
Probability Revisions 2

Exercise 1. Let X a random variable with a binomial distribution $\mathcal{B}(p, n)$. Show that for all $\lambda > 0$ and $\varepsilon > 0$:

$$\mathbb{P}(X - np > n\varepsilon) \leq \mathbb{E}(\exp(\lambda(X - np - n\varepsilon)))$$

Exercise 2. We toss a coin. The probability to have a head is p .

- 1) What is the probability to have no tail for the first n tosses ?
- 2) Estimate the probability to get a tail exactly at the n^{th} toss.
- 3) Calculate the expectancy of the number of toss to the first tail.

Exercise 3. Calculate $\mathbb{E}(X(X-1)\cdots(X-r+1))$ for $r \in \mathbb{N}$

- 1) When X is a random variable with a poisson distribution $\mathcal{P}(\lambda)$.
- 2) When X is a random variable with a geometric distribution $\mathcal{G}(p)$.

Exercise 4. Prove the Bienaymé-Chebyshev formula :

$$\forall a > 0, \quad \mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}(X^2)}{a^2}$$

Exercise 5. Let X be a random variable with a gaussian distribution $\mathcal{N}(\mu, \sigma^2)$. Calculate the density function of $Y = e^X$.

Exercise 6. Let X be a random variable and F_X its cumulative distribution function. What is the law of $Y = F_X(X)$?

- Exercise 7.**
- 1) Let X be a random vector of length m , φ_X its characteristic function, $A \in \mathcal{M}_m(\mathbb{R})$ and $a \in \mathbb{R}^m$. Determine the law and the characteristic function of $Y = a + AX$.
 - 2) Determine the characteristic function of $X = \exp(\lambda)$ for $\lambda \in \mathbb{R}$.

Exercise 8. Let X_1, \dots, X_n be n random vectors independant from each other. Consider $X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$. Show that X is a gaussian vector if and only if for all $i \in \llbracket 1; n \rrbracket$, X_i is a gaussian vector.

Exercise 9. Let $(X_n)_{n \geq 1}$ be a sequence of independant random variables with a poisson distribution : for all $n \geq 1$, $X_n \sim \mathcal{P}(n)$. Let consider $Z_n = \frac{1}{n}(X_n - n)$. Show why Z_n converge in law to Z where $Z \sim \mathcal{N}(0, 1)$.