
Solutions: Markov Chains

Exercise 1. a) $\{0\}$ et $\{1\}$ are linked, thus the chain is irreducible. Since $E = \{0, 1\}$ is finite, the chain is recurrent.

b) The equation $\pi P = \pi$ can be rewritten :

$$\begin{cases} (1 - \alpha)\pi_0 + \beta\pi_1 = \pi_0 \\ \alpha\pi_0 + (1 - \beta)\pi_1 = \pi_1 \end{cases}$$

Considering the fact that $\pi_0 + \pi_1 = 1$, then

$$\boxed{\pi_0 = \frac{\beta}{\alpha + \beta} \text{ and } \pi_1 = \frac{\alpha}{\alpha + \beta}}$$

c) Let n be an integer. Then

$$\begin{aligned} \mathbb{P}_0(T_1 = n) &= \mathbb{P}_0(X_1 = 0; \dots; X_{n-1} = 0; X_n = 1) \\ &= P(0, 0)^{n-1} P(0, 1) \\ &= \alpha(1 - \alpha)^{n-1}. \end{aligned}$$

The same kind of calculation leads to

$$\mathbb{P}_0(T_0 = 1) = \mathbb{P}(X_1 = 0) = 1 - \alpha$$

and for all $n \geq 2$,

$$\mathbb{P}_0(T_0 = n) = \mathbb{P}_0(X_1 = 1; \dots; X_{n-1} = 1; X_n = 0) = \alpha(1 - \beta)^{n-2}\beta.$$

Then one can find the stationary probability with the following relation : for all $x \in \{0, 1\}$, $\pi(x) = \mathbb{E}_x(T_x)^{-1}$. Since

$$\begin{aligned} \mathbb{E}_0(T_0) &= \sum_{n=1}^{+\infty} n \mathbb{P}_0(T_0 = n) \\ &= \mathbb{P}_0(T_0 = 1) + \sum_{n=2}^{+\infty} n \mathbb{P}_0(T_0 = n) \\ &= 1 - \alpha + \sum_{n=2}^{+\infty} n \alpha (1 - \beta)^{n-2} \beta \\ &= 1 - \alpha + \alpha \beta \sum_{n=1}^{+\infty} (n + 1) (1 - \beta)^{n-1} \\ &= 1 - \alpha + \alpha \beta \left(\frac{1}{\beta^2} + \frac{1}{\beta} \right) \\ &= \frac{\alpha + \beta}{\beta}, \end{aligned}$$

we have $\pi_0 = \frac{\beta}{\alpha + \beta}$. The same calculation for $\mathbb{E}_0(T_0)$ brings $\pi_1 = \frac{\alpha}{\alpha + \beta}$.

d) A first approach for the calculation of P^n can be the diagonalization of P , but here, we will use the theorem of Cayley-Hamilton. The real 1 is an eigenvalue of P (this is true for any stochastic matrix).

Let λ be the other eigenvalue. Then,

$$\text{Tr}(P) = 1 + \lambda = 2 - \alpha - \beta.$$

Thus, $\lambda = 1 - \alpha - \beta$.

The characteristic polynom of P is

$$\chi_P(t) = (t - 1)(t - 1 + \alpha + \beta).$$

By euclidian division,

$$X^n = (X - 1)(X - 1 + \alpha + \beta)Q(X) + aX + b$$

where $a, b \in \mathbb{R}$ must be determined. Evaluating at 1 and at λ the above expression, we get

$$1 = a + b \quad \text{and} \quad (1 - \alpha - \beta)^n = a(1 - \alpha - \beta) + b.$$

Thus,

$$a = \frac{1 - (1 - \alpha - \beta)^n}{\alpha + \beta} \quad \text{and} \quad b = \frac{\alpha + \beta - 1 + (1 - \alpha - \beta)^n}{\alpha + \beta}.$$

Applying the Cayley-Hamilton theorem,

$$P^n = \chi_P(P)Q(P) + aP + b = aP + b = a(P - I_2) + I_2,$$

that is to say,

$$\begin{aligned} P^n &= \frac{1}{\alpha + \beta} \begin{pmatrix} \beta & \alpha \\ \beta & \alpha \end{pmatrix} + \frac{(1 - \alpha - \beta)^n}{\alpha + \beta} \begin{pmatrix} \alpha & -\alpha \\ -\beta & \beta \end{pmatrix} \\ &= \begin{pmatrix} \pi_0 & \pi_1 \\ \pi_0 & \pi_1 \end{pmatrix} + (1 - \alpha - \beta)^n \begin{pmatrix} \pi_1 & -\pi_1 \\ -\pi_0 & \pi_0 \end{pmatrix} \end{aligned}$$

$\rho = 1 - \alpha - \beta$ et $A = 1$ work.

Exercise 2. For all $n \geq 1$,

$$\{\tau_x = n\} = \{X_0 = x; \dots; X_{n-1} = x; X_n \neq x\} \in \sigma(X_0, \dots, X_n)$$

and

$$\{\tau_x = 0\} = \{X_0 \neq x\} \in \sigma(X_0).$$

Thus τ_x is a stopping time relative to the canonical filtration of $(X_n)_{n \geq 0}$. Using the definition of $\{\tau_x = n\}$,

$$\begin{aligned} \mathbb{P}_x(\tau_x = n) &= \sum_{y \neq x} \mathbb{P}_x(X_0 = x; \dots; X_{n-1} = x; X_n = y) \\ &= \sum_{y \neq x} P(x, x)^{n-1} P(x, y) \\ &= P(x, x)^{n-1} (1 - P(x, x)). \end{aligned}$$

When $P(x, x) > 0$ the random variable $\tau_x - 1$ have a geometric distribution $\mathcal{G}(P(x, x))$.

When $P(x, x) = 0$, $\tau_x = 1$ almost surely.

Exercise 3. a)

$$P = \begin{pmatrix} 1/2 & 1/2 \\ 3/4 & 1/4 \end{pmatrix}$$

b) The chain takes its values in $E = \{1, 2\}$. For all $n \geq 0$,

$$\begin{aligned} \mathbb{P}(X_{n+1} = 1) &= \mathbb{P}(X_{n+1} = 1; X_n = 1) + \mathbb{P}(X_{n+1} = 1; X_n = 2) \\ &= \mathbb{P}(X_n = 1)\mathbb{P}(X_{n+1} = 1|X_n = 1) + \mathbb{P}(X_n = 2)\mathbb{P}(X_{n+1} = 1|X_n = 2), \end{aligned}$$

thus

$$p_{n+1} = p_n P(1, 1) + (1 - p_n) P(2, 1) = \frac{3}{4} - \frac{1}{4} p_n.$$

Its fixed point is $l = \frac{3}{5}$ and so for all $n \geq 0$,

$$p_n = \frac{3}{5} + \left(\frac{-1}{4}\right)^n \left(p_0 - \frac{3}{5}\right)$$

and $\lim_{n \rightarrow +\infty} p_n = 3/5$.

c) The aim is to calculate $\mathbb{P}(X_{n+7} = 1|X_n = 1)$. Using the homogeneity assumption and assuming $p_0 = 1$, we have

$$\mathbb{P}(X_{n+7} = 1|X_n = 1) = \mathbb{P}(X_7 = 1|X_0 = 1) = P^7(1, 1) = p_7.$$

Thus,

$$\mathbb{P}(X_{n+7} = 1|X_n = 1) = \frac{3}{5} + \frac{2}{5} \left(\frac{-1}{4}\right)^7 \approx \frac{3}{5}.$$

d) The recurrence relation indicates that $p_n = 3/5$ implies $p_{n+1} = 3/5$.

e) Let C_n be the random cost in euros of the day n :

$$C_n = 10 \mathbf{1}_{\{X_{n-1}=1; X_n=1\}} + 25 \mathbf{1}_{\{X_{n-1}=1; X_n=2\}} + 15 \mathbf{1}_{\{X_{n-1}=2; X_n=1\}} + 20 \mathbf{1}_{\{X_{n-1}=2; X_n=2\}},$$

thus

$$C_n = 10p_{n-1}P(1, 1) + 25p_{n-1}P(1, 2) + 15(1 - p_{n-1})P(2, 1) + 20(1 - p_{n-1})P(2, 2).$$

If we are on an "asymptotic" day such that $p_n = 3/5$, then $\mathbb{E}(C_n) = 17 \text{ €}$.