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## Martingales

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**Exercise 1.** Let  $(X_n)_{n \geq 1}$  be a sequence of random variables independent identically distributed. We define the random process  $S$  by

$$\begin{cases} S_0 &= 0 \\ S_n &= X_1 + \dots + X_n \quad \text{if } n \geq 1 \end{cases}$$

$S$  is called a **random walk**. Let  $\mathcal{F}_0 = \{\emptyset, \Omega\}$  and  $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$  if  $n \geq 1$ .

- 1) Prove that  $\mathcal{F}_n = \sigma(S_1, \dots, S_n)$ .
- 2) If  $\mathbb{E}[|X_1|] < +\infty$  and  $\mathbb{E}[X_1] = 0$ , prove that  $S$  is a  $(\mathcal{F}_n)$ -martingale.

**Exercise 2.** With the same notation as in the previous exercise, we assume that  $\mathbb{E}[X_1^2] < +\infty$  and  $\mathbb{E}[X_1] = 0$ . We define  $V$  by  $V_n = S_n^2 - n\mathbb{E}[X_1^2]$  for  $n \in \mathbb{N}$ . Prove that the random process  $V$  is a  $(\mathcal{F}_n)$ -martingale.

*Hint :* Calculate  $\mathbb{E}((S_{n+1} - S_n)^2 | \mathcal{F}_n)$ .

**Exercise 3.** With the same notation as in the first exercise, let  $\lambda$  be a real such that  $\varphi(\lambda) = \log \mathbb{E}(e^{\lambda X_1}) < +\infty$ . Then we define a random process  $Z^\lambda$  by  $Z_n^\lambda = \exp(\lambda S_n - n\varphi(\lambda))$  for  $n \in \mathbb{N}$ . Prove that  $Z^\lambda$  is a  $(\mathcal{F}_n)$ -martingale.

**Exercise 4.** Let  $M = (M_n)_{n \geq 0}$  a  $(\mathcal{F}_n)$ -martingale and  $H = (H_n)_{n \geq 0}$  a random process such that for all  $n$ ,  $H_n$  is  $\mathcal{F}_n$ -measurable and bounded. Let  $G_0$  be a random variable  $\mathcal{F}_0$ -measurable and integrable. For  $n \geq 1$  we define  $G_n = G_0 + \sum_{k=0}^{n-1} H_k(M_{k+1} - M_k)$  or equivalently :

$$\forall n \geq 0, \quad G_{n+1} - G_n = H_n(M_{n+1} - M_n)$$

- 1) Prove that  $G = (G_n)_{n \geq 0}$  is a  $(\mathcal{F}_n)$ -martingale.
- 2) Application : gain of a betting strategy in a fair game. We note  $H = (H_n)_{n \geq 0}$  the sequence of bets on a sequence of results bringing  $(X_{n+1})_{n \geq 0}$  by unit bet. we suppose that for each  $n$ ,  $H_n$  is a function of  $(X_1, \dots, X_n)$  and is a bounded random variable.  $H_0$  is deterministic. The profit realized at time  $n$  is then defined by the following recurrent relation :  $G_{n+1} - G_n = H_n X_{n+1}$ .

- a) Using 1), prove that  $(G_n)_{n \geq 0}$  is a  $(\mathcal{F}_n)$ -martingale with  $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ .
- b) Prove that it is impossible to earn money in a fair coin tossing game.
- c) Explain why it is impossible to win money without taking risk, i.e.  $\mathbb{P}(G_n \geq G_0) = 1$  and  $\mathbb{P}(G_n > G_0) > 0$ .

**Exercise 5.** We note  $\wedge$  the mathematical operator defined by  $x \wedge y = \inf\{x, y\}$ . Let  $M$  be a  $(\mathcal{F}_n)$  random process, i.e. for each  $n$ ,  $M_n$  is  $\mathcal{F}_n$ -measurable, and  $\tau$  a  $\mathcal{F}_n$ -stopping time. We define another random process  $N$  by  $N_n = M_{\tau \wedge n}$ . Prove that  $N$  is a  $(\mathcal{F}_n)$ -martingale.

**Exercise 6.** Stopped martingale theorem.  $\mathcal{F}_n$  is a filtration of  $(\Omega, \mathcal{T}, \mathbb{P})$ . Let  $\tau$  be a bounded  $\mathcal{F}_n$ -stopping time (i.e. there exists  $C \geq 0$  such that  $\tau \leq C$  a.s.) and  $M = (M_n)_{n \geq 0}$  a  $(\mathcal{F}_n)$ -martingale. Show that  $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$ .

**Exercise 7.** **[\*\*]** Let  $(X_n)_{n \geq 1}$  be a sequence of independent random variables with the same law,  $\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = 1/2$ . We define the random walk generated by  $(X_n)_{n \geq 1}$ ,  $S$ , defined by :

$$\begin{cases} S_0 &= 0 \\ S_n &= X_1 + \dots + X_n \quad \text{if } n \geq 1 \end{cases}$$

For  $a \in \mathbb{N}^*$ , we define  $\tau_a = \inf\{n \geq 0, S_n = a\}$ . ( $\tau_a = +\infty$  if  $\{n \geq 0, S_n = a\} = \emptyset$ )

- 1) Prove that  $\tau_a$  is a  $(\mathcal{F}_n)$ -stopping time with  $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$ .
- 2) Show that  $\mathbb{P}(\tau_a < +\infty) = 1$ .
- 3) Explain why  $\mathbb{E}(S_{\tau_a}) = a > 0$ .
- 4) Interpret this result in terms of coin-flipping.
- 5) Prove  $\mathbb{E} \left[ \sup_{n \leq \tau_a} S_n^- \right] = +\infty$  where  $x^-$  is the negative part of  $x$ . Interpret this result for a fair coin-flipping game.