{cases} p & \text{if } i = S-1 \\ 0 & \text{if } 0 \leq i \leq S-2 \end{cases}}$$

3.2) In fact we remark that  $\{T_S < T_0; X_1 = j\} = \{T'_S < T'_0; X_1 = j\}$ :

$$\mathbb{P}(T_S < T_0; X_1 = j \mid X_0 = i) = \mathbb{P}(T'_S < T'_0; X_1 = j \mid X_0 = i)$$

$$= \mathbb{P}(T'_S < T'_0 \mid X_1 = j; X_0 = i) \mathbb{P}(X_1 = j \mid X_0 = i)$$

(Definition of condexp.)

Problem

Part 1

then:

$$3.2) \quad \mathbb{P}(T_S' < T_0' \mid X_1 = j, X_0 = i) = \mathbb{P}(T_S' < T_0' \mid X_1 = j)$$

( $(X_n)$  is a Markov chain)

So that  $\underline{\mathbb{P}(T_S < T_0; X_1 = j \mid X_0 = i) = \mathbb{P}(T_S' < T_0' \mid X_1 = j) \mathbb{P}(X_1 = j \mid X_0 = i)}$

3.3) • By definition,  $\mathbb{P}(X_1 = j \mid X_0 = i) = P(i, j)$

• Using the definition of  $T_0, T_S, T_0'$  and  $T_S'$ :

$$\mathbb{P}(T_S' < T_0' \mid X_1 = j) = \mathbb{P}(T_S < T_0 \mid X_0 = j) = \varphi_j$$

Finally:  $\underline{\mathbb{P}(T_S < T_0; X_1 = j \mid X_0 = i) = \varphi_j P(i, j)} \quad (1 \leq j \leq S-1)$

3.4) For  $i \geq 1$ :

$$\begin{aligned} \varphi_i &= \mathbb{P}(T_S < T_0 \mid X_0 = i) = \sum_{j=1}^S \mathbb{P}(T_S < T_0; X_1 = j \mid X_0 = i) \\ &= \sum_{j=1}^{S-1} \varphi_j P(i, j) + \mathbb{P}(T_S < T_0; X_1 = S \mid X_0 = i) \\ &= \varphi_{i-1} q + \varphi_{i+1} p \quad (\text{using 3.1), 3.2) and 3.3}) \end{aligned}$$

then:  $\underline{\text{For } 1 \leq i \leq S-1: \quad \varphi_i = q \cdot \varphi_{i-1} + p \cdot \varphi_{i+1}}$

3.5) We follow the proposed hint:

$$\varphi_{i+1} - \varphi_i = \frac{q}{p} (\varphi_i - \varphi_{i-1}) \quad \forall 1 \leq i \leq S-1$$

$$\Rightarrow \varphi_{i+1} - \varphi_i = \left(\frac{q}{p}\right)^i (\varphi_1 - \varphi_0) \quad \forall 0 \leq i \leq S-1$$

$$\Rightarrow \sum_{j=0}^{i-1} (\varphi_{j+1} - \varphi_j) = \sum_{j=0}^{i-1} \left(\frac{q}{p}\right)^j (\varphi_1 - \varphi_0)$$

but  $\varphi_0 = 0$  (cf. Part 1, 2)) so  $\underline{\varphi_i = \varphi_1 \times \sum_{j=0}^{i-1} \left(\frac{q}{p}\right)^j}$

Problem  
Part 1

3.5) \* if  $p=q=\frac{1}{2}$ , then  $\varphi_i = \varphi_1 \times i$   
but cf. 2)  $\varphi_S = 1$  and  $\varphi_S = \varphi_1 \times S \Rightarrow \boxed{\varphi_1 = \frac{1}{S}}$

and  $\boxed{\varphi_i = \frac{i}{S} \text{ for all } 0 \leq i \leq S}$

\* if  $p \neq q$  ( $p \neq \frac{1}{2}$ ) then  $\varphi_i = \varphi_1 \times \frac{1 - (\frac{q}{p})^i}{1 - \frac{q}{p}}$

with the same idea as in the previous case:

$1 = \varphi_S = \varphi_1 \times \frac{1 - (\frac{q}{p})^S}{1 - \frac{q}{p}}$  then:  $\boxed{\varphi_1 = \frac{1 - \frac{q}{p}}{1 - (\frac{q}{p})^S}}$

and  $\boxed{\varphi_i = \frac{1 - (\frac{q}{p})^i}{1 - (\frac{q}{p})^S} \text{ for all } 0 \leq i \leq S}$

Part II

4) 4.1).  $T$  is a random variable in  $\mathbb{N} \cup \{+\infty\}$ .

• For  $n \geq 0$ :

$$\{T \leq n\} = \left( \bigcup_{k=0}^n \{S_k = -a\} \right) \cup \left( \bigcup_{k=0}^n \{S_k = b\} \right)$$

so that  $\{T \leq n\} \in \sigma(S_0, \dots, S_n)$ .

As we have already seen,  $\sigma(S_0, \dots, S_n) = \sigma(X_0, \dots, X_n) = \mathcal{T}_n$

so that  $\underline{\{T \leq n\} \in \mathcal{T}_n}$

4.2) •  $\mathbb{E}(S_{n+1} - S_n | \mathcal{T}_n) = \mathbb{E}(X_{n+1} | \mathcal{T}_n) = \frac{1}{2} \times 1 + \frac{1}{2} (-1) = 0$

•  $\mathbb{E}(|S_n|) \leq \sum_{k=0}^n \mathbb{E}(|X_k|) \leq n < +\infty$

$\Rightarrow (S_n)$  is a  $\mathcal{T}_n$ -martingale.

Part II

4.2)  $\rightarrow (S_n)$  is not convergent since  $(X_n)$  is not converging to 0.

$\rightarrow S^T = (S_{T \wedge n})$ . this is a martingale (using stopped-martingale theorem). As it is bounded:  $-a \leq S_{T \wedge n} \leq b \forall n \in \mathbb{N}$ , this martingale converges almost surely.

But over  $\{T = +\infty\}$ ,  $S^T = (S_n)_{n \geq 0}$  !

And  $(S_n)_{n \geq 0}$  is not convergent which is not allowed with the first result. Then  $\mathbb{P}(T = +\infty) = 0$ .

4.3) If  $p \neq q$ , we use the strong Law of Large Numbers:

$$\frac{S_n}{n} = \frac{X_0 + X_1 + \dots + X_n}{n} \xrightarrow[n \rightarrow +\infty]{a.s.} \mathbb{E}(X_1) = p - q \neq 0.$$

$$\text{So that } \begin{cases} S_n \xrightarrow{a.s.} +\infty & \checkmark \quad p - q > 0 \\ S_n \xrightarrow{a.s.} -\infty & \checkmark \quad p - q < 0 \end{cases} \Rightarrow |S_n| \xrightarrow{a.s.} +\infty \Leftrightarrow \mathbb{P}(|S_n| \xrightarrow{a.s.} +\infty) = 1$$

But  $\{T = +\infty\} = \{(S_n) \text{ is bounded}\}$ .

$$\text{So that } \mathbb{P}(T = +\infty) = 1 - \mathbb{P}(|S_n| \xrightarrow{a.s.} +\infty) = 0$$

$$\Rightarrow \underline{\mathbb{P}(T = +\infty) = 0}$$

5) Using the Stopped-martingale theorem: to  $(S_n)$

$$\underline{\mathbb{E}(S_T) = \mathbb{E}(S_0) = 0} \quad (\text{ES}) \text{ (if } p=q=\frac{1}{2}, \text{ martingale)}$$

$$\text{But } \mathbb{E}(S_T) = (-a) \cdot \mathbb{P}(S_T = -a) + b \cdot \mathbb{P}(S_T = b) = -a r_a + b r_b.$$

$$\text{but } r_b = 1 - r_a \Rightarrow \mathbb{E}(S_T) = -a r_a + b(1 - r_a)$$

$$\Rightarrow \underline{\mathbb{E}(S_T) = b - (a+b) r_a}$$



5.1) If  $p = q = \frac{1}{2}$  we can remark  $(S_n)$  is a martingale

we apply the stopped-Martingale theorem since it is dominated

Note: (one more hypothesis should be needed since we don't have

$$0 \leq T < M < +\infty \quad \text{a.s.}$$

$$\text{but } \mathbb{P}(T < +\infty) = 1.$$

Anyway we have  $\mathbb{E}(\sup_{n \geq 0} |S_{T \wedge n}|) < +\infty$

(stronger version of the stopped Martingale theorem)

here

we apply the theorem  $\Rightarrow \mathbb{E}(S_T) = 0$

$$\Rightarrow b - (a+b)r_a = 0$$

$$\Rightarrow \boxed{r_a = \frac{b}{a+b}}$$

5.2)  $p \neq q$  :  $(S_n)$  is not a martingale :

$$\left. \begin{array}{l} \mathbb{E}(S_1) = p - q \\ \mathbb{E}(S_0) = 0 \end{array} \right\} \Rightarrow \mathbb{E}(S_1) \neq \mathbb{E}(S_0)$$

$$\rightarrow \mathbb{E}(Z_{n+1} | \mathcal{T}_n) = \mathbb{E}\left(Z_n \cdot \left(\frac{q}{p}\right)^{X_{n+1}} \mid \mathcal{T}_n\right)$$

$$= Z_n \cdot \mathbb{E}\left(\left(\frac{q}{p}\right)^{X_{n+1}} \mid \mathcal{T}_n\right) \quad \text{since } Z_n \text{ is } \mathcal{T}_n\text{-adapted.}$$

$$= Z_n \cdot \left(\left(\frac{q}{p}\right)^1 \times p + \left(\frac{q}{p}\right)^{-1} \times q\right)$$

$$= Z_n (q + p) = Z_n.$$

$$\text{and } \mathbb{E}(|Z_n|) \leq \left(\frac{q}{p}\right)^n < +\infty$$

so that  $Z_n$  is a  $\mathcal{T}_n$ -martingale.

$$\rightarrow \mathbb{E}(Z_T) = \mathbb{E}(Z_0) = 1 \quad (\text{Stopped Martingale th.})$$

Problem

Part 2 5.1) case  $p \neq q$

$$\begin{aligned} \mathbb{E}(Z_T) &= \left(\frac{q}{p}\right)^{-a} r_a + \left(\frac{q}{p}\right)^b r_b \quad \text{by definition.} \\ &= \left( \left(\frac{q}{p}\right)^{-a} - \left(\frac{q}{p}\right)^b \right) r_a + \left(\frac{q}{p}\right)^b \end{aligned}$$

with the previous equation:  $1 = \left(\left(\frac{q}{p}\right)^{-a} - \left(\frac{q}{p}\right)^b\right) r_a + \left(\frac{q}{p}\right)^b$

$$\Rightarrow \boxed{r_a = \frac{1 - \left(\frac{q}{p}\right)^b}{\left(\frac{q}{p}\right)^{-a} - \left(\frac{q}{p}\right)^b}}$$

$$\text{then: } r_a = \frac{\left(\frac{q}{p}\right)^b \left(\left(\frac{p}{q}\right)^b - 1\right)}{\left(\frac{q}{p}\right)^b \left(\left(\frac{q}{p}\right)^{-a-b} - 1\right)} = \frac{1 - \left(\frac{p}{q}\right)^b}{1 - \left(\frac{p}{q}\right)^{a+b}}$$

$$\text{then: } \boxed{r_a = \frac{1 - \left(\frac{p}{q}\right)^b}{1 - \left(\frac{p}{q}\right)^{a+b}}}$$

6) 6.1) To prove that  $M_n$  is a martingale, we study:

$$\begin{aligned} \mathbb{E}((S_{n+1} - S_n)^2 | \mathcal{T}_n) &= \mathbb{E}(S_{n+1}^2 + S_n^2 - 2S_n S_{n+1} | \mathcal{T}_n) \\ &= \mathbb{E}(S_{n+1}^2 | \mathcal{T}_n) + S_n^2 - 2S_n \mathbb{E}(S_{n+1} | \mathcal{T}_n) \end{aligned}$$

because  $S_n$  is  $\mathcal{T}_n$ -adapted

$$= \mathbb{E}(S_{n+1}^2 | \mathcal{T}_n) + S_n^2 - 2S_n^2$$

since  $(S_n)$  is a  $\mathcal{T}_n$ -martingale

$$\text{so that } \mathbb{E}((S_{n+1} - S_n)^2 | \mathcal{T}_n) = \mathbb{E}(S_{n+1}^2 | \mathcal{T}_n) - S_n^2$$

$$\text{But: } \mathbb{E}((S_{n+1} - S_n)^2 | \mathcal{T}_n) = \mathbb{E}(X_{n+1}^2 | \mathcal{T}_n)$$

$$= \mathbb{E}(X_{n+1}^2) \quad \text{because } X_{n+1} \text{ is independent from } \mathcal{T}_n.$$

$$= 1$$

$$\Rightarrow \mathbb{E}(S_{n+1}^2 | \mathcal{T}_n) = S_n^2 + 1$$

# Problem

Part 2

$$6.1) \quad \mathbb{E}(M_{n+1} | \mathcal{T}_n) = \mathbb{E}(S_{n+1}^2 - (n+1) | \mathcal{T}_n) = \mathbb{E}(S_{n+1}^2 | \mathcal{T}_n) - (n+1) \\ = S_n^2 + 1 - (n+1) = S_n^2 - n = M_n$$

$$\text{and } \mathbb{E}(|M_n|) \leq n^2 < +\infty$$

So that  $(M_n)$  is a  $\mathcal{T}_n$ -martingale.

- We apply the stopped-martingale theorem to  $(M_n)$  and  $T \wedge n$ :

$$\underline{\mathbb{E}(M_{T \wedge n}) = \mathbb{E}(M_0) = 0}$$

$$\text{and } \underline{\mathbb{E}(S_{T \wedge n}^2) = \mathbb{E}(T \wedge n)}$$

- We use Lebesgue theorem to  $(S_{T \wedge n}) \rightarrow S_T$

$$\text{and then } \lim_{n \rightarrow +\infty} \mathbb{E}(S_{T \wedge n}^2) = \mathbb{E}(S_T^2)$$

$$\text{and } \lim_{n \rightarrow +\infty} \mathbb{E}(T \wedge n) = \mathbb{E}(T)$$

$$\text{So that } \mathbb{E}(S_T^2) = \mathbb{E}(T).$$

$$\bullet \quad \mathbb{E}(S_T^2) = (-a)^2 r_a + b^2 r_b = a^2 \frac{b}{a+b} + b^2 \frac{a}{a+b} \\ = ab \times \frac{a+b}{a+b}$$

$$\text{then } \underline{\mathbb{E}(S_T^2) = ab}$$

$$\text{and } \underline{\mathbb{E}(T) = ab}$$

$$6.2) \quad \bullet \quad R_n \text{ is } L^1 \text{ and } \mathbb{E}(R_{n+1} | \mathcal{T}_n) = R_n + \mathbb{E}(X_{n+1} | \mathcal{T}_n) - (p-q)$$

$$\text{then } \mathbb{E}(R_{n+1} | \mathcal{T}_n) = R_n + (p-q) - (p-q) \quad \text{bc. } X_{n+1} \perp \mathcal{T}_n$$

$$\Rightarrow \underline{\mathbb{E}(R_{n+1} | \mathcal{T}_n) = R_n}$$

$$\bullet \quad \text{S.M. Theorem: } \mathbb{E}(R_{T \wedge n}) = \mathbb{E}(R_0) = 0$$

$$\Rightarrow \underline{\mathbb{E}(S_{T \wedge n}) = (p-q) \mathbb{E}(T \wedge n)}$$

$$\bullet \quad \text{Using monotone convergence: } \mathbb{E}(S_T) = (p-q) \mathbb{E}(T)$$

$$\text{and } \underline{\mathbb{E}(T) = \frac{1}{p-q} \mathbb{E}(S_T)}$$

Problem  
Part 2

$$\mathbb{E}(S_T) ? : \mathbb{E}(S_T) = -a r_a + b r_b = b - (a+b) r_a$$

with question 5)  $r_a = \frac{1 - (\frac{p}{q})^b}{1 - (\frac{p}{q})^{a+b}}$

$$\Rightarrow \boxed{\mathbb{E}(S_T) = b - (a+b) \frac{1 - (\frac{p}{q})^b}{1 - (\frac{p}{q})^{a+b}}}$$

and  $\boxed{\mathbb{E}(T) = \frac{b}{p-q} - \frac{a+b}{p-q} \times \frac{1 - (\frac{p}{q})^b}{1 - (\frac{p}{q})^{a+b}}}$

We can reverse the situation of  $(a, p)$  and  $(b, q)$ :

$$\boxed{\mathbb{E}(T) = \frac{a}{q-p} - \frac{a+b}{q-p} \times \frac{1 - (\frac{q}{p})^a}{1 - (\frac{q}{p})^{a+b}}}$$

Part 3

7) B is infinitely rich.

7.1) A casino can be considered as being infinitely rich. In the same fashion, a state, organizing lotteries have the same characteristic. This can also be used when one gambler is significantly richer than the other.

For example in a casino,  $a \approx 1000 \text{€}$ ,  $p = \frac{18}{37}$  (roulette)

7.2) 1) if  $b \rightarrow +\infty$   $r_a(b) \rightarrow 1$   $\boxed{r_a(+\infty) = 1}$

It means that A will always be broke in this case, where neither A nor B have an edge.

2) if  $p < q$ ,  $(\frac{p}{q}) < 1 \Rightarrow \lim_{b \rightarrow +\infty} (\frac{p}{q})^b = 0 \Rightarrow \boxed{r_a(+\infty) = 1}$

This case happens when the casino has an edge over the player A (roulette, for example). Obviously, A goes broke in this case.

3) if  $q < p$ , A has an advantage.

$$r_a = \left(\frac{q}{p}\right)^a \times \frac{\left(\frac{q}{p}\right)^b - 1}{\left(\frac{q}{p}\right)^{a+b} - 1} \quad \text{then} \quad \boxed{r_a(+\infty) = \left(\frac{q}{p}\right)^a}$$

As  $r_a < 1$ , A doesn't always go broke.

Problem  
Part III

7.3)  $E(T)$  is the time to bust.

1)  $E(T) = +\infty$  in this case.

It means that A is going to lose all his fortune. But this can take an infinitely long time.

2) if  $p < q$  we use  $E(T) = \frac{a}{q-p} - \frac{a+b}{q-p} \times \frac{1 - (q/p)^a}{1 - (q/p)^{a+b}}$ .

so that  $E(T) \xrightarrow{b \rightarrow +\infty} \boxed{\frac{a}{q-p} = E(T_\infty)}$

A goes broke in a finite time on average.

3) if  $q < p$   $E(T) = \frac{b}{p-q} - \frac{a+b}{p-q} \times \frac{1 - (p/q)^b}{1 - (p/q)^{a+b}}$ .

then:  $E(T) = -\frac{a}{p-q} - \frac{a+b}{p-q} \times \frac{(p/q)^{a+b} - (p/q)^b}{1 - (p/q)^{a+b}}$   
 $= -\frac{a}{p-q} + \frac{b+a}{p-q} \times \frac{(p/q)^{a+b} - (p/q)^b}{(p/q)^{a+b} - 1}$   
 (Note:  $p-q \rightarrow +\infty$  and  $\frac{(p/q)^{a+b} - (p/q)^b}{(p/q)^{a+b} - 1} \rightarrow 1$ )

$\Rightarrow \boxed{E_\infty(T) = +\infty}$  The time to bust is infinite in this case.

• Example: advantage of 5% :  $p = 0,55$  then  $S_n = (p-q)n = 0,10 \times n \times (\text{betting unit})$

Considering the scheme 3 days / week, 30 weeks / year, 30 bets / day and 40 years.

$\Rightarrow n = 3 \times 30 \times 30 \times 40 = 108000$

$\Rightarrow E(S_{\text{Life}}) = 10800$  units of bets.

Playing with a unit of 100 €  $\Rightarrow E(S_{\text{Life}}) = 1080000$  € !