
Conditional Expectation

Exercise 1. Let $(\Omega, \mathcal{T}, \mathbb{P})$ be a probabilized space. Let \mathcal{B} be a sub- σ algebra of \mathcal{T} . Let X et Y be two random variables. Y has a Poisson distribution of parameter λ . We suppose that X is \mathcal{B} -measurable and Y and \mathcal{B} are independent. Calculate $\mathbb{E}^{\mathcal{B}}[\cos(XY)]$

Hint : Use $\mathbb{E}^{\mathcal{B}}[f(X, Y)] = \hat{f}(X)$ \mathbb{P} -a.s. where $\hat{f}(x) = \mathbb{E}(f(x, Y))$ (proved on exercise 5).

Exercise 2. Let X, Y be jointly continuous variables with the probability distribution function

$$f_{(X,Y)}(x, y) = \begin{cases} x + y & \text{if } 0 \leq x, y \leq 1 \\ 0 & \text{else} \end{cases}$$

Calculate $\mathbb{E}[X|Y]$ using $f_{X|Y}(x|y)$ and $f_Y(y)$.

Exercise 3. Let Y be a random variable defined by $Y = XU$ where X and U are independent random variables and X has the Rayleigh density :

$$f_X(x) = \begin{cases} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} & \text{if } 0 \leq x \\ 0 & \text{else} \end{cases}$$

and U is uniformly distributed on the interval $[0, 1]$. The aim of this exercise is to calculate $\mathbb{E}[X|Y]$.

- 1) Calculate $\mathbb{E}(X)$, $\mathbb{E}(Y)$, $\mathbb{E}(X^2)$, $\text{Var}(Y)$, $\text{Cov}(X, Y)$.
- 2) Determine $f_{(X,Y)}$ and f_Y .
- 3) Estimate $\mathbb{E}[X|Y = y]$ and then $\mathbb{E}[X|Y]$.

Exercise 4. Let B_1, \dots, B_n be a partition of the universe Ω and $\mathcal{B} = \sigma(B_1, \dots, B_n)$ the σ -algebra generated by $\{B_1, \dots, B_n\}$. Let Y be a random variable \mathcal{B} -measurable.

- 1) Prove that $Y = \sum_{i=1}^n y_i \mathbb{1}_{B_i}$ where $y_i = Y(\omega_i)$ for an element $\omega_i \in B_i$.
- 2) Using this formula, calculate $\mathbb{E}(Y|\mathcal{B})$.

Exercise 5. Let $(\Omega, \mathcal{T}, \mathbb{P})$ be a probabalized space. Let \mathcal{B} be a sub σ -algebra of \mathcal{T} , X and Y two real random variables such that X is independent from \mathcal{B} . We choose Y \mathcal{B} -measurable and finally, we consider a bounded measurable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. We define

$$\hat{f} : \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ y \longmapsto \mathbb{E}(f(X, Y)) \end{array}$$

Show that $\mathbb{E}(f(X, Y)|\mathcal{B}) = \hat{f}(Y)$.

Exercise 6. Let X be a Poisson random variable of parameter λ . Let S be another random variable defined this way : if $X = n$, we toss a coin n times with a probability p to get a head : S is the number of heads obtained during those n throws. The aim of this exercise is to calculate $\mathbb{E}[X|S]$ and $\mathbb{E}[S|X]$.

1) $\mathbb{E}[S|X]$

a) If $X = n$, show that $\mathbb{E}[S|X = n] = pn$.

b) Prove that $\mathbb{E}[S|X] = pX$.

2) $\mathbb{E}[X|S]$

a) Calculate $\mathbb{P}(X = n|S = k)$ and then $\mathbb{E}[X|S = k]$.

b) Prove that $\mathbb{E}[X|S] = S + (1 - p)\lambda$.

Exercise 7. Gaussian couple. Let $Z = (X, Y)$ be a gaussian random vector of dimension 2 with $\mathbb{E}(Z) = 0$. We note σ_X^2 (resp. σ_Y^2) the variance of X (resp. Y) and ρ_{XY} the correlation coefficient of X and Y defined by

$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

Calculate $\mathbb{E}[X|Y]$.

Exercise 8. Let $(X_n)_{n \geq 0}$ be a sequence of independent identically distributed random variables with $\mathbb{E}(X_n) = \mu$ and $\text{Var}(X_n) = v$. We consider $S_n = X_0 + \dots + X_n$. Let N be a random variable which takes its values in \mathbb{N} , independent from the X_n , with $\mathbb{E}(N) = \nu$ and $\text{Var}(N) = w$.

1) Show that $\mathbb{E}(S_N|N = n) = \mathbb{E}(S_n)$.

2) Deduce from it $\mathbb{E}(S_N)$ and $\text{Var}(S_N)$.