
Markov Chains

Exercise 1. Two states Markov Chain.

Let $(X_n)_{n \geq 0}$ be a Markov chain on the set $E = \{0, 1\}$. We note its transition matrix P , so that : $P = \begin{pmatrix} 1 - \alpha & \alpha \\ \beta & 1 - \beta \end{pmatrix}$, where α, β are reals in the interval $]0, 1[$.

- Draw the chain's graph and prove that it is irreducible and positive recurrent.
- Figure out the stationary distribution¹ π .
- Calculate $\mathbb{P}_0(T_1 = n)$, $\mathbb{P}_0(T_0 = n)$ and find another way to get the invariant measure.
- Calculate P^n . Find $A > 0$ and $\rho \in]0, 1[$ such that :

$$\forall x, y \in E^2, \quad |P^n(x, y) - \pi(y)| \leq A\rho^n$$

Exercise 2. Length of stay in a state.

Let $(X_n)_{n \geq 0}$ be a Markov chain with state space E .
We note τ_x the length of stay in the state x , the random variable :

$$\tau_x = \inf\{n \in \mathbb{N} \mid X_n \neq x\}$$

(we assume $\inf \emptyset = +\infty$)

Prove that τ_x is a stopping-time for the filtration relative to (X_n) (the σ -algebra $\sigma(X_0, X_1, \dots)$), and then determine the distribution of τ_x under \mathbb{P}_x .

Exercise 3. Energy management. in a house with two heaters, a main one and an additional one. We say "we are in the state 1" if only the main heating supply is working, and "in the state 2", if both main and additionnal heaters are on.

- If one day we only use the main heater, we will do so the next day with probability $\frac{1}{2}$.
- On the contrary, if we use both heaters, the next day the house is hotter and we turn off the additionnal one with probability $\frac{3}{4}$.

Let's note X_n the state of the heating supplies on the n^{th} day. We assume $(X_n)_{n \geq 0}$ is a Markov chain.

1. π is also called "invariant measure"

- a) Determine the chain's transition matrix and its graph.
- b) We define $p_n = \mathbb{P}(X_n = 1)$.
Give a recurrent relation satisfied by p_n , then express p_n as a function of p_0 . What is the value of $\lim_{n \rightarrow \infty} p_n$?
- c) Assuming we are in the first state a Sunday, find the probability to stay in the same state the next Sunday.
- d) Prove that if a day we are on the first state with a probability $\frac{3}{5}$, then it is the same for all the days that follows.
- e) Each day on the first state costs 10€, the second 20€ and each transition from 1 to 2 and inversely cost 5€. Calculate the average cost of one day in the situation described in d).