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## FINAL TEST

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The present test is composed of an exercise and a problem. The problem is divided in three parts all independent from each other. All documents are allowed. Calculators, or computers are forbidden.

**Exercise.** *Three States Markov Chain.*

Let  $(X_n)_{n \geq 0}$  be a Markov chain on the set  $E = \{1, 2, 3\}$ . We note its transition matrix  $P$ , so that :

$$P = \begin{pmatrix} 0 & 1 - \alpha & \alpha \\ 1 - \alpha & 0 & \alpha \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix},$$

where  $0 < \alpha < 1$ .

- Draw the chain's graph and verify the chain is irreducible, positive recurrent.
- Verify that  $1$ ,  $\alpha - 1$  and  $\frac{1}{3} - \alpha$  are eigenvalues of  $P$ .  
From now, we assume  $\alpha \neq \frac{2}{3}$ , so that the eigenvalues are all different from each other.
- Show that  $(1, 1, 1)$ ,  $(1, -1, 0)$  and  $(3\alpha, 3\alpha, -2)$  are eigenvectors related to the eigenvalues and recall the eigenvalue for each vector.
- We define the set of eigenvectors :

$$\mathcal{B} = ((1, 1, 1), (1, -1, 0), (3\alpha, 3\alpha, -2)).$$

Briefly recall why it is a basis of  $\mathbb{R}^3$ . Then give the change of basis matrix noted  $A$  from the canonical basis  $((1, 0, 0), (0, 1, 0), (0, 0, 1))$  to  $\mathcal{B}$ . Calculate  $A^{-1}$ .

- Express  $P$  as the product of  $A$ ,  $A^{-1}$ , and a diagonal matrix  $D$ . Then, calculate  $P^n$  based on this expression.
- Isolate the first line vector of  $\lim_{n \rightarrow \infty} P^n$ , noted  $\pi$ , such as  $\pi(i) = \lim_{n \rightarrow \infty} P^n(1, i)$ . If all your calculations are correct, you should find :

$$\pi = \left( \frac{1}{2 + 3\alpha}, \frac{1}{2 + 3\alpha}, \frac{3\alpha}{2 + 3\alpha} \right).$$

Then check that  $\pi$  is a stationary distribution of the Markov Chain, meaning :

$$\pi \cdot P = \pi.$$

**Problem.** *Ruin of the two gamblers.*

Two gamblers,  $A$  and  $B$ , play an unfair coin tossing game, with an infinite number of independent tosses. The total amount of their fortune is worth  $S = a + b$  euros where  $a \in \mathbb{N}$  (resp.  $b \in \mathbb{N}$ ) is the fortune of  $A$  (resp.  $B$ ). At each toss, the gambler  $A$  wins with probability  $p$  and loses with probability  $q = 1 - p$  with  $0 < p < 1$ . At each step, the loser gives one euro to the winner. The game ends as soon as one of the gambler is broke.

## Part 1

To go further, we will note  $X_n$  the gambler's  $A$  fortune after the  $n^{\text{th}}$  toss.

- 1) Explain why  $(X_n)_{n \geq 0}$  is a Markov chain on the set of states  $E = \{0, 1, \dots, S\}$  and give its transition matrix.

Give an interpretation for the states  $0$  and  $S$ . Verify that those states are absorbing.

This Markov Chain has two recurrent classes,  $\{S\}$  and  $\{0\}$  and one transient class,  $\{1, 2, \dots, S - 1\}$ .

For  $i \in \{1, \dots, S - 1\}$ , let us note  $\varphi_i$  the probability that the chain hits the states  $S$  before the state  $0$ , starting on the state  $i$ , meaning, the probability of ruin for the gambler  $B$ , when gambler  $A$  has a fortune of  $i$  euros. We then define the standard *hitting times*  $T_S$  and  $T_0$  by

$$T_S = \inf\{n \geq 1, X_n = S\}, \quad \text{and} \quad T_0 = \inf\{n \geq 1, X_n = 0\}.$$

- 2) Express  $\varphi_i$  as function of  $T_0, T_S, X_0$  and  $i$ . Remark and explain why  $\varphi_0 = 0$  and  $\varphi_S = 1$ . Give the solution of the two gambler's game ruin, if  $S = 1$ .

For now and then, we assume  $a, b \geq 1$  and then  $S \geq 2$ .

- 3) In this question, our aim is to show the recurrence relation satisfied by  $(\varphi_i)$  :

$$\forall i \in \llbracket 1, S - 1 \rrbracket, \quad \varphi_i = q\varphi_{i-1} + p\varphi_{i+1}.$$

obtained in conditioning regarding the state at  $n = 1$ .

We define two other hitting times :

$$T'_S = \inf\{n \geq 2, X_n = S\}, \quad \text{and} \quad T'_0 = \inf\{n \geq 2, X_n = 0\}.$$

- 3.1) Explain why  $\mathbb{P}(T_S < T_0, X_1 = 0 \mid X_0 = i)$  equals to  $0$ . In a same fashion show that  $\mathbb{P}(T_S < T_0, X_1 = S \mid X_0 = i)$  equals to  $p$  if  $i = S - 1$  and to  $0$  if else.

For  $j \in \llbracket i, S - 1 \rrbracket$ , we can remark the events  $\{T_S < T_0, X_1 = j\}$  and  $\{1 + T'_S < 1 + T'_0, X_1 = j\}$  are the same.

- 3.2) Using the Markov Chain definition and the previous remark, give a relation between  $\mathbb{P}(T_S < T_0, X_1 = j \mid X_0 = i)$ ,  $\mathbb{P}(T'_S < T'_0 \mid X_1 = j)$  and  $\mathbb{P}(X_1 = j \mid X_0 = i)$ .
- 3.3) Using the previous question, prove the following equality :

$$\mathbb{P}(T_S < T_0, X_1 = j \mid X_0 = i) = \varphi_j P(i, j)$$

where  $P(i, j)$  is the  $i^{\text{th}}$  line,  $j^{\text{th}}$  column element of the transition matrix  $P$  of the Markov chain.

3.4) Then prove the following equality :

$$\mathbb{P}(T_S < T_0 | X_0 = i) = \sum_{j=1}^S \mathbb{P}(T_S < T_0, X_1 = j | X_0 = i)$$

then using 3.1, 3.2, 3.3 and this formula, finally conclude on the recurrence relation satisfied by  $\varphi_i$ .

3.5) Give the explicit formula of  $\varphi_i$ , whether  $p = \frac{1}{2}$  or  $p \neq \frac{1}{2}$ .

To solve this question you can either use the classical technique for recurrence relation with the discriminant, or remark,  $\varphi_{i+1} - \varphi_i = \frac{q}{p}(\varphi_i - \varphi_{i-1})$ .

## Part 2

Now, we are going to use another path to the solution with Martingales. We suppose  $a, b \geq 1$ . We introduce another process,  $S_n$ , defined by  $S_n = X_0 + \dots + X_n$ , and as a convention,  $X_0 = 0$ . In this framework,  $S_n$  represents the profit made by  $A$  at the  $n^{\text{th}}$  step of the game, so that  $a + S_n$  (resp.  $b - S_n$ ) represents  $A$  (resp.  $B$ ) fortune at time  $n$ , as long as those sums stay positive. We then define a random variable  $T$  :

$$T = \min\{n \geq 1, S_n = -a \text{ or } S_n = b\}.$$

Finally, we note  $\mathcal{T}_n$ , the filtration generated by  $\sigma(X_0, \dots, X_n)$ .

4) In this question we prove that  $T$  is almost surely finite.

4.1) Verify that  $T$  is a stopping time relative to the filtration  $(\mathcal{T}_n)_{n \geq 0}$ .

4.2) Assuming  $p = q = \frac{1}{2}$ , check that  $(S_n)_{n \geq 0}$  is a  $(\mathcal{T}_n)_{n \geq 0}$ -martingale. Then, remark that the sequence  $(X_n)$  is not convergent (explain why); now consider the stopped martingale  $S^T = (S_{T \wedge n})$  (recall why  $S^T$  is a martingale). Since  $S^T$  is bounded, we concede that  $S^T$  is convergent (no proof to give). Then considering  $(S_n) = S^T$  on the event  $\{T = +\infty\}$ , what can you say about  $\mathbb{P}(T = +\infty)$ ? Then conclude that  $T$  is almost surely finite in this case.

4.3) If we assume  $p \neq q$ , and the event  $\{T = +\infty\}$ , use the Law of Large Numbers to prove the convergence of  $\frac{S_n}{n}$ . Calculate the limit.

Then explain why we have two cases :  $S_n \rightarrow +\infty$  a.s., or  $S_n \rightarrow -\infty$  a.s., depending on the sign of  $p - q$ . Finally, explain why  $\mathbb{P}(T < \infty) = 1$ , and thus why  $T$  is almost surely finite.

5) In this question we evaluate the ruin probability.

Since  $T$  is almost surely finite, we can consider the random variable  $S_T$ . With this definition,  $S_T$  can only be equal to  $-a$  (ruin of  $A$ ) or  $b$  (ruin of  $B$ ). We define :

$$r_a = \mathbb{P}(S_T = -a) \quad \text{and} \quad r_b = \mathbb{P}(S_T = b),$$

and then  $r_a + r_b = 1$ . Calculate  $\mathbb{E}(S_T)$  as function of  $r_a$ .

5.1) Case  $p = q = \frac{1}{2}$

Recall why the process  $(S_n)_{n \geq 0}$  is a martingale.

Apply the Stopped Martingale Theorem to find :

$$r_a = \frac{b}{a+b}.$$

- Case  $p \neq q$

Explain why the process  $(S_n)_{n \geq 0}$  is not anymore a  $(\mathcal{T}_n)_{n \geq 0}$ -martingale, using  $\mathbb{E}(S_1)$ . We need to use another technique to achieve this valuation of  $r_a$ . We introduce a new stochastic process, noted  $Z$  defined by :

$$Z_0 = 1, \quad \forall n \geq 1 \quad Z_n = \left(\frac{q}{p}\right)^{S_n}.$$

- Prove that the process  $(Z_n)_{n \geq 0}$  is a  $(\mathcal{T}_n)_{n \geq 0}$ -martingale.
- Apply the Stopped Martingale Theorem to the process  $Z$  and the stopping time  $T$ .
- Calculate  $\mathbb{E}(Z_T)$  directly as a function of  $r_a, q, p, a$  and  $b$ .
- Using the last two questions prove that the ruin probability of  $A$  in the case  $p \neq q$  equals to :

$$r_a = \frac{1 - \left(\frac{p}{q}\right)^b}{1 - \left(\frac{p}{q}\right)^{a+b}}.$$

6) Length of the game : evaluation of the expectancy of  $T$ .

6.1) Case  $p = q = \frac{1}{2}$

To calculate  $\mathbb{E}(T)$ , we still want to use the stopped martingale theorem. As already done during the course, we introduce another martingale  $M$  :

$$M_0 = 0 \quad \text{and} \quad \forall n \geq 1, \quad M_n = S_n^2 - n.$$

- Let  $n$  be an integer, apply the stopped martingale theorem to the martingale  $M$  and the stopping time  $T \wedge n$ .
- Using the classical convergence theorem for the expectancy, prove :

$$\mathbb{E}(S_T^2) = \mathbb{E}(T).$$

- Give a direct calculation of  $\mathbb{E}(S_T^2)$  using 5.1, then with 5.1, prove that :

$$\mathbb{E}(T) = ab.$$

6.2) Case  $p \neq q$ .

As previously said in this test,  $(S_n)$  is not a  $(\mathcal{T}_n)_{n \geq 0}$ -martingale. However, if we slightly modify  $S$ , it can be made into a martingale. Since  $\mathbb{E}(X_1) = p - q$ , we define the process  $R$  :

$$R_n = S_n - (p - q) \cdot n.$$

- Prove that the stochastic process  $R$  is a  $(\mathcal{T}_n)_{n \geq 0}$ -martingale.
- For a fixed integer  $n$ , apply the stopped martingale theorem to  $R$ , and  $T \wedge n$  to prove :

$$\mathbb{E}(S_{T \wedge n}) = \mathbb{E}(T \wedge n)(p - q).$$

- Using the classical convergence theorem, prove that :

$$\mathbb{E}(T) = \frac{1}{p - q} \mathbb{E}(S_T).$$

- Calculate directly  $\mathbb{E}(S_T)$ , as a function of  $r_a$ ,  $p$  and  $q$ . Then using the expression of  $r_a$ , prove :

$$\mathbb{E}(T) = \frac{b}{p-q} - \frac{a+b}{p-q} \cdot \frac{1 - \left(\frac{p}{q}\right)^b}{1 - \left(\frac{p}{q}\right)^{a+b}}.$$

Then deduce :

$$\mathbb{E}(T) = \frac{a}{q-p} - \frac{a+b}{q-p} \cdot \frac{1 - \left(\frac{q}{p}\right)^a}{1 - \left(\frac{q}{p}\right)^{a+b}}.$$

### 6.3) Numerical application

If the game is fair, how many throws are needed before one gambler goes broke, if  $A$  has 50€ and  $B$  200€? Comment on this result.

## Part 3

7) In this part, we consider the case where  $B$  is infinitely rich.

7.1) Explain why this case can be useful for actual modelization at least as an approximation. Give two examples at least of such situations, and give orders of magnitude for  $a$  and  $p$ .

To get the formulas in this situation we only need to make  $b$  goes to  $+\infty$  in all the formulas.

7.2) Ruin probability of  $A$ .

7.2.1) Case  $p = q = \frac{1}{2}$

Give  $r_a$  with these assumptions ( $B$  infinitely rich and  $p = q$ ). Give an interpretation. Is it always sure that  $A$  goes broke against an infinitely rich opponent?

7.2.2) Case  $p < q$

Give  $r_a$  with these assumptions ( $B$  infinitely rich and  $p < q$ ). Give an interpretation. Is it always sure that  $A$  goes broke against an infinitely rich opponent?

7.2.3) Case  $p > q$

Give  $r_a$  with these assumptions ( $B$  infinitely rich and  $p > q$ ). Give an interpretation. Is it always sure that  $A$  goes broke against an infinitely rich opponent?

7.3) Time to busto. For all the cases below, give the value of  $\mathbb{E}(T)$ .

7.3.1) Case  $p = q$

Give  $\mathbb{E}(T)$  and an interpretation.

7.3.2) Case  $p < q$

Give  $\mathbb{E}(T)$  and an interpretation.

7.3.3) Case  $p > q$

Give  $\mathbb{E}(T)$  and interpretes this result.

Give an example of casino game where the gambler has an advantage over the house. Propose a numerical application for his life expected earnings using all the material of this test. (try to be creative).