
Markov Chains 2

Exercise 1. Generic Markov chain.

Let S, C be two finite or countable sets and $f : S \times C \rightarrow S$ an application. We consider a random variable X_0 in S , and a sequence of random variables $(U_n)_{n \geq 1}$ in C and we define the sequence $(X_n)_{n \geq 0}$ by induction

$$X_{n+1} = f(X_n, U_{n+1}).$$

We suppose that the random variables $(X_0, U_1, U_2, \dots, U_k, \dots)$ are independants and that all the $(U_n)_{n \geq 1}$ have the same distribution on C , noted ν . The goal of this exercise is to prove that $(X_n)_{n \geq 0}$ is a Markov chain and to calculate its transition matrix.

- Prove by induction that the random vector (X_0, X_1, \dots, X_n) is $\sigma(X_0, U_1, \dots, U_n)$ -measurable. Deduce from this that U_{n+1} is independant from (X_0, X_1, \dots, X_n) .
- Calculate $\mathbb{P}(X_{n+1} = s \mid X_0 = x_0, X_1 = x_1, \dots, X_n = x_n)$ and prove it equals to

$$\mathbb{P}(f(x_n, U_1) = s).$$

- Conclude on the nature of $(X_n)_{n \geq 0}$.
- Finally, prove that for two states s, t of S , the coefficient of the transition matrix M is given by

$$M(s, t) = \sum_{\substack{k \in C \\ f(s, k) = t}} \nu(k) = \mathbb{P}(f(s, U_1) = t).$$

Exercise 2. Invariant distribution, average return time to a state.

Let $E = \{1, 2, 3, 4, 5\}$ and $X = (X_n)_{n \geq 0}$ a process such that for each $x \in E$, $(X_n)_{n \geq 0}$ is an homogeneous Markov chain over E , on a probabilised space $(\Omega, \mathcal{F}, (\mathcal{F})_{n \geq 0}, \mathbb{P}_x)$.

Its transition matrix M is

$$M = \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

- a) Determine the graph associated to the chain. Isolate and give its communication classes.
- b) Specify the nature and the period of the communication classes.
- c) Using b), justify the existence of a unique invariant distribution for the Markov chain. Calculate it and deduce $\mathbb{E}_x(T_x)$ where $T_x = \inf\{n \in \mathbb{N}^*, X_n = x\}$ is the time of first transit at x (and $\inf \emptyset = \infty$).
- d) Determine the potential matrix, noted R , of this Markov chain.
- e) Calculate the probabilities $\mathbb{P}_4(T_5 < +\infty)$ and $\mathbb{P}_5(T_5 < +\infty)$.
- f) Let us note for each $y \in E$, $N_y = \sum_{j \in \mathbb{N}} \mathbf{1}_{\{X_j=y\}}$ the number of transit to the state y . Calculate the probabilities $\mathbb{P}_4(N_5 = m)$ and $\mathbb{P}_5(N_5 = m)$ for each $m \in \mathbb{N}$.
- g) Show the following equality for every $x, y \in E$,

$$\mathbb{E}_x[T_y \mathbf{1}_{\{T_y < +\infty\}}] = M(x, y) + \sum_{x \neq y} M(x, z) [\mathbb{E}_z(T_z \mathbf{1}_{\{T_y < +\infty\}}) + \mathbb{P}_z(T_y < +\infty)].$$

- h) Deduce the value of $\mathbb{E}_x(T_1)$ for every $x \in E$.

Exercise 3. Random walk on $\{0, 1, 2, \dots, N\}$

Let $X = (X_n)_{n \geq 0}$ a process taking its values on $E = \{0, 1, 2, \dots, N\}$. For each $x \in E$, X is an homogeneous Markov chain with transition matrix M , given by

$$\begin{cases} M(x, x+1) = p, & M(x, x-1) = q, & \text{if } 1 \leq x \leq N-1, \\ M(0, 1) = 1, & \text{and } M(N, N-1) = 1. \end{cases}$$

- a) Draw the graph of the transition matrix and determine its communication classes.
- b) Justify the existence and unicity of an invariant probability ν for the Markov chain. Calculate it as a function of p , q and N .
- c) Deduce the value of $\mathbb{E}_0(T_0)$.