HIDDEN FUNCTIONAL EQUATIONS FOR RANKIN-SELBERG INTEGRALS ASSOCIATED TO REAL QUADRATIC FIELDS

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David Lecomte
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I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Daniel Bump   Principal Adviser

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Benjamin Brubaker

I certify that I have read this dissertation and that, in my opinion, it is fully adequate in scope and quality as a dissertation for the degree of Doctor of Philosophy.

Paul Cohen

Approved for the University Committee on Graduate Studies.

Abstract

In 1981, Zagier explained how the Rankin-Selberg method, originally shown to work with $\mathrm{SL}_2(\mathbb{Z})$-automorphic forms that decay rapidly at infinity, can be naturally extended to automorphic forms that behave like $\sum_{i=1}^{\ell} y_i^{\alpha_i} \log^{n_i} y$. The technique used is called renormalization.

In this thesis, we identify the full group of functional equations for the renormalized Rankin-Selberg transform of a product of an Eisenstein series and a Hilbert modular Eisenstein series associated to a real quadratic field.

From Zagier's theory, 16 functional equations are trivially expected. The work presented here shows that this object has exactly 48 functional equations.

Preface

The present document is the final achievement of my work as a Ph.D. student at Stanford University and I take great pride in writing it.

In the winter quarter of 2004, my second year at Stanford, I was at a loss in my search for an advisor, as I could not find someone to work with, who met my interest for abstract functional analysis and Banach space theory. I finally decided to change fields, learn new material, and however difficult a decision this might have been at first, it brought me together with Professor BUMP.

While we had never met before, he welcomed me unquestioningly and proposed that I start learning automorphic forms. Having explained him my strong interest for "anything that mixes algebra and analysis", however vague that may be, Professor BUMP very promptly told me about a problem he might have for me.

Twenty years ago, with Professor GOLDFELD, in [4], he investigated the renormalized Rankin-Selberg transform

$$\mathrm{R}(s_0, s_1) = \mathrm{RN} \int\limits_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathscr{H}} \mathrm{E}^\star(z, s_0)\mathrm{E}^\star_{\mathrm{K}}(z, s_1)\,\frac{\mathrm{d}z}{y^2}$$

as defined by Zagier in [9], of a Hilbert modular Eisenstein series $\mathrm{E}^\star_{\mathrm{K}}(z, s_1)$ for a totally real cubic field. They were able to show that this function actually is a period of the $\mathrm{SL}_3(\mathbb{Z})$ Eisenstein series, and inherits from it functional equations that would be undetectable from Zagier's sole definition for R.

In 2002, Professors BEINEKE and BUMP considered this time the renormalized Rankin-Selberg transform of a product of three $\mathrm{SL}_2(\mathbb{Z})$ Eisenstein series and showed that such a phenomenon happened again.

My mission, would I accept it, would be to investigate the last remaining case

$$\mathrm{RN} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathscr{H}} \mathrm{E}^\star(z, s_0)\mathrm{E}^\star(z, s_1)\mathrm{E}^\star_{\mathrm{K}}(z, s_2)\, \frac{\mathrm{d}z}{y^2}$$

where K is a totally real quadratic field. The group of functional equations for this function is of order 48, while Zagier's theory only predicts 16. And this is the main result in the present thesis.

People who know me know that I like to present material that is self-contained. Doing this here would be a difficult task given the amount of prerequisites to understand what I have been working on. The volume, in the present thesis, devoted to prerequirements would also overwhelm the part where original work is actually presented. Thus I will usually direct the reader to the appropriate references. This to the exception of a couple chapters devoted to the Hilbert modular group and the Hilbert modular Eisenstein series.

Indeed, the only elementary reference that I found on the subject is Siegel's *Lectures Notes on Advanced Analytic Number Theory* [7]. This is a very old book, out of publication, very well written but still hard to read because typewritten. The parts relevant to the Eisenstein series are also pretty much scattered in many places throughout the book. It is after reading it that I was finally able to understand all objects involved in my problem, and therefore get started solving it. Since it meant so much to my work, I am reproducing the relevant parts here. The reader who is already comfortable with this material can skip directly to the third chapter, containing my original work.

Acknowledgements

First and foremost, my thanks go to my advisor, Daniel Bump. He was kind enough to accept me as a student, and was patient enough to let me learn at my pace. Which is very slow.

He is also so knowledgeable that it is frustrating at times. Like that one time when, with very rough knowledge of the details of what I was stuck on, he told me that at some point, I might need results from a paper written by Zagier [8]. Back then, this seemed unrelated to what I was specifically working on. But three months later, after finally solving my problem, I looked again at that paper and it turns out I was doing something described there, I had just failed to see how I could use it.

I would like to express my gratitude to my readers, Professors Ben Brubaker and Paul Cohen, and to Professors Mihran Papikian and Zhenia Khassina who did me the honor of being on my defense commitee.

Ultimately, my parents, Camille and Michel Lecomte are the reason I became interested in mathematics. When my brother and I were approximately 6 and 8, we were a pretty terrible bunch and my parents had to find a way to keep us quiet in the car when we would go on vacation. So they decided to teach us some math stuff and have us extract square roots and logarithms of the licence plate numbers from other cars. It worked, and from thereon, math has been the only discipline that appealed to me. Thanks to both of you for pushing me in this direction. Thank you also for coming all the way from France to attend my graduation.

Finally, I would like to thank my brother and close friends, simply for being my friends: Grégory, Olivier, Blandine, Vincent, Olivia, Julien, Ouille, Dana, Léo and Pikachu.

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Chapter 1

The Hilbert modular group

In this chapter, we present the theory of the Hilbert modular group, as a group acting on copies of the complex upper half-plane. This group generalizes the traditional modular group $\mathrm{SL}_2(\mathbb{Z})$, whose theory is presented for example in the beginning of Bump's *Automorphic Forms and Representations* [3]. We will define this action, study it and show that it actually is discontinuous, and identify fundamental domains for particular subgroups. Those results, already interesting as such, will be fundamental to defining the Hilbert modular Eisenstein series studied in the next chapter.

1.1 First notations

In this chapter, K is a totally real algebraic number field of degree n over $\mathbb{Q}$. This means that all the conjugates of K are actually subfields of $\mathbb{R}$. For example, $\mathbb{Q}(\sqrt{2})$ is a totally real quadratic field since its only conjugate is itself; while $\mathbb{Q}(\sqrt[3]{2})$ is not a totally real cubic field, since its conjugates are $\mathbb{Q}(\sqrt[3]{2})$, $\mathbb{Q}(\mathrm{j}\sqrt[3]{2})$ and $\mathbb{Q}(\mathrm{j}^2\sqrt[3]{2})$ with $\mathrm{j} = \mathrm{e}^{2\pi\mathrm{i}/3}$. If α is in K, its conjugates are denoted by $\alpha^{(1)} = \alpha, \alpha^{(2)}, \ldots, \alpha^{(n)}$.

For our purposes, we need to add a point at infinity to K, thus forming

$$\widehat{\mathrm{K}} = \mathrm{K} \cup \{\infty\}$$

as a subset of the completed complex plane $\widehat{\mathbb{C}}$. Operations in $\widehat{\mathrm{K}}$ between elements of K work just as in K. And the element ∞ satisfies the following conditions:

$$\begin{aligned} \forall a \in \mathbb{C} \qquad & a + \infty = \infty \\ \forall a \in \mathbb{C} \setminus \{0\} \qquad & a \times \infty = \infty \\ & \infty \times \infty = \infty \\ & \frac{1}{0} = \infty \\ & \frac{1}{\infty} = 0 \\ \forall i \in \{1, \dots, n\} \qquad & \infty^{(i)} = \infty \end{aligned}$$

$\widehat{\mathrm{K}}$ can be embedded in n copies of the extended complex plane as follows:

$$\begin{aligned} \Phi : \quad \widehat{\mathrm{K}} & \longrightarrow \widehat{\mathbb{C}}^n \\ x & \longmapsto \Phi x = \begin{bmatrix} x^{(1)} \\ \vdots \\ x^{(n)} \end{bmatrix} \end{aligned}$$

An element x in K will often be identified with its image by Φ in what follows.

In $\widehat{\mathbb{C}}^n$, we define the following maps :

$$\forall \mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \in \widehat{\mathbb{C}}^n \qquad \operatorname{Tr} \mathbf{z} = z_1 + \cdots + z_n \qquad \text{and} \qquad \mathbb{N}(\mathbf{z}) = z_1 \cdots z_n$$

so that $\operatorname{Tr} \circ \Phi$ and $\mathbb{N} \circ \Phi$ are respectively the usual trace and norm on $\widehat{\mathrm{K}}$. These new notations are thus consistent with the identification we make between $x \in \mathrm{K}$ and $\Phi(x)$.

Now, if $\mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$ is in $\widehat{\mathbb{C}}^n$, we will note :

$$\mathbf{x} = \operatorname{Re} \mathbf{z} = \begin{bmatrix} \operatorname{Re} z_1 \\ \vdots \\ \operatorname{Re} z_n \end{bmatrix} \qquad \text{and} \qquad \mathbf{y} = \operatorname{Im} \mathbf{z} = \begin{bmatrix} \operatorname{Im} z_1 \\ \vdots \\ \operatorname{Im} z_n \end{bmatrix}$$

Let $\mathscr{G}$ be the group of 2×2 matrices with coefficients in K and determinant 1. $\mathscr{G}$ naturally acts on $\widehat{\mathscr{H}^n}$, the n-fold product of copies of the extended upper-half plane $\widehat{\mathscr{H}}$ as follows:

$$\forall \mathrm{M} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \quad \forall \mathbf{z} \in \mathscr{H}^n \qquad \mathrm{M}\mathbf{z} = \begin{bmatrix} \dfrac{\alpha^{(1)} z_1 + \beta^{(1)}}{\gamma^{(1)} z_1 + \delta^{(1)}} \\ \vdots \\ \dfrac{\alpha^{(n)} z_n + \beta^{(n)}}{\gamma^{(n)} z_n + \delta^{(n)}} \end{bmatrix} = \mathbf{x}_\mathrm{M} + \mathrm{i}\mathbf{y}_\mathrm{M}$$

and

$$\mathrm{M}\infty = \begin{bmatrix} \dfrac{\alpha^{(1)}}{\gamma^{(1)}} \\ \vdots \\ \dfrac{\alpha^{(n)}}{\gamma^{(n)}} \end{bmatrix}$$

Finally, the *Hilbert modular group* is the set of matrices in $\mathscr{G}$ with coefficients in $\mathfrak{o}$, quotiented by the subgroup $\{\pm \mathrm{I}_2\}$ of matrices that act trivially on $\mathscr{H}^n$. This group will be noted $\mathscr{M} = \mathrm{SL}_2(\mathfrak{o})/\{\pm \mathrm{I}_2\}$.

1.2 Cusps of $\mathscr{M}$ and class number

Definition 1 Two elements λ and μ in $\widehat{\mathrm{K}}$ are equivalent, and we note $\lambda \sim \mu$, if and only if

$$\exists \mathrm{M} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathscr{M} \qquad \mu = \mathrm{M}\lambda = \frac{\alpha\lambda + \beta}{\gamma\lambda + \delta}$$

A cusp for $\mathscr{M}$ is an equivalence class in $\widehat{\mathrm{K}}$ for $\sim$, and their collection is denoted by $\mathscr{C}$.

Theorem 2
The number of cusps is the class number of K.

Proof: Let $\mathcal{I}$ be the group of fractional ideals of $\mathfrak{o}$ and $\mathcal{P}$ be the group of principal ideals. We know, from basic theory of Dedekind domains, that any fractional ideal $\mathfrak{a}$ can be generated by two elements.

Our first step is to show that if $\mathfrak{a} = \langle \alpha, \beta \rangle = \langle \alpha^\star, \beta^\star \rangle$, then $\frac{\alpha}{\beta} \sim \frac{\alpha^\star}{\beta^\star}$. We know that there exist elements γ, δ, $\gamma^\star$, $\delta^\star$ in $\mathfrak{a}^{-1}$ such that

$$\alpha\gamma - \beta\delta = 1 \qquad \text{and} \qquad \alpha^\star\gamma^\star - \beta^\star\delta^\star = 1$$

Define then $$\mathrm{A} = \begin{bmatrix} \alpha & \delta \\ \beta & \gamma \end{bmatrix} \qquad \text{and} \qquad \mathrm{A}^\star = \begin{bmatrix} \alpha^\star & \delta^\star \\ \beta^\star & \gamma^\star \end{bmatrix}$$

These matrices have coefficients in K and determinant 1. Thus

$$\mathrm{A}^\star \mathrm{A}^{-1} = \begin{bmatrix} \alpha^\star & \delta^\star \\ \beta^\star & \gamma^\star \end{bmatrix} \begin{bmatrix} \gamma & -\delta \\ -\beta & \alpha \end{bmatrix} = \begin{bmatrix} \alpha^\star\gamma - \delta^\star\beta & \delta^\star\alpha - \alpha^\star\delta \\ \beta^\star\gamma - \gamma^\star\beta & \gamma^\star\alpha - \beta^\star\delta \end{bmatrix}$$

has determinant 1. Remember that the α's and β's are in $\mathfrak{a}$, while the γ's and δ's are in $\mathfrak{a}^{-1}$. Thus $\mathrm{A}^\star\mathrm{A}^{-1}$ has coefficients in $\mathfrak{o}$ and is in the Hilbert modular group $\mathscr{M}$. It is now a simple computation to check that

$$\mathrm{A}^\star \mathrm{A}^{-1} \frac{\alpha}{\beta} = \frac{\alpha^\star}{\beta^\star}$$

which proves that $$\frac{\alpha}{\beta} \sim \frac{\alpha^\star}{\beta^\star}$$

Thus the function $\varphi : \mathcal{I} \longrightarrow \mathscr{C}$, that associates to a fractional ideal $\mathfrak{a} = \langle \alpha, \beta \rangle$ the cusp $\frac{\alpha}{\beta}$, is well defined: its value at an ideal $\mathfrak{a}$ does not depend on the choice of generators for this ideal.

Next, we check that φ induces actually a function on $\mathcal{I}/\mathcal{P}$. Indeed, if A is an ideal class represented by fractional ideals $\mathfrak{a}$ or $\mathfrak{b}$, there exists an integer c such that $\mathfrak{b} = c\mathfrak{a}$. If (α, β) is a set of generators for $\mathfrak{a}$, then $\mathfrak{b}$ is generated by $(c\alpha, c\beta)$. Thus

$$\varphi(\mathfrak{b}) = \frac{c\alpha}{c\beta} = \frac{\alpha}{\beta} = \varphi(\mathfrak{a})$$

So φ projects to a function $\overline{\varphi} : \mathcal{I}/\mathcal{P} \longrightarrow \mathscr{C}$, by defining $\overline{\varphi}(\mathrm{A})$ to be $\varphi(\mathfrak{a})$ for any fractional ideal $\mathfrak{a} \in \mathrm{A}$.

Of course, $\overline{\varphi}$ is surjective since if λ is a cusp represented by the algebraic number $\frac{\alpha}{\beta}$, then $\overline{\varphi}(\mathrm{A}) = \lambda$ where A is the ideal class of $\langle \alpha, \beta \rangle$. If $\lambda = \infty$, just take $\mathrm{A} = \mathcal{P}$.

The last step consists in checking that $\overline{\varphi}$ is injective. Suppose that two fractional ideals $\mathfrak{a}$ and $\mathfrak{a}^\star$ are such that $\varphi(\mathfrak{a})$ and $\varphi(\mathfrak{a}^\star)$ define the same cusp. Let (ρ, σ) and $(\rho^\star, \sigma^\star)$ be generators for $\mathfrak{a}$ and $\mathfrak{a}^\star$ respectively. Because the cusps $\frac{\rho}{\sigma}$ and $\frac{\rho^\star}{\sigma^\star}$ are the same,

$$\exists \mathrm{M} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathscr{M} \qquad \frac{\rho^\star}{\sigma^\star} = \mathrm{M}\frac{\rho}{\sigma} = \frac{\alpha\frac{\rho}{\sigma} + \beta}{\gamma\frac{\rho}{\sigma} + \delta} = \frac{\alpha\rho + \beta\sigma}{\gamma\rho + \delta\sigma}$$

This is turn implies that, for some algebraic number $c \in \mathrm{K}$, we have

$$\rho^\star = c(\alpha\rho + \beta\sigma) \qquad \text{and} \qquad \sigma^\star = c(\gamma\rho + \delta\sigma) \tag{1}$$

Of course, the ideal $\langle \alpha\rho + \beta\sigma, \gamma\rho + \delta\sigma \rangle$ is included in $\langle \rho, \sigma \rangle = \mathfrak{a}$. Conversely, because $\alpha\delta - \beta\gamma = 1$, we have

$$\begin{cases} \rho = \delta(\alpha\rho + \beta\sigma) - \beta(\gamma\rho + \delta\sigma) \\ \sigma = -\gamma(\alpha\rho + \beta\sigma) + \alpha(\gamma\rho + \delta\sigma) \end{cases}$$

which proves that $\mathfrak{a} = \langle \rho, \sigma \rangle = \langle \alpha\rho + \beta\sigma, \gamma\rho + \delta\sigma \rangle$. This, together with (**1**), shows that $\langle \rho^\star, \sigma^\star \rangle = c\langle \rho, \sigma \rangle$, or in other words, $\mathfrak{a}^\star$ and $\mathfrak{a}$ are in the same ideal class. Thus $\overline{\varphi}$ is injective. □

We now know that $\mathscr{M}$ has finitely many cusps, since the class number of K is finite. Let h be this number. We fix for the remainder of the chapter a complete set of representatives $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ of ideal classes, such that $\mathfrak{a}_i$ has minimum norm in its class, and note the corresponding cusps $\lambda_1, \ldots, \lambda_h$. We convene that $\mathfrak{a}_1 = \mathfrak{o}$, so that $\lambda_1 = \infty$.

Also, each $\mathfrak{a}_i$ can be written as $\langle \rho_i, \sigma_i \rangle$ and we have $\lambda_i \sim \frac{\rho_i}{\sigma_i}$. Because ρ_i and σ_i are in $\mathfrak{a}_i$, there exist ξ_i, η_i in $\mathfrak{a}_i^{-1}$ such that $\rho_i\eta_i - \sigma_i\xi_i = 1$. Those algebraic numbers provide us with a matrix

$$\mathrm{A}_i = \begin{bmatrix} \rho_i & \xi_i \\ \sigma_i & \eta_i \end{bmatrix} \in \mathrm{SL}_2(\mathrm{K})$$

1.3 The stabilizer of algebraic numbers

For $\lambda \in \widehat{\mathrm{K}}$, we want to identify its stabilizer Γ_λ in $\mathscr{M}$, that is

$$\Gamma_\lambda = \left\{ \mathrm{M} \in \mathscr{M} \;\middle|\; \mathrm{M}\lambda = \lambda \right\}$$

Since $\lambda_1, \ldots, \lambda_h$ are a complete set of representatives for $\mathrm{K}/\sim$, there exists $\mathrm{M} \in \mathscr{M}$ and $i \in \{1, \ldots, h\}$ such that $\lambda = \mathrm{M}\lambda_i$. Thus

$$\Gamma_\lambda = \left\{ \mathrm{N} \in \mathscr{M} \;\middle|\; \mathrm{N}\mathrm{M}\lambda_i = \mathrm{M}\lambda_i \right\} = \left\{ \mathrm{N} \in \mathscr{M} \;\middle|\; \mathrm{M}^{-1}\mathrm{N}\mathrm{M}\lambda_i = \lambda_i \right\} = \mathrm{M}\Gamma_{\lambda_i}\mathrm{M}^{-1}$$

and it is sufficient to know $\Gamma_{\lambda_1}, \ldots, \Gamma_{\lambda_n}$ in order to know all Γ_λ's.

Theorem 3
If $i \in \{1, \ldots, h\}$,

$$\Gamma_{\lambda_i} = \left\{ \mathrm{A}_i \begin{bmatrix} \varepsilon & \zeta\varepsilon^{-1} \\ 0 & \varepsilon^{-1} \end{bmatrix} \mathrm{A}_i^{-1} \;\middle|\; \varepsilon \in \mathfrak{o}^\times \quad \zeta \in \mathfrak{a}_i^{-2} \right\}$$

Proof: Let
$$\mathrm{M} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \Gamma_{\lambda_i}$$

Then
$$\frac{\alpha\rho_i + \beta\sigma_i}{\gamma\rho_i + \delta\sigma_i} = \frac{\rho_i}{\sigma_i}$$

and
$$\sigma_i(\alpha\rho_i + \beta\sigma_i) = \rho_i(\gamma\rho_i + \delta\sigma_i)$$

Just like in the proof of **Theorem 2**, the fact that M has determinant 1 and coefficients in $\mathfrak{o}$ implies that $(\alpha\rho_i + \beta\sigma_i, \gamma\rho_i + \delta\sigma_i)$ is another system of generators for $\mathfrak{a}_i$:

$$\mathfrak{a}_i = \langle \rho_i, \sigma_i \rangle = \langle \alpha\rho_i + \beta\sigma_i, \gamma\rho_i + \delta\sigma_i \rangle \tag{2}$$

This implies that $\frac{\langle \rho_i \rangle}{\mathfrak{a}_i}$ and $\frac{\langle \sigma_i \rangle}{\mathfrak{a}_i}$ are relatively prime; as well as $\frac{\langle \alpha\rho_i + \beta\sigma_i \rangle}{\mathfrak{a}_i}$ and $\frac{\langle \gamma\rho_i + \delta\sigma_i \rangle}{\mathfrak{a}_i}$.

> This comes again from the theory of Dedekind domains, and I believe a quick proof is required. The fundamental relationship needed is a consequence of the unique factorization of ideals as a product of primes:
>
> $$\forall \mathfrak{a}, \mathfrak{b} \in \mathcal{I} \quad \forall \mathfrak{p} \text{ prime} \qquad v_\mathfrak{p}(\mathfrak{a} + \mathfrak{b}) = \mathrm{Min}\big(v_\mathfrak{p}(\mathfrak{a}), v_\mathfrak{p}(\mathfrak{b})\big) \tag{3}$$
>
> where $v_\mathfrak{p}$ is the $\mathfrak{p}$-adic valuation. This can be found in Frohlich's *Algebraic Number Theory* [5], for example. Because $\mathfrak{a} \subset \mathfrak{a} + \mathfrak{b}$, we have for every prime $\mathfrak{p}$:
>
> $$v_\mathfrak{p}\Big(\frac{\mathfrak{a}}{\mathfrak{a} + \mathfrak{b}}\Big) \geqslant 0$$
>
> Similarly
> $$v_\mathfrak{p}\Big(\frac{\mathfrak{b}}{\mathfrak{a} + \mathfrak{b}}\Big) \geqslant 0$$
>
> But also, because of (**3**),
>
> $$v_\mathfrak{p}(\mathfrak{a} + \mathfrak{b}) = v_\mathfrak{p}(\mathfrak{a}) \text{ or } v_\mathfrak{p}(\mathfrak{b})$$
>
> Thus
> $$v_\mathfrak{p}\Big(\frac{\mathfrak{a}}{\mathfrak{a} + \mathfrak{b}}\Big) \cdot v_\mathfrak{p}\Big(\frac{\mathfrak{b}}{\mathfrak{a} + \mathfrak{b}}\Big) = 0$$
>
> This being true for every prime $\mathfrak{p}$ implies that $\frac{\mathfrak{a}}{\mathfrak{a}+\mathfrak{b}}$ and $\frac{\mathfrak{b}}{\mathfrak{a}+\mathfrak{b}}$ are relatively prime.

Dividing both sides of (**2**) by $\mathfrak{a}_i^2$ yields

$$\frac{\langle\sigma_i\rangle}{\mathfrak{a}_i}\frac{\langle\alpha\rho_i+\beta\sigma_i\rangle}{\mathfrak{a}_i}=\frac{\langle\rho_i\rangle}{\mathfrak{a}_i}\frac{\langle\gamma\rho_i+\delta\sigma_i\rangle}{\mathfrak{a}_i}$$

The coprimality relationships stated above imply that

$$\frac{\langle\sigma_i\rangle}{\mathfrak{a}_i}=\frac{\langle\gamma\rho_i+\delta\sigma_i\rangle}{\mathfrak{a}_i}\qquad\text{and}\qquad\frac{\langle\rho_i\rangle}{\mathfrak{a}_i}=\frac{\langle\alpha\rho_i+\beta\sigma_i\rangle}{\mathfrak{a}_i}$$

and in turn, from this and the equality

$$\sigma_i(\alpha\rho_i+\beta\sigma_i)=\rho_i(\gamma\rho_i+\delta\sigma_i)$$

presented earlier, there exists a unit ε such that

$$\alpha\rho_i+\beta\sigma_i=\varepsilon\rho_i\qquad\text{and}\qquad\gamma\rho_i+\delta\sigma_i=\varepsilon\sigma_i$$

Therefore $$\mathrm{MA}_i=\begin{bmatrix}\alpha&\beta\\\gamma&\delta\end{bmatrix}\begin{bmatrix}\rho_i&\xi_i\\\sigma_i&\eta_i\end{bmatrix}=\begin{bmatrix}\varepsilon\rho_i&\alpha\xi_i+\beta\eta_i\\\varepsilon\sigma_i&\gamma\xi_i+\delta\eta_i\end{bmatrix}$$

Defining $\quad\xi_i^\star=\varepsilon(\alpha\xi_i+\beta\eta_i)\in\mathfrak{a}_i^{-1}\quad$ and $\quad\eta_i^\star=\varepsilon(\gamma\xi_i+\delta\eta_i)\in\mathfrak{a}_i^{-1}$

we have $$\mathrm{MA}_i=\begin{bmatrix}\rho_i&\xi_i^\star\\\sigma_i&\eta_i^\star\end{bmatrix}\begin{bmatrix}\varepsilon&0\\0&\varepsilon^{-1}\end{bmatrix}\tag{4}$$

Furthermore $$\rho_i\eta_i^\star-\sigma_i\xi_i^\star=1$$

Since we had already $$\rho_i\eta_i-\sigma_i\xi_i=1$$

it follows that $$\rho_i(\eta_i-\eta_i^\star)=\sigma_i(\xi_i-\xi_i^\star)\tag{5}$$

or, in terms of ideals, $$\frac{\langle\rho_i\rangle}{\mathfrak{a}_i}\frac{\langle\eta_i-\eta_i^\star\rangle}{\mathfrak{a}_i^{-1}}=\frac{\langle\sigma_i\rangle}{\mathfrak{a}_i}\frac{\langle\xi_i-\xi_i^\star\rangle}{\mathfrak{a}_i^{-1}}$$

But $\frac{\langle\rho_i\rangle}{\mathfrak{a}_i}$ and $\frac{\langle\sigma_i\rangle}{\mathfrak{a}_\mathrm{i}}$ are relatively prime, therefore

$$\frac{\langle\rho_i\rangle}{\mathfrak{a}_i}\Big|\frac{\langle\xi_i-\xi_i^\star\rangle}{\mathfrak{a}_i^{-1}}\qquad\text{and}\qquad\frac{\langle\sigma_i\rangle}{\mathfrak{a}_i}\Big|\frac{\langle\eta_i-\eta_i^\star\rangle}{\mathfrak{a}_i^{-1}}$$

In particular $\quad\exists\mathfrak{b}$ integral ideal $\quad\langle\xi_i-\xi_i^\star\rangle\subset\mathfrak{a}_i^{-2}\langle\rho_i\rangle\mathfrak{b}\subset\mathfrak{a}_i^{-2}\langle\rho_i\rangle$

and it follows that $\quad\exists\zeta\in\mathfrak{a}_i^{-2}\qquad\xi_i^\star=\xi_i+\zeta\rho_i$

Because of (**5**), we get as well

$$\eta_i^\star = \eta_i + \zeta\sigma_i$$

Putting this information back in (**4**):

$$\mathrm{MA}_i = \begin{bmatrix} \rho_i & \xi_i + \zeta\rho_i \\ \sigma_i & \eta_i + \zeta\sigma_i \end{bmatrix} \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix} = \begin{bmatrix} \rho_i & \xi_i \\ \sigma_i & \eta_i \end{bmatrix} \begin{bmatrix} 1 & \zeta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix} = \mathrm{A}_i \begin{bmatrix} \varepsilon & \zeta\varepsilon^{-1} \\ 0 & \varepsilon^{-1} \end{bmatrix}$$

and
$$\mathrm{M} = \mathrm{A}_i \begin{bmatrix} \varepsilon & \zeta\varepsilon^{-1} \\ 0 & \varepsilon^{-1} \end{bmatrix} \mathrm{A}_i^{-1}$$

Conversely, one easily checks by computation that such a matrix M, with $\varepsilon \in \mathfrak{o}^\times$ and $\zeta \in \mathfrak{a}_i^{-2}$ stabilizes $\lambda_i = \frac{\rho_i}{\sigma_i}$, has integral coefficients and determinant 1. This achieves the proof of **Theorem 3**. $\square$

1.4 Discontinuity of the action of $\mathscr{M}$

We investigate in more details the action of $\mathscr{M}$ on the n-fold upper half-plane $\mathscr{H}^n$ defined in the first section of this chapter. We will find properties very similar to those of $\mathrm{SL}_2(\mathbb{Z})$ acting on $\mathscr{H}$, which should help us describe a fundamental domain. First, let's define some notations.

Definition 4 If V is a subset of $\mathscr{H}^n$ and M is a Hilbert modular transformation, V_M is the image M(V) of V under M.

Definition 5 For any $\mathbf{z} \in \mathscr{H}^n$, the *isotropy group* of $\mathbf{z}$ is the subgroup $\Gamma_\mathbf{z}$ of $\mathscr{M}$, consisting of those Hilbert modular transformations fixing $\mathbf{z}$.

We start with a functional analysis lemma:

Lemma 6
Let B *be a compact subset of the upper half-plane* $\mathscr{H}$. *There exist positive numbers* c_1, c_2, *depending only on* B, *such that*

$$\forall (u,v) \in \mathbb{R}^2 \qquad c_1\big(u^2+v^2\big) \leqslant |uz+v|^2 \leqslant c_2\big(u^2+v^2\big)$$

Proof: Let S be the unit sphere in $\mathbb{R}^2$ for the euclidean norm:

$$\mathrm{S} = \left\{(u,v) \in \mathbb{R}^2 \mid u^2+v^2=1\right\}$$

and define $$\forall z \in \mathrm{B} \quad \forall (u,v) \in \mathrm{S} \qquad \varphi(u,v,z) = |uz+v|^2$$

This function is continuous on the compact set $\mathrm{B} \times \mathrm{S}$ and therefore reaches its minimum and maximum, that we call c_1 and c_2. These numbers are nonnegative since φ is a nonnegative function. Because c_1 is attained, there exist $z \in \mathrm{B}$, $(u,v) \in \mathrm{S}$ so that

$$c_1 = |uz+v|^2$$

If $c_1 = 0$, then $uz+v=0$. Writing $z = x + \mathrm{i}y$, it follows that

$$ux+v=0 \qquad \text{and} \qquad uy=0$$

B is a compact subset of the upper half-plane, therefore stays away from the real axis. Which implies that $y \neq 0$, $u=0$ and $v=0$. This contradicts the fact that $u^2+v^2=1$. Therefore, $c_1 > 0$ and as a consequence, $c_2 > 0$ as well.

Now, let $(u,v) \neq 0$ be any couple of real numbers. Then $\frac{(u,v)}{\sqrt{u^2+v^2}}$ is in S and therefore

$$c_1 \leqslant \frac{|uz+v|^2}{u^2+v^2} \leqslant c_2$$

which proves the lemma. □

Theorem 7
Let B *and* B′ *be any two compact sets in* $\mathscr{H}^n$. *There are only finitely many Hilbert modular transformations* M, *such that* $\mathrm{B_M} \cap \mathrm{B}' \neq \emptyset$.

Proof: Since B and B′ are closed and bounded in $\mathscr{H}^n$, there exists a positive real number c big enough such that

$$\forall \mathbf{z} \in \mathrm{B} \quad \forall j \in \{1,\ldots,n\} \qquad \frac{1}{c} < y_j < c \tag{6}$$

and $$\forall \mathbf{z} \in \mathrm{B}' \quad \forall j \in \{1,\ldots,n\} \qquad \frac{1}{c} < y_j < c \tag{7}$$

Let
$$\Lambda = \{\mathrm{M} \in \mathscr{M} \mid \mathrm{B_M} \cap \mathrm{B}' \neq \emptyset\}$$
and take
$$\mathrm{M} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \Lambda$$
There exists $\mathbf{z} \in \mathrm{B}$ such that $\mathrm{M}\mathbf{z} \in \mathrm{B}'$. Then we have in particular:
$$\frac{1}{c} < \mathrm{Im}\,(\mathrm{M}\mathbf{z})_1 = \frac{y_1}{\left|\gamma^{(1)} z_1 + \delta^{(1)}\right|^2} < c$$
Thus
$$\frac{y_1}{c} < |\gamma z_1 + \delta|^2 < y_j c$$
and by (**6**)
$$\frac{1}{c^2} < |\gamma z_1 + \delta|^2 < c^2$$
Since z_1 is in a compact subset of the upper half-plane, **Lemma 6** implies that there exists a positive constant c_1, depending only on B, such that
$$c_1\left(\gamma^2 + \delta^2\right) \leqslant |\gamma z_1 + \delta|^2 \leqslant c^2$$
Hence
$$\gamma^2 + \delta^2 \leqslant \frac{c^2}{c_1}$$
But γ and δ are in $\mathfrak{o}$, which is discrete. So there are only finitely many possibilities for those two integers. What we showed so far can be summarized as: *Let* B, B' *be two compact sets in* $\mathscr{H}^n$*. There are only finitely many possibilities for the bottom rows of matrices* $\mathrm{M} \in \mathscr{M}$ *such that* $\mathrm{B_M} \cap \mathrm{B}' \neq \emptyset$*.*

Let's show now that the same happens with the top row. Let $\mathrm{I} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. This matrix is in $\mathscr{M}$ and the sets $\mathrm{B_I}$ and B'_I are both compact. Consider
$$\Lambda_\mathrm{I} = \{\mathrm{M} \in \mathscr{M} \mid \mathrm{B_{MI}} \cap \mathrm{B}'_\mathrm{I} \neq \emptyset\}$$
We know then that bottom rows of elements of Λ_I are taken from a finite set. But, using the fact that $\mathrm{I}^{-1} = \mathrm{I}$,
$$\Lambda_\mathrm{I} = \{\mathrm{M} \in \mathscr{M} \mid \mathrm{B_{IMI}} \cap \mathrm{B}' \neq \emptyset\} = \mathrm{I}\Lambda\mathrm{I}$$
Observe that
$$\forall \mathrm{M} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \Lambda \qquad \mathrm{IMI} = \begin{bmatrix} -\delta & \gamma \\ \beta & -\alpha \end{bmatrix}$$

Thus, up to some sign changes, bottom rows of elements of Λ_{I} are top rows of elements of Λ. Hence there are, as well, only finitely many possibilities for the top rows of elements of Λ. □

Corollary 8

For every $\boldsymbol{z} \in \mathscr{H}^n$, the isotropy group $\Gamma_{\boldsymbol{z}}$ is finite.

Proof: $\Gamma_{\mathbf{z}}$ is the set of $\mathrm{M} \in \mathscr{M}$ such that $\{\mathbf{z}\}_{\mathrm{M}} = \{\mathbf{z}\}$, thus finite by **Theorem 7**.□

Theorem 9

The Hilbert modular group $\mathscr{M}$ acts properly discontinuously on the n-fold upper half-plane $\mathscr{H}^n$. In other words, for any $\boldsymbol{z} \in \mathscr{H}^n$, there exists a neighbourhood U of $\boldsymbol{z}$ such that the set of $\mathrm{M} \in \mathscr{M}$ such that $\mathrm{U}_{\mathrm{M}} \cap \mathrm{U} \neq \emptyset$ is the isotropy group $\Gamma_{\boldsymbol{z}}$.

Proof: Let V be a relatively compact neighbourhood of $\mathbf{z}$. By **Theorem 7**, the set Λ of matrices $\mathrm{M} \in \mathscr{M}$ such that $\mathrm{V}_{\mathrm{M}} \cap \mathrm{V} \neq \emptyset$ is finite. Note that Λ also contains $\Gamma_{\mathbf{z}}$, so let's write Λ as a disjoint union

$$\Lambda = \{\mathrm{M}_1, \ldots, \mathrm{M}_r\} \cup \Gamma_{\mathbf{z}} \tag{8}$$

Suppose that, for some i, there is no neighbourhood O of $\mathbf{z}$ such that $\mathrm{O}_{\mathrm{M}_i} \cap \mathrm{O} = \emptyset$. Then

$$\forall p \in \mathbb{N}^\star \qquad \mathcal{B}\Big(\mathbf{z}, \frac{1}{p}\Big)_{\mathrm{M}_i} \cap \mathcal{B}\Big(\mathbf{z}, \frac{1}{p}\Big) \neq \emptyset$$

and thus contains a point $\mathbf{w}_p = \mathrm{M}_i(\mathbf{z}_p)$ with

$$\|\mathbf{z}_p - \mathbf{z}\| \leqslant \frac{1}{p} \qquad \text{and} \qquad \|\mathbf{w}_p - \mathbf{z}\| \leqslant \frac{1}{p}$$

Since M_i is continuous, letting p go to ∞ yields $\mathbf{z} = \mathrm{M}_i(\mathbf{z})$, which contradicts the fact that M_i is not $\mathbf{z}$-isotropic.

Hence, for every i, there is a neighbourhood W_i of $\mathbf{z}$ such that $\mathrm{M}_i(\mathrm{W}_i) \cap \mathrm{W}_i = \emptyset$. Then $\mathrm{W} = \mathrm{W}_1 \cap \cdots \cap \mathrm{W}_r$ is a neighbourhood of $\mathbf{z}$ such that

$$\forall i \in \{1, \ldots, r\} \qquad \mathrm{W}_{\mathrm{M}_i} \cap \mathrm{W} = \emptyset$$

Finally, let $\mathrm{U} = \mathrm{V} \cap \mathrm{W}$. This is a neighbourhood of $\mathbf{z}$. Let $\mathrm{M} \in \mathscr{M}$ be such that $\mathrm{U}_{\mathrm{M}} \cap \mathrm{U} \neq \emptyset$. Then $\mathrm{V}_{\mathrm{M}} \cap \mathrm{V} \neq \emptyset$, so M is in Λ. Given the disjoint union (**8**), either $\mathrm{M} \in \Gamma_{\mathbf{z}}$ or $\mathrm{M} = \mathrm{M}_i$ for some i. The latter case is impossible, since $\mathrm{U} \subset \mathrm{W}$ and $\mathrm{W}_{\mathrm{M}_i} \cap \mathrm{W} = \emptyset$. Therefore M is $\mathbf{z}$-isotropic and U satisfies the requirements for the theorem. $\square$

Corollary 10

If $\boldsymbol{z} \in \mathscr{H}^n$ is such that $\Gamma_{\boldsymbol{z}} = \{\mathrm{I}_2\}$, there exists a neighbourhood U *of $\boldsymbol{z}$ such that*

$$\forall \mathrm{M} \in \mathscr{M} \setminus \{\mathrm{I}_2\} \qquad \mathrm{U}_{\mathrm{M}} \cap \mathrm{U} = \emptyset$$

1.5 A fundamental domain for Γ_λ

Let $\lambda = \frac{\rho}{\sigma}$ represent a cusp of $\mathscr{H}^n$ for the Hilbert modular group $\mathscr{M}$, and $\mathfrak{a} = \langle \rho, \sigma \rangle$ be the ideal, among $\mathfrak{a}_1, \ldots, \mathfrak{a}_h$ representing λ. The fractional ideal $\mathfrak{a}^{-2}$ is a free $\mathbb{Z}$-module of rank n and therefore has $\mathbb{Z}$-bases; let $(\alpha_1, \ldots, \alpha_n)$ be one of them.

As explained in **section 1.2**, we associate with the cusp λ a matrix $\mathrm{A} = \begin{bmatrix} \rho & \xi \\ \sigma & \eta \end{bmatrix}$ of determinant 1 with ξ and η in $\mathfrak{a}^{-1}$.

As a consequence of the units theorem, applied to the special case of totally real fields, there exist *fundamental units* $\varepsilon_1, \ldots, \varepsilon_{n-1}$ in $\mathfrak{o}^\times$, such that

$$\mathfrak{o}^\times = \left\{ \pm \varepsilon_1^{k_1} \cdots \varepsilon_{n-1}^{k_{n-1}} \mid k_1, \ldots, k_{n-1} \in \mathbb{Z} \right\}$$

In this section, we will construct a fundamental domain in $\mathscr{H}^n$ for the group Γ_λ identified in **section 1.3**. This set will be easier to describe in a different coordinate system, which we shall present now.

Let $\mathbf{z} = (z_1, \ldots, z_n)$ be any point in $\mathscr{H}^n$. We define $\mathbf{z}^\star = \mathbf{x}^\star + \mathrm{i}\mathbf{y}^\star$ to be simply $\mathbf{z}_{\mathrm{A}^{-1}}$ and we write

$$\mathbf{z}^\star = (z_1^\star, \ldots, z_n^\star) \qquad \mathbf{x}^\star = (x_1^\star, \ldots, x_n^\star) \qquad \mathbf{y}^\star = (y_1^\star, \ldots, y_n^\star)$$

We define the local coordinates of $\mathbf{z}$ at λ to be the $2n$ quantities

$$q, \mathrm{Y}_1, \ldots, \mathrm{Y}_{n-1}, \mathrm{X}_1, \ldots, \mathrm{X}_n$$

determined by the relationships

$$q = \mathbb{N}(\mathbf{y}^\star)$$

$$\begin{bmatrix} \ln\left|\varepsilon_1^{(1)}\right| & \cdots & \ln\left|\varepsilon_{n-1}^{(1)}\right| \\ \vdots & & \vdots \\ \ln\left|\varepsilon_1^{(n-1)}\right| & \cdots & \ln\left|\varepsilon_{n-1}^{(n-1)}\right| \end{bmatrix} \begin{bmatrix} \mathrm{Y}_1 \\ \vdots \\ \mathrm{Y}_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\ln \frac{y_1^\star}{\sqrt[n]{\mathbb{N}(\mathbf{y}^\star)}} \\ \vdots \\ \frac{1}{2}\ln \frac{y_{n-1}^\star}{\sqrt[n]{\mathbb{N}(\mathbf{y}^\star)}} \end{bmatrix} \tag{\textbf{9}}$$

and

$$\begin{bmatrix} \alpha_1^{(1)} & \cdots & \alpha_n^{(1)} \\ \vdots & & \vdots \\ \alpha_1^{(n)} & \cdots & \alpha_n^{(n)} \end{bmatrix} \begin{bmatrix} \mathrm{X}_1 \\ \vdots \\ \mathrm{X}_n \end{bmatrix} = \mathbf{x}^\star \tag{\textbf{10}}$$

The matrix in (**9**) is invertible as a consequence of the units theorem; the matrix in (**10**) is invertible because $(\alpha_1, \dots, \alpha_n)$ is a $\mathbb{Z}$-basis for $\mathfrak{a}^{-2}$, which is an n-dimensional lattice in K (see chapters 3 and 5 of Lang's *Algebraic Number Theory* [6]). Therefore, $\mathrm{Y}_1, \dots, \mathrm{Y}_{n-1}, \mathrm{X}_1, \dots, \mathrm{X}_n$ are well defined.

Furthermore, the correspondance between usual coordinates $x_1, \dots, x_n, y_1, \dots, y_n$ for $\mathbf{z}$ and local coordinates is bijective. Indeed, if the local coordinates are known, (**10**) lets us recovers $\mathbf{x}^\star$, while (**9**) lets us recover $y_1^\star, \dots, y_{n-1}^\star$. The last number $y_n^\star$ is then obtained, using the fact that

$$q = \mathbb{N}(\mathbf{y}^\star) = y_1^\star \cdots y_{n-1}^\star y_n^\star$$

We wish to know how local coordinates change, when $\mathbf{z}$ is changed through particular transformations.

Proposition 11

Let $\boldsymbol{z} \in \mathscr{H}$, $\varepsilon = \pm\varepsilon_1^{k_1} \cdots \varepsilon_{n-1}^{k_{n-1}} \in \mathfrak{o}^\times$ and $\zeta = m_1\alpha_1 + \cdots + m_n\alpha_n \in \mathfrak{a}^{-2}$.

1. *Under the modular transformation $\mathrm{A}\begin{bmatrix} 1 & \zeta \\ 0 & 1 \end{bmatrix}\mathrm{A}^{-1}$, the local coordinates of $\boldsymbol{z}$ become*

$$\mathbb{N}(\boldsymbol{y}^\star), \mathrm{Y}_1, \dots, \mathrm{Y}_{n-1}, \mathrm{X}_1 + m_1, \dots, \mathrm{X}_n + m_n$$

In other words, the first n local coordinates are unchanged, the last n are translated by $(m_1, \dots, m_n)$.

2. *Under the modular transformation* $\mathrm{A}\begin{bmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{bmatrix}\mathrm{A}^{-1}$, *the first* n *local coordinates of* $\boldsymbol{z}$ *become*

$$\mathbb{N}(\boldsymbol{y}^\star), \mathrm{Y}_1 + k_1, \ldots, \mathrm{Y}_{n-1} + k_{n-1}$$

In other words, the first local coordinate does not change and the next $n-1$ *are translated by* $(k_1, \ldots, k_{n-1})$.

Proof: Let $\mathrm{M} = \mathrm{A}\begin{bmatrix}1 & \zeta\\ 0 & 1\end{bmatrix}\mathrm{A}^{-1}$, so that

$$\mathrm{A}^{-1}\mathrm{M}\mathbf{z} = \begin{bmatrix}1 & \zeta\\ 0 & 1\end{bmatrix}\mathrm{A}^{-1}$$

Thus $$(\mathrm{M}\mathbf{z})^\star = \begin{bmatrix}1 & \zeta\\ 0 & 1\end{bmatrix}\mathbf{z}^\star = \mathbf{z}^\star + \zeta = \begin{bmatrix}x_1^\star + \zeta^{(1)}\\ \vdots\\ x_n^\star + \zeta^{(n)}\end{bmatrix} + \mathrm{i}\begin{bmatrix}y_1^\star\\ \vdots\\ y_n^\star\end{bmatrix}$$

Let $q', \mathrm{Y}'_1, \ldots, \mathrm{Y}'_{n-1}, \mathrm{X}'_1, \ldots, \mathrm{X}'_n$ be the local coordinates of $\mathrm{M}\mathbf{z}$. Since $q', \mathrm{Y}'_1, \ldots, \mathrm{Y}'_{n-1}$ are determined by $\operatorname{Im}(\mathrm{M}\mathbf{z})^\star$, which is equal to $\operatorname{Im}\mathbf{z}^\star$, we get

$$q' = q \qquad \mathrm{Y}'_1 = \mathrm{Y}_1 \qquad \cdots \qquad \mathrm{Y}'_{n-1} = \mathrm{Y}_{n-1}$$

The numbers $\mathrm{X}'_1, \ldots, \mathrm{X}'_n$ satisfy

$$\begin{bmatrix}\alpha_1^{(1)} & \cdots & \alpha_n^{(1)}\\ \vdots & & \vdots\\ \alpha_1^{(n)} & \cdots & \alpha_n^{(n)}\end{bmatrix}\begin{bmatrix}\mathrm{X}'_1\\ \vdots\\ \mathrm{X}'_n\end{bmatrix} = \begin{bmatrix}x_1^\star + \zeta^{(1)}\\ \vdots\\ x_n^\star + \zeta^{(n)}\end{bmatrix} = \begin{bmatrix}x_1^\star\\ \vdots\\ x_n^\star\end{bmatrix} + \begin{bmatrix}\zeta^{(1)}\\ \vdots\\ \zeta^{(n)}\end{bmatrix}$$
$$= \begin{bmatrix}x_1^\star\\ \vdots\\ x_n^\star\end{bmatrix} + \begin{bmatrix}m_1\alpha_1^{(1)} + \cdots + m_n\alpha_n^{(1)}\\ \vdots\\ m_1\alpha_1^{(n)} + \cdots + m_n\alpha_n^{(n)}\end{bmatrix}$$
$$= \begin{bmatrix}x_1^\star\\ \vdots\\ x_n^\star\end{bmatrix} + \begin{bmatrix}\alpha_1^{(1)} & \cdots & \alpha_n^{(1)}\\ \vdots & & \vdots\\ \alpha_1^{(n)} & \cdots & \alpha_n^{(n)}\end{bmatrix}\begin{bmatrix}m_1\\ \vdots\\ m_n\end{bmatrix}$$

and therefore
$$\begin{bmatrix} \alpha_1^{(1)} & \cdots & \alpha_n^{(1)} \\ \vdots & & \vdots \\ \alpha_1^{(n)} & \cdots & \alpha_n^{(n)} \end{bmatrix} \begin{bmatrix} \mathrm{X}_1' - m_1 \\ \vdots \\ \mathrm{X}_n' - m_n \end{bmatrix} = \begin{bmatrix} x_1^\star \\ \vdots \\ x_n^\star \end{bmatrix}$$

It follows that
$$\begin{bmatrix} \mathrm{X}_1' - m_1 \\ \vdots \\ \mathrm{X}_n' - m_n \end{bmatrix} = \begin{bmatrix} \mathrm{X}_1 \\ \vdots \\ \mathrm{X}_n \end{bmatrix}$$
which proves the first part of the proposition.

The second part is proved along the same lines. □

Of course, it is not for random reasons that we restricted our attention to matrices such as $\mathrm{A}\begin{bmatrix} 1 & \zeta \\ 0 & 1 \end{bmatrix}\mathrm{A}^{-1}$ and $\mathrm{A}\begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix}\mathrm{A}^{-1}$. For one, according to **Theorem 3**, they belong to Γ_λ. But also, any element of Γ_λ can be decomposed as a product of those, as was seen during the proof of that theorem:
$$\mathrm{A}\begin{bmatrix} \varepsilon & \varepsilon^{-1}\zeta \\ 0 & \varepsilon^{-1} \end{bmatrix}\mathrm{A}^{-1} = \mathrm{A}\begin{bmatrix} 1 & \zeta \\ 0 & 1 \end{bmatrix}\mathrm{A}^{-1} \times \mathrm{A}\begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix}\mathrm{A}^{-1} \qquad \textbf{(11)}$$
This decomposition, together with **Proposition 11**, are key to constructing a fundamental domain for Γ_λ, and prompt us to introduce the notion of a point *reduced with respect to* λ:

Definition 12 A point $\mathbf{z}$ in $\mathscr{H}^n$ is called reduced with respect to λ if and only if all local coordinates, except the first one, are between $-\frac{1}{2}$ and $\frac{1}{2}$:
$$-\frac{1}{2} \leqslant \mathrm{Y}_1, \ldots, \mathrm{Y}_{n-1}, \mathrm{X}_1, \ldots, \mathrm{X}_n < \frac{1}{2}$$

Proposition 13
Any $\mathbf{z}$ in $\mathscr{H}^n$ is Γ_λ-equivalent to a reduced point for λ.

Proof: $\mathbf{z}$ has local coordinates $q, \mathrm{Y}_1, \ldots, \mathrm{Y}_{n-1}, \mathrm{X}_1, \ldots, \mathrm{X}_n$. We first find integers $k_1, \ldots, k_{n-1}$ such that
$$\forall i \in \{1, \ldots, n-1\} \qquad -\frac{1}{2} \leqslant \mathrm{Y}_i + k_i < \frac{1}{2} \qquad \textbf{(12)}$$

and then define $\mathrm{M}_1 = \mathrm{A}\begin{bmatrix}\varepsilon & 0\\ 0 & \varepsilon^{-1}\end{bmatrix}\mathrm{A}^{-1}$ with $\varepsilon = \varepsilon_1^{k_1}\cdots\varepsilon_{n-1}^{k_{n-1}}$

Let's call $q', \mathrm{Y}'_1, \ldots, \mathrm{Y}'_{n-1}, \mathrm{X}'_1, \ldots, \mathrm{X}'_n$ the local coordinates of $\mathrm{M}_1\mathbf{z}$. According to **Proposition 11** and (**12**), $\mathrm{Y}'_1, \ldots, \mathrm{Y}'_{n-1}$ are reduced.

Next, find integers $m_1, \ldots, m_n$ such that

$$\forall i \in \{1, \ldots, n-1\} \qquad -\frac{1}{2} \leqslant \mathrm{X}'_i + m_i < \frac{1}{2} \tag{\textbf{13}}$$

and define $\mathrm{M}_2 = \mathrm{A}\begin{bmatrix}1 & \zeta\\ 0 & 1\end{bmatrix}\mathrm{A}^{-1}$ with $\zeta = m_1\alpha_1 + \cdots + m_n\alpha_n$

Let's call $q'', \mathrm{Y}''_1, \ldots, \mathrm{Y}''_{n-1}, \mathrm{X}''_1, \ldots, \mathrm{X}''_n$ the local coordinates of the point $\mathrm{M}_2\mathrm{M}_1\mathbf{z}$. **Proposition 11** tells us that the Y'''s are equal to the Y''s, and together with (**13**), shows as well that the X'''s are reduced. Hence $\mathrm{M}_2\mathrm{M}_1\mathbf{z}$ is reduced with respect to λ and Γ_λ-equivalent to $\mathbf{z}$ since $\mathrm{M}_2\mathrm{M}_1 \in \Gamma_\lambda$. □

Proposition 14

Any two reduced points which are Γ_λ-equivalent are equal.

Proof: Let $\mathbf{z}$ and $\mathbf{w}$ be reduced with respect to λ, and such that for some $\mathrm{M} \in \Gamma_\lambda$, we have $\mathbf{z} = \mathrm{M}\mathbf{w}$. According to **Theorem 3**, M can be written as

$$\mathrm{M} = \mathrm{A}\begin{bmatrix}\varepsilon & \varepsilon^{-1}\zeta\\ 0 & \varepsilon^{-1}\end{bmatrix}\mathrm{A}^{-1} \qquad \zeta \in \mathfrak{a}^{-2} \quad \varepsilon \in \mathfrak{o}^\times$$

Let $q, \mathrm{Y}_1, \ldots, \mathrm{Y}_{n-1}, \mathrm{X}_1, \ldots, \mathrm{X}_n$ be the local coordinates of $\mathbf{z}$; let $p, \mathrm{U}_1, \ldots, \mathrm{U}_{n-1}, \mathrm{V}_1, \ldots, \mathrm{V}_n$ be the local coordinates of $\mathbf{w}$; write $\varepsilon = \pm\varepsilon_1^{k_1}\cdots\varepsilon_n^{k_n}$. Because of the decomposition (**11**) and **Proposition 11**, we have

$$\forall i \in \{1, \ldots, n-1\} \qquad \mathrm{Y}_i = \mathrm{U}_i + k_i$$

Since the Y's and the U's are all in $[-\frac{1}{2}, \frac{1}{2})$, it follows that all the k's are equal to 0. Therefore $\varepsilon = \pm 1$.

If we write $\zeta = \sum_{k=1}^{n} m_k\alpha_k$, we obtain through the same reasoning that all the m's are 0. Therefore $\mathrm{M} = \pm\mathrm{I}_2$ and it follows that $\mathbf{z} = (\pm\mathrm{I}_2)\mathbf{w} = \mathbf{w}$. □

As a consequence,

Theorem 15
The set $\mathscr{G}_\lambda$ of reduced points in $\mathscr{H}^n$ is a fundamental domain for Γ_λ.

As we can see, $\mathscr{G}_\lambda$ is easy to describe in terms of local coordinates at λ. One can check that, if the field K is actually $\mathbb{Q}$, our construction provides us with the strip $[-\frac{1}{2}, \frac{1}{2}) \times [0, \infty)$, the usual fundamental domain for Γ_∞, the stabilizer of the only cusp for $\mathrm{PSL}_2(\mathbb{Z})$.

1.6 Distance of a point to a cusp

In this final section, we want to associate to every cusp λ and $\mathbf{z} \in \mathscr{H}^n$ a number $\Delta(\mathbf{z}, \lambda)$ that indicates how close $\mathbf{z}$ is from λ. This intuitive notion of a distance should be such that, if $\mathbf{z}$ is "close" to a cusp λ, than it cannot be "close" to another cusp μ. In some sense, this will indicate that cusps are well separated from each other.

Definition 16 For every $\mathsf{z} \in \mathscr{H}^n$ and every cusp $\lambda = \frac{\rho}{\sigma}$, the distance of z to λ is defined as

$$\Delta(\mathsf{z}, \lambda) = \frac{1}{\sqrt{\mathbb{N}(\mathrm{Im}\,\mathrm{A}^{-1}\mathsf{z})}} = \mathbb{N}\frac{|-\sigma\mathsf{z} + \rho|}{\sqrt{\mathsf{y}}}$$

For every positive number r, the r-neighbourhood of λ is

$$\mathcal{U}_{\lambda,r} = \left\{\mathsf{z} \in \mathscr{H} \mid \Delta(\mathsf{z}, \lambda) < r\right\}$$

Proposition 17
For any $\mathrm{M} \in \mathscr{M}$, $\lambda = \frac{\rho}{\sigma}$ and $\boldsymbol{z} \in \mathscr{H}^n$, we have $\Delta(\boldsymbol{z}, \lambda) = \Delta(\boldsymbol{z}_{\mathrm{M}}, \lambda_{\mathrm{M}})$.

Proof: This is really just a computation. Let $\mathrm{A}_\lambda = \begin{bmatrix} \rho & \star \\ \sigma & \star \end{bmatrix}$ be the matrix associated to λ as usual, and let $\mathrm{M} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathscr{M}$. We have

$$\lambda_{\mathrm{M}} = \frac{\alpha\rho + \beta\sigma}{\gamma\rho + \delta\sigma}$$

and
$$\mathrm{MA}_\lambda = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \rho & \star \\ \sigma & \star \end{bmatrix} = \begin{bmatrix} \alpha\rho + \beta\sigma & \star \\ \gamma\rho + \delta\sigma & \star \end{bmatrix}$$

Since the numerator and denominator of λ_{M} form the first column of MA_λ, this matrix is a good candidate for $\mathrm{A}_{\lambda_{\mathrm{M}}}$. This done, the computation unfolds easily:

$$\mathrm{Im}\,(\mathrm{A}_{\lambda_{\mathrm{M}}}^{-1}\mathbf{z}_{\mathrm{M}}) = \mathrm{Im}\,\big((\mathrm{MA}_\lambda)^{-1}\mathbf{z}_{\mathrm{M}}\big) = \mathrm{Im}\,(\mathrm{A}_\lambda^{-1}\mathrm{M}^{-1}\mathbf{z}_{\mathrm{M}}) = \mathrm{Im}\,(\mathrm{A}_\lambda^{-1}\mathbf{z})$$

and it follows that
$$\Delta(\mathbf{z}_{\mathrm{M}}, \lambda_{\mathrm{M}}) = \Delta(\mathbf{z}, \lambda)$$
□

Proposition 18

There exists a positive number d, depending only on K, *such that*

$$\forall \boldsymbol{z} \in \mathscr{H}^n \quad \forall \lambda, \mu \in \mathrm{K} \qquad \big(\Delta(\boldsymbol{z}, \lambda) < d \quad \textit{and} \quad \Delta(\boldsymbol{z}, \mu) < d\big) \implies \lambda = \mu$$

Proof: Let $\lambda = \frac{\rho}{\sigma}$ and $\mu = \frac{\rho_1}{\sigma_1}$ be in K. Let $\mathbf{z} \in \mathscr{H}^n$ and let d be any positive real number. Assume that

$$\Delta(\mathbf{z}, \lambda) < d \qquad \text{and} \qquad \Delta(\mathbf{z}, \mu) < d$$

This means that
$$\mathbb{N}\frac{(-\sigma\mathbf{x} + \rho)^2 + \sigma^2\mathbf{y}^2}{\mathbf{y}} < d^2 \tag{\textbf{14}}$$

and
$$\mathbb{N}\frac{(-\sigma_1\mathbf{x} + \rho_1)^2 + \sigma_1^2\mathbf{y}^2}{\mathbf{y}} < d^2 \tag{\textbf{15}}$$

It is a consequence of the units theorem (see Lang's *Algebraic Number Theory* [6], chapter 5) that there exists a constant C depending only on K such that, for every $\mathbf{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n$ with $\mathbb{N}\mathbf{r} \neq 0$, there exists a unit ε that satisfies

$$\forall i \in \{1, \ldots, n\} \qquad \left|r_i\varepsilon^{(i)}\right| \leqslant \mathrm{C}\sqrt[n]{\mathbb{N}\mathbf{r}}$$

Of course, multiplying ρ and σ by the same unit won't change the expression (**14**), since units have norm ± 1. Nor will it change the ratio $\frac{\rho}{\sigma}$. Therefore, up to such an operation, we can assume that

$$\forall i \in \{1, \ldots, n\} \qquad \frac{\big(-\sigma^{(i)}x_i + \rho^{(i)}\big)^2 + {\sigma^{(i)}}^2 y_i^2}{y_i} < \mathrm{C}d^{\frac{2}{n}}$$

Similarly,
$$\forall i \in \{1, \ldots, n\} \qquad \frac{\big(-\sigma_1^{(i)}x_i + \rho_1^{(i)}\big)^2 + {\sigma_1^{(i)}}^2 y_i^2}{y_i} < \mathrm{C}d^{\frac{2}{n}}$$

can be assumed as well. This in turn implies that for every i,

$$\begin{aligned}\frac{\big(-\sigma^{(i)}x_i+\rho^{(i)}\big)^2}{y_i} &< \mathrm{C}d^{\frac{2}{n}}\\ {\sigma^{(i)}}^2 y_i &< \mathrm{C}d^{\frac{2}{n}}\\ \frac{\big(-\sigma_1^{(i)}x_i+\rho_1^{(i)}\big)^2}{y_i} &< \mathrm{C}d^{\frac{2}{n}}\\ {\sigma_1^{(i)}}^2 y_i &< \mathrm{C}d^{\frac{2}{n}}\end{aligned}$$

But $\qquad \rho^{(i)}\sigma_1^{(i)}-\rho_1^{(i)}\sigma^{(i)}=\dfrac{-\sigma^{(i)}x_i+\rho^{(i)}}{\sqrt{y_i}}\times\sigma_1^{(i)}\sqrt{y_i}-\dfrac{-\sigma_1^{(i)}x_i+\rho_1^{(i)}}{\sqrt{y_i}}\times\sigma^{(i)}\sqrt{y_i}$

which implies $\qquad \Big|\rho^{(i)}\sigma_1^{(i)}-\rho_1^{(i)}\sigma^{(i)}\Big|<2\mathrm{C}d^{\frac{2}{n}}$

Hence $\qquad \Big|\mathbb{N}(\rho\sigma_1-\rho_1\sigma)\Big|<2^n\mathrm{C}^n d^2$

Now, if we choose d small enough so that the righthandside is less than 1, we get

$$\Big|\mathbb{N}(\rho\sigma_1-\rho_1\sigma)\Big|<1$$

Since the norm of an algebraic integer is an integer, it follows that $\rho\sigma_1-\rho_1\sigma=0$, and in turn

$$\lambda=\frac{\rho}{\sigma}=\frac{\rho_1}{\sigma_1}=\mu \qquad \square$$

Chapter 2

The Hilbert modular Eisenstein series

In this chapter, we define the Hilbert modular Eisenstein series for the totally real field K, show that it has analytic continuation and functional equation.

Simply put, if A is an ideal class for the field K, the Hilbert modular Eisenstein series associated to A is the function defined by the series expansion

$$\forall \mathbf{z} \in \mathscr{H}^n \quad \forall s \in \mathbb{C} \quad \operatorname{Re} s > 1 \qquad \mathrm{E}_{\mathrm{K,A}}(\mathbf{z}, s) = \mathbb{N}(\mathfrak{a})^{2s} \sum_{\substack{(\gamma,\delta) \in \mathfrak{a}^2/\mathfrak{o}^\times \\ \langle \gamma,\delta \rangle = \mathfrak{a}}} \prod_{j=1}^{n} \frac{(\operatorname{Im} z_j)^s}{\left|\gamma^{(j)} z_j + \delta^{(j)}\right|^{2s}}$$

where $\mathfrak{a}$ is any ideal in A^{-1}, and the sum is over nonassociated pairs (γ, δ) generating $\mathfrak{a}$ as an ideal. Of course, this definition raises issues: why does this series converge? Does it depend on the choice for $\mathfrak{a} \in \mathrm{A}^{-1}$, or on the choice for representatives $(\gamma, \delta) \in \mathfrak{a}^2/\mathfrak{o}^\times$?

We will start by answering these questions, and the previous chapter will be key in proving convergence for $\mathrm{E}_{\mathrm{K,A}}(\mathbf{z}, s)$. Then we proceed to computing the Fourier expansion of $\mathrm{E}_{\mathrm{K,A}}$ in terms of characters of $\mathbb{R}^n/\mathfrak{o}$, which will prove at the same time that this function has analytic continuation and functional equation under $s \longmapsto 1-s$.

2.1 Convergence and automorphicity

We let A be an ideal class, $\mathfrak{a} = \langle \rho, \sigma \rangle$ be an integral ideal in A^{-1} and $\lambda = \frac{\rho}{\sigma}$ the corresponding cusp for $\mathscr{M}$. We choose a $\mathbb{Z}$-basis $(\alpha_1, \ldots, \alpha_n)$ of $\mathfrak{a}^{-2}$, and a system of fundamental units $\varepsilon_1, \ldots, \varepsilon_{n-1}$. We also find ξ, η in $\mathfrak{a}^{-1}$ such that $\rho\eta - \sigma\xi = 1$ and form the matrix $\mathrm{A} = \begin{bmatrix} \rho & \xi \\ \sigma & \eta \end{bmatrix}$.

The Hilbert modular Eisenstein series will be shown to converge by a very clever argument: its sum is comparable to the volume of a subset of the fundamental domain $\mathscr{G}_\lambda$ constructed in the previous chapter.

Of course, as we saw, $\mathscr{G}_\lambda$ is easily described in terms of the local coordinates $q, \mathrm{Y}_1, \ldots, \mathrm{Y}_{n-1}, \mathrm{X}_1, \ldots, \mathrm{X}_n$ as some sort of vertical strip:

$$\mathscr{G}_\lambda = \left\{ \mathbf{z} \in \mathscr{H}^n \mid q > 0 \quad -\frac{1}{2} \leqslant \mathrm{Y}_i, \mathrm{X}_i < \frac{1}{2} \text{ for all } i \right\}$$

and therefore we would like to privilege this coordinate system. This requires the Jacobian computation for the transformation from usual coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ to local coordinates. We remind the reader that we have:

$$q = \mathbb{N}(\mathbf{y}^\star) = y_1^\star \cdots y_n^\star \qquad (\mathbf{1})$$

$$\begin{bmatrix} \ln\left|\varepsilon_1^{(1)}\right| & \cdots & \ln\left|\varepsilon_{n-1}^{(1)}\right| \\ \vdots & & \vdots \\ \ln\left|\varepsilon_1^{(n-1)}\right| & \cdots & \ln\left|\varepsilon_{n-1}^{(n-1)}\right| \end{bmatrix} \begin{bmatrix} \mathrm{Y}_1 \\ \vdots \\ \mathrm{Y}_{n-1} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \ln \frac{y_1^\star}{\sqrt[n]{\mathbb{N}(\mathbf{y}^\star)}} \\ \vdots \\ \frac{1}{2} \ln \frac{y_{n-1}^\star}{\sqrt[n]{\mathbb{N}(\mathbf{y}^\star)}} \end{bmatrix} \qquad (\mathbf{2})$$

and

$$\begin{bmatrix} \alpha_1^{(1)} & \cdots & \alpha_n^{(1)} \\ \vdots & & \vdots \\ \alpha_1^{(n)} & \cdots & \alpha_n^{(n)} \end{bmatrix} \begin{bmatrix} \mathrm{X}_1 \\ \vdots \\ \mathrm{X}_n \end{bmatrix} = \mathbf{x}^\star \qquad (\mathbf{3})$$

where

$$\mathbf{z}^\star = \mathbf{x}^\star + \mathrm{i}\mathbf{y}^\star = \mathrm{A}^{-1}\mathbf{z}$$

Proposition 19

Let D *and* R *be respectively the discriminant and the regulator of* K. *We have*

$$dq\, d\mathrm{Y}_1 \cdots d\mathrm{Y}_{n-1} d\mathrm{X}_1 \cdots d\mathrm{X}_n = \frac{\mathbb{N}\mathfrak{a}^2}{2^{n-1}\mathrm{R}\sqrt{\mathrm{D}}} \left|\mathbb{N}(-\sigma \boldsymbol{z} + \rho)\right|^{-4} d\boldsymbol{x} d\boldsymbol{y}$$

Proof: We first find the relationship between $\mathrm{d}\mathbf{x}^\star \mathrm{d}\mathbf{y}^\star$ and $\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}$. Since $\mathbf{z}^\star = \mathrm{A}^{-1}\mathbf{z}$, we have for every $i \in \{1, \ldots, n\}$:

$$z_i^\star = x_i^\star + \mathrm{i}y_i^\star = \mathrm{A}^{(i)^{-1}} z_i = \begin{bmatrix} \eta^{(i)} & -\xi^{(i)} \\ -\sigma^{(i)} & \rho^{(i)} \end{bmatrix} z_i = \frac{\eta^{(i)} z_i - \xi^{(i)}}{-\sigma^{(i)} z_i + \rho^{(i)}}$$

In particular $$y_i^\star = \frac{y_i}{\left| -\sigma^{(i)} z_i + \rho^{(i)} \right|^2}$$

Now, we know from the theory of $\mathrm{SL}_2(\mathbb{R})$ acting on the upper half-plane that the measure $\frac{\mathrm{d}x\mathrm{d}y}{y^2}$ is $\mathrm{SL}_2(\mathbb{R})$-invariant. Thus

$$\frac{\mathrm{d}x_i \mathrm{d}y_i}{y_i^2} = \frac{\mathrm{d}x_i^\star \mathrm{d}y_i^\star}{y_i^{\star 2}}$$

and $$\mathrm{d}x_i \mathrm{d}y_i = \frac{y_i^2}{y_i^{\star 2}} \mathrm{d}x_i^\star \mathrm{d}y_i^\star = \left| -\sigma^{(i)} z_i + \rho^{(i)} \right|^4 \mathrm{d}x_i^\star \mathrm{d}y_i^\star$$

Taking the product over i of such measures yields:

$$\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} = \left| \mathbb{N}(-\sigma \mathbf{z} + \rho) \right|^4 \mathrm{d}\mathbf{x}^\star \mathrm{d}\mathbf{y}^\star \tag{\textbf{4}}$$

Next, we relate $\mathrm{d}q\mathrm{d}\mathrm{Y}_1 \cdots \mathrm{d}\mathrm{Y}_{n-1} \mathrm{d}\mathrm{X}_1 \cdots \mathrm{d}\mathrm{X}_n$ to $\mathrm{d}\mathbf{x}^\star \mathrm{d}\mathbf{y}^\star$. Because of (**3**), which is simply a linear relationship between $\mathbf{x}^\star$ and the X's, we have

$$\begin{vmatrix} \alpha_1^{(1)} & \cdots & \alpha_n^{(1)} \\ \vdots & & \vdots \\ \alpha_1^{(n)} & \cdots & \alpha_n^{(n)} \end{vmatrix} \mathrm{d}\mathrm{X}_1 \cdots \mathrm{d}\mathrm{X}_n = \mathrm{d}\mathbf{x}^\star$$

The determinant on the left is simply $\mathbb{N}(\mathfrak{a}^{-2})\sqrt{\mathrm{D}}$, from known results in algebraic number theory (see chapter 3 of Lang's *Algebraic Number Theory* [6]). Thus

$$\mathrm{d}\mathrm{X}_1 \cdots \mathrm{d}\mathrm{X}_n = \frac{\mathbb{N}\mathfrak{a}^2}{\sqrt{\mathrm{D}}} \mathrm{d}\mathbf{x}^\star \tag{\textbf{5}}$$

The definition (**2**) tells us that

$$\forall i \in \{1, \ldots, n-1\} \qquad \sum_{k=1}^{n-1} \mathrm{Y}_k \ln \left| \varepsilon_k^{(i)} \right| = \frac{1}{2} \ln \frac{y_i^\star}{\sqrt[n]{\mathbb{N}(\mathbf{y}^\star)}} = \frac{1}{2} \ln y_i^\star - \frac{1}{2n} \ln q$$

Therefore $$\forall i \in \{1, \ldots, n-1\} \qquad \ln y_i^\star = \frac{1}{n} \ln q + \sum_{k=1}^{n-1} 2\mathrm{Y}_k \ln \left| \varepsilon_k^{(i)} \right| \tag{\textbf{6}}$$

Summing these relations for all values of i between 1 and $n-1$ gives:

$$\ln(y_1^\star \cdots y_{n-1}^\star) = \frac{n-1}{n}\ln q + \sum_{k=1}^{n-1} 2\mathrm{Y}_k \sum_{i=1}^{n-1} \ln\left|\varepsilon_k^{(i)}\right| \qquad (\mathbf{7})$$

But
$$y_1^\star \cdots y_{n-1}^\star = \frac{q}{y_n^\star}$$

and
$$\sum_{i=1}^{n-1} \ln\left|\varepsilon_k^{(i)}\right| = -\ln\left|\varepsilon_k^{(n)}\right|$$

due to the product formula $\prod_{i=1}^{n}\left|\varepsilon_k^{(i)}\right| = 1$. So, pluging this back into (**7**) yields

$$\ln q - \ln y_n^\star = \frac{n-1}{n}\ln q - \sum_{k=1}^{n-1} 2\mathrm{Y}_k \ln\left|\varepsilon_k^{(n)}\right|$$

so that
$$\ln y_n^\star = \frac{1}{n}\ln q + \sum_{k=1}^{n-1} 2\mathrm{Y}_k \ln\left|\varepsilon_k^{(n)}\right|$$

This formula, together with (**6**) allow us to recover completely the $y^\star$'s in terms of q and the Y's:

$$\forall i \in \{1,\ldots,n\} \qquad y_i^\star = q^{\frac{1}{n}} \mathrm{e}^{\sum\limits_{k=1}^{n-1} 2\mathrm{Y}_k \ln\left|\varepsilon_k^{(i)}\right|}$$

Thus
$$\frac{\partial y_i^\star}{\partial q} = \frac{1}{n} q^{\frac{1}{n}-1} \mathrm{e}^{\sum\limits_{k=1}^{n-1} 2\mathrm{Y}_k \ln\left|\varepsilon_k^{(i)}\right|} = \frac{y_i^\star}{nq}$$

and
$$\frac{\partial y_i^\star}{\partial \mathrm{Y}_j} = 2\ln\left|\varepsilon_j^{(i)}\right| q^{\frac{1}{n}} \mathrm{e}^{\sum\limits_{k=1}^{n-1} 2\mathrm{Y}_k \ln\left|\varepsilon_k^{(i)}\right|} = 2y_i^\star \ln\left|\varepsilon_j^{(i)}\right| y_i^\star$$

Finally
$$\mathrm{d}\mathbf{y}^\star = \begin{vmatrix} \frac{y_1^\star}{nq} & 2y_1^\star \ln\left|\varepsilon_1^{(1)}\right| & \cdots & 2y_1^\star \ln\left|\varepsilon_{n-1}^{(1)}\right| \\ \vdots & \vdots & & \vdots \\ \frac{y_n^\star}{nq} & 2y_n^\star \ln\left|\varepsilon_1^{(n)}\right| & \cdots & 2y_n^\star \ln\left|\varepsilon_{n-1}^{(n)}\right| \end{vmatrix} \mathrm{d}q\mathrm{d}\mathrm{Y}_1 \cdots \mathrm{d}\mathrm{Y}_{n-1}$$

$$\mathrm{d}\mathbf{y}^\star = \frac{2^{n-1} y_1^\star \cdots y_n^\star}{nq} \begin{vmatrix} 1 & \ln\left|\varepsilon_1^{(1)}\right| & \cdots & \ln\left|\varepsilon_{n-1}^{(1)}\right| \\ \vdots & \vdots & & \vdots \\ 1 & \ln\left|\varepsilon_1^{(n)}\right| & \cdots & \ln\left|\varepsilon_{n-1}^{(n)}\right| \end{vmatrix} \mathrm{d}q\mathrm{d}\mathrm{Y}_1 \cdots \mathrm{d}\mathrm{Y}_{n-1}$$

and this determinant is precisely $n\mathrm{R}$ (see Lang's [6], chapter 5). Putting this result together with formulas (**5**) and (**4**) yields the announced result:

$$\mathrm{d}q\mathrm{dY}_1\cdots\mathrm{dY}_{n-1}\mathrm{dX}_1\cdots\mathrm{dX}_n = \frac{\mathbb{N}\mathfrak{a}^2}{2^{n-1}\mathrm{R}\sqrt{\mathrm{D}}}\left|\mathbb{N}(-\sigma\mathbf{z}+\rho)\right|^{-4}\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y} \qquad \square$$

For convenience, in the remainder of the chapter, the volume element in the local coordinates system will be denoted $\mathrm{d}\tau$

$$\mathrm{d}\tau = \mathrm{d}q\mathrm{dY}_1\cdots\mathrm{dY}_{n-1}\mathrm{dX}_1\cdots\mathrm{dX}_n$$

and we let

$$c = \frac{\mathbb{N}\mathfrak{a}^2}{2^{n-1}\mathrm{R}\sqrt{\mathrm{D}}}$$

To see how $\mathrm{d}\tau$ is modified under a Hilbert modular substitution, it will be convenient to first relate it to the invariant measure $\mathrm{d}\omega = \frac{\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}}{\mathbb{N}(\mathbf{y})^2}$:

Proposition 20

$$d\tau = c\mathbb{N}(\boldsymbol{y}^\star)^2\, d\omega$$

The measure $\frac{d\tau}{q^2}$ *is* $\mathscr{M}$*-invariant.*

Proof: Everything we need has already been proven in **Proposition 19**. We already know that

$$\mathrm{d}\omega = \frac{\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}}{\mathbb{N}(\mathbf{y})^2} = \frac{\mathrm{d}\mathbf{x}^\star\mathrm{d}\mathbf{y}^\star}{\mathbb{N}(\mathbf{y}^\star)^2}$$

and it has been shown that

$$\mathrm{d}\tau = c\,\mathrm{d}\mathbf{x}^\star\mathrm{d}\mathbf{y}^\star \qquad \square$$

This done, we are almost ready to show the existence of the Hilbert modular Eisenstein series. We announced earlier that it would be related to an integral computation over a piece of the fundamental domain $\mathscr{G}_\lambda$ for Γ_λ constructed in the first chapter. To see that, we need somehow to relate the Eisenstein series to the group Γ_λ. This is done through the following lemma:

Lemma 21

Let $(\mathrm{M}_j)_{j\in\mathbb{N}}$ *be a complete set of representatives for the quotient* $\Gamma_\lambda\backslash\mathscr{M}$*. Define*

$$\forall j \in \mathbb{N} \qquad \mathrm{P}_j = \mathrm{A}^{-1}\mathrm{M}_j = \begin{bmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{bmatrix}$$

The collection $\mathcal{P} = \{(\gamma_j, \delta_j) \mid j \in \mathbb{N}\}$ *is a complete set of nonassociated pairs of generators for the ideal* $\mathfrak{a}$*.*

Proof: We recall that, saying that $(\mathrm{M}_j)_{j\in\mathbb{N}}$ is a complete set of representatives for $\Gamma_\lambda\backslash\mathscr{M}$ means that we have the disjoint union

$$\mathscr{M} = \bigcup_{j\in\mathbb{N}} \Gamma_\lambda \mathrm{M}_j$$

We also recall that two pairs (γ, δ), (γ', δ') that generate $\mathfrak{a}$ as an ideal are associated if they can be obtained one from the other through a unit, that is

$$\exists \varepsilon \in \mathfrak{o} \qquad \gamma' = \varepsilon\gamma \qquad \text{and} \qquad \delta' = \varepsilon\delta$$

We first show that each pair in $\mathcal{P}$ generates $\mathfrak{a}$ as an ideal. Let $\mathrm{M} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ be any matrix in $\mathscr{M}$. We have

$$\mathrm{A}^{-1}\mathrm{M} = \begin{bmatrix} \eta & -\xi \\ -\sigma & \rho \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} \eta\alpha - \xi\gamma & \eta\beta - \xi\delta \\ \rho\gamma - \alpha\sigma & \rho\delta - \beta\sigma \end{bmatrix} \tag{8}$$

Let $$u = \rho\gamma - \alpha\sigma \qquad \text{and} \qquad v = \rho\delta - \beta\sigma$$

Since $\rho, \sigma \in \mathfrak{a}$ and $\alpha, \beta, \gamma, \delta \in \mathfrak{o}$, the integers u and v are in $\mathfrak{a}$. Therefore

$$\langle u, v \rangle \subset \mathfrak{a}$$

For the converse, using the fact that $\alpha\delta - \beta\gamma = 1$, we get

$$\begin{cases} \beta u - \alpha v = \rho \\ \gamma v - \delta u = \sigma \end{cases}$$

so that $$\mathfrak{a} = \langle \rho, \sigma \rangle \subset \langle u, v \rangle$$

Hence $$\langle u, v \rangle = \mathfrak{a}$$

The matrix M was an arbitrary Hilbert modular function. So in particular, each pair in $\mathcal{P}$ generates $\mathfrak{a}$ as an ideal.

Now we show that those pairs are not associated. Suppose that for some non-negative integers i and j, there exists a unit ε such that

$$\gamma_j = \varepsilon\gamma_i \qquad \text{and} \qquad \delta_j = \varepsilon\delta_i \tag{\textbf{9}}$$

Because P_i has determinant 1, either γ_i or δ_i is not zero. Assume it is γ_i and let

$$\zeta = \frac{\alpha_j - \varepsilon^{-1}\alpha_i}{\varepsilon\gamma_i}$$

This number has been chosen so that

$$\varepsilon^{-1}\alpha_i + \varepsilon\zeta\gamma_i = \alpha_j \tag{\textbf{10}}$$

It also satisfies another nice relation, though this one requires more work to see:

$$\begin{aligned} \varepsilon^{-1}\beta_i + \varepsilon\zeta\delta_i &= \varepsilon^{-1}\beta_i + \frac{\alpha_j\delta_i - \varepsilon^{-1}\alpha_i\delta_i}{\gamma_i} \\ &= \frac{\alpha_j\delta_i - \varepsilon^{-1}(\alpha_i\delta_i - \beta_i\gamma_i)}{\gamma_i} \end{aligned}$$

Using the facts that

$$\alpha_i\delta_i - \beta_i\gamma_i = 1 \qquad \gamma_i = \varepsilon^{-1}\gamma_j \qquad \text{and} \qquad \delta_i = \varepsilon^{-1}\delta_j$$

we get
$$\varepsilon^{-1}\beta_i + \varepsilon\zeta\delta_i = \frac{\alpha_j\delta_j - 1}{\gamma_j}$$

But we also have $\alpha_j\delta_j - \beta_j\gamma_j = 1$ so that $\alpha_j\delta_j - 1 = \beta_j\gamma_j$

Finally:
$$\varepsilon^{-1}\beta_i + \varepsilon\zeta\delta_i = \beta_j \tag{\textbf{11}}$$

Formulas (**9**), (**10**) and (**11**) can be summarized in the following matrix form:

$$\begin{bmatrix} \varepsilon^{-1} & \varepsilon\zeta \\ 0 & \varepsilon \end{bmatrix} \mathrm{P}_i = \begin{bmatrix} \varepsilon^{-1} & \varepsilon\zeta \\ 0 & \varepsilon \end{bmatrix} \begin{bmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{bmatrix} = \begin{bmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{bmatrix} = \mathrm{P}_j$$

Since $\mathrm{P}_i = \mathrm{A}^{-1}\mathrm{M}_i$ and $\mathrm{P}_j = \mathrm{A}^{-1}\mathrm{M}_j$, we see from the computation (**8**) that α_i and α_j are both in $\mathfrak{a}^{-1}$ (remember that ξ and η are in $\mathfrak{a}^{-1}$). Therefore, ζ belongs to $\mathfrak{a}^{-2}$. Thus

$$\begin{bmatrix} \varepsilon^{-1} & \varepsilon\zeta \\ 0 & \varepsilon \end{bmatrix} \mathrm{A}^{-1}\mathrm{M}_i = \mathrm{A}^{-1}\mathrm{M}_j$$

or
$$\mathrm{A}\begin{bmatrix}\varepsilon^{-1} & \varepsilon\zeta\\ 0 & \varepsilon\end{bmatrix}\mathrm{A}^{-1}\mathrm{M}_i = \mathrm{M}_j$$

In view of **Theorem 3**, it appears that M_i and M_j are in the same right coset for Γ_λ. It follows that $\mathrm{M}_i = \mathrm{M}_j$, $\mathrm{P}_i = \mathrm{P}_j$ and therefore $\gamma_i = \gamma_j$, $\delta_i = \delta_j$: if two pairs in $\mathcal{P}$ are associated, they must be equal. □

The following, long-awaited theorem, is the achievement of everything done so far:

Theorem 22

The Hilbert modular Eisenstein series

$$\mathrm{E}_{\mathrm{K,A}}(\boldsymbol{z}, s) = \mathbb{N}(\mathfrak{a})^{2s} \sum_{\substack{(\gamma,\delta)\in\mathfrak{a}^2/\mathfrak{o}^\times\\ \langle\gamma,\delta\rangle=\mathfrak{a}}} \prod_{j=1}^{n} \frac{y_j^s}{\left|\gamma^{(j)} z_j + \delta^{(j)}\right|^{2s}} = \mathbb{N}(\mathfrak{a})^{2s} \sum_{\substack{(\gamma,\delta)\in\mathfrak{a}^2/\mathfrak{o}^\times\\ \langle\gamma,\delta\rangle=\mathfrak{a}}} \frac{\mathbb{N}(\boldsymbol{y})^2}{\left|\mathbb{N}(\gamma\boldsymbol{z}+\delta)\right|^{2s}}$$

converges absolutely for every $\boldsymbol{z} \in \mathscr{H}^n$ and $s \in \mathbb{C}$ with real part bigger than 1. The convergence is uniform in s in half planes $\{\sigma_1 > \mathrm{Re}\, s > \sigma_0 > 1\}$, and uniform in $\boldsymbol{z}$ in every compact subset of $\mathscr{H}^n$.

Furthermore, this definition does not depend on the choice of non-associated pairs (γ, δ) of generators of $\mathfrak{a}$, nor does it depend on the choice for $\mathfrak{a}$ in A^{-1}.

Proof: Fix an ideal $\mathfrak{a} = \langle\rho, \sigma\rangle$ in A^{-1} and a complete set of representatives $(\mathrm{M}_j)_{j\in\mathbb{N}}$ for the right cosets in $\Gamma_\lambda\backslash\mathscr{M}$. Up to renaming them, we can assume that $\mathrm{M}_0 = \mathrm{I}_2$.

According to **Proposition 18**, there is a positive number d such that, for every $\mathbf{z} \in \mathscr{H}$ and every cusp μ, if $\Delta(\mathbf{z}, \lambda)$ and $\Delta(\mathbf{z}, \mu)$ are both less than d, then $\mu = \lambda$.

The Hilbert modular group $\mathscr{M}$ is countable, since $\mathfrak{o}$ is countable. Besides, every nontrivial modular transformation has at most two fixed points. Therefore, the set of points $\mathbf{z} \in \mathscr{H}^n$ such that $\mathrm{M}\mathbf{z} = \mathbf{z}$ for some $\mathrm{M} \in \mathscr{M}$ different than I_2 is countable. Since the neighbourhood

$$\mathcal{U}_{\lambda,d} \cap \mathring{\mathscr{G}}_\lambda = \left\{\mathbf{z} \in \mathscr{H}^n \mid \frac{1}{\sqrt{q}} = \Delta(\mathbf{z}, \lambda) < d \quad -\frac{1}{2} < \mathrm{X}_i, \mathrm{Y}_i < \frac{1}{2} \text{ for all } i\right\}$$

is uncountable, we can certainly find $\mathbf{z}_0$ in there that is only fixed by I_2. According to **Corollary 10**, there is a neighbourhood U of $\mathbf{z}_0$, such that

$$\forall \mathrm{M} \in \mathscr{M} \setminus \{\mathrm{I}_2\} \qquad \mathrm{U}_\mathrm{M} \cap \mathrm{U} = \emptyset \tag{12}$$

And up to choosing U smaller, we can make it relatively compact and suppose that it is entirely contained in $\mathcal{U}_{\lambda,d} \cap \mathscr{G}_\lambda$. There is also a positive number d_0 such that $q < d_0^2$ for all $\mathbf{z}$ in U, which can also be restated as $\Delta(\mathbf{z}, \lambda) \geqslant \frac{1}{d_0}$.

Because of (**12**), the sets $(\mathrm{U}_{\mathrm{M}_j})_{j\in\mathbb{N}}$ are pairwise disjoint. Indeed, suppose that $\mathrm{M}_i(\mathrm{U}) \cap \mathrm{M}_j(\mathrm{U})$ is not empty. Then $\mathrm{M}_j^{-1}\mathrm{M}_i(\mathrm{U}) \cap \mathrm{U}$ is not empty as well, which implies that $\mathrm{M}_j^{-1}\mathrm{M}_i = \mathrm{I}_2$, and in turn that $i = j$.

Next, up to multiplying each M_j on the left by an element of Γ_λ, we can suppose that $\mathrm{U}_{\mathrm{M}_j}$ sits entirely in the fundamental domain $\mathscr{G}_\lambda$.

Notice now that by **Proposition 17** and by definition of d, if $\mathbf{z}$ is in U, we have for every positive integer j,

$$\Delta(\mathbf{z}_{\mathrm{M}_j}, \lambda) = \Delta(\mathbf{z}, \lambda_{\mathrm{M}_j^{-1}}) \geqslant d \geqslant \frac{1}{d_0}$$

Indeed, $\mathrm{M}_j \neq \mathrm{I}_2$, so $\mathrm{M}_j \notin \Gamma_\lambda$ and therefore $\lambda_{\mathrm{M}_j^{-1}} \neq \lambda$. Thus

$$\forall \mathbf{z} \in \bigcup_{j\in\mathbb{N}} \mathrm{U}_{\mathrm{M}_j} \qquad q(\mathbf{z}) = \frac{1}{\Delta(\mathbf{z}, \lambda)^2} \leqslant d_0^2$$

Let's consider the following integral, where s is a real number:

$$\mathrm{J}_s = \sum_{j=0}^{\infty} \int_{\mathrm{U}_{\mathrm{M}_j}} \mathbb{N}(\mathbf{y}^\star)^{s-2}\, \mathrm{d}\tau$$

Since the $(\mathrm{U}_{\mathrm{M}_j})_{j\in\mathbb{N}}$ are pairwise disjoint,

$$\mathrm{J}_s = \int_{\bigcup_{j\in\mathbb{N}} \mathrm{U}_{\mathrm{M}_j}} q^{s-2} \mathrm{d}\tau$$

We were careful enough to make sure that each $\mathrm{U}_{\mathrm{M}_j}$ is inside $\mathscr{G}_\lambda$. And the q-coordinates of elements in those sets are bounded by d_0^2. Thus if s is a real number bigger than 1,

$$\mathrm{J}_s \leqslant \int_0^{d_0^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \cdots \int_{-\frac{1}{2}}^{\frac{1}{2}} q^{s-2}\, \mathrm{d}q \mathrm{dY}_1 \cdots \mathrm{dY}_{n-1} \mathrm{dX}_1 \cdots \mathrm{dX}_n = \int_0^{d_0^2} q^{s-2}\, \mathrm{d}q = \frac{d_0^{2s-2}}{s-1} < \infty$$

The convergence of this integral will imply the convergence of the Eisenstein series.

Let's fix for now real numbers $s_1 > s > s_0 > 1$. Since U is compact, **Lemma 6** provides us with positive constants c_1 and c_2 such that

$$\forall \mathbf{z} \in \mathrm{U} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^n \qquad c_1 \left|\mathbb{N}(\mathbf{u}\mathrm{i} + \mathbf{v})\right|^2 \leqslant \left|\mathbb{N}(\mathbf{u}\mathbf{z} + \mathbf{v})\right|^2 \leqslant c_2 \left|\mathbb{N}(\mathbf{u}\mathrm{i} + \mathbf{v})\right|^2$$

Also, because U is included in $\mathcal{U}_{\lambda,d}$,

$$\forall \mathbf{z} \in \mathrm{U} \qquad \mathbb{N}(\mathbf{y}) = \frac{1}{\Delta(\mathbf{z}, \lambda)^2} \geqslant \frac{1}{d^2}$$

Thus, for every integer j,

$$\frac{1}{\left|\mathbb{N}(\gamma_j \mathrm{i} + \delta_j)\right|^{2s}} \leqslant \frac{\mathbb{N}(\mathbf{y})^s}{\left|\mathbb{N}(\gamma_j \mathbf{z} + \delta_j)\right|^{2s}} c_2^s d^{2s} = c_2^s d^{2s} \mathbb{N}(\operatorname{Im} \mathrm{P}_j \mathbf{z})^s = c_2^s d^{2s} \mathbb{N}(\operatorname{Im} \mathrm{A}^{-1} \mathrm{M}_j \mathbf{z})^s$$

Integrating this inequality over U for the invariant measure gives

$$\frac{1}{\left|\mathbb{N}(\gamma_j \mathrm{i} + \delta_j)\right|^{2s}} \int_{\mathrm{U}} \mathrm{d}\omega \leqslant c_2^s d^{2s} \int_{\mathrm{U}} \mathbb{N}(\operatorname{Im} \mathrm{A}^{-1} \mathrm{M}_j \mathbf{z})^s \, \mathrm{d}\omega$$

The integral on the left is simply the volume of U, which is finite since U is compact; we simply denote it by $\mathcal{V}$. In the integral on the righthandside, we make the substitution $\mathrm{M}_j \mathbf{z} \longrightarrow \mathbf{z}$ and use the invariance of the measure $\mathrm{d}\omega$:

$$\frac{\mathcal{V}}{\left|\mathbb{N}(\gamma_j \mathrm{i} + \delta_j)\right|^{2s}} \leqslant c_2^s d^{2s} \int_{\mathrm{U}_{\mathrm{M}_i}} \mathbb{N}(\mathbf{y}^\star)^s \, \mathrm{d}\omega$$

We proved with **Lemma 20** that $\mathrm{d}\omega = \frac{\mathrm{d}\tau}{cq^2}$ where c is just a constant depending on the field K. Thus

$$\frac{\mathcal{V}}{\left|\mathbb{N}(\gamma_j \mathrm{i} + \delta_j)\right|^{2s}} \leqslant \frac{c_2^s d^{2s}}{c} \int_{\mathrm{U}_{\mathrm{M}_i}} q^{s-2} \, \mathrm{d}\tau$$

Finally, we sum over all values of j:

$$\sum_{(\gamma_j, \delta_j) \in \mathcal{P}} \frac{1}{\left|\mathbb{N}(\gamma_j \mathrm{i} + \delta_j)\right|^{2s}} \leqslant \frac{c_2^s d^{2s}}{c\mathcal{V}} \mathrm{J}_s = \frac{c_2^s d^{2s} d_0^{2s-2}}{c(s-1)\mathcal{V}} \leqslant \frac{c_2^{s_1} d^{2s_1} d_0^{2s_1-2}}{c(s_0-1)\mathcal{V}}$$

But we proved in **Lemma 21** that our sum is precisely over all non-associated pairs (γ, δ) generating $\mathfrak{a}$. Thus, we have just shown that the Hilbert modular Eisenstein series converges at $(\mathrm{i}, \ldots, \mathrm{i})$.

Getting the convergence at every $\mathbf{z}$ is now straightforward, thanks to **Lemma 6**. Indeed, if K is compact in $\mathscr{H}^n$, there exist positive constants c_3 and c_4 such that

$$\forall \mathbf{z} \in \mathrm{K} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^n \qquad c_3 \big|\mathbb{N}(\mathbf{u}\mathrm{i} + \mathbf{v})\big|^2 \leqslant \big|\mathbb{N}(\mathbf{u}\mathbf{z} + \mathbf{v})\big|^2 \leqslant c_4 \big|\mathbb{N}(\mathbf{u}\mathrm{i} + \mathbf{v})\big|^2$$

Thus $$\sum_{(\gamma_j,\delta_j)\in\mathcal{P}} \frac{1}{\big|\mathbb{N}(\gamma_j \mathbf{z} + \delta_j)\big|^{2s}} \leqslant \frac{1}{c_3^s} \sum_{(\gamma_j,\delta_j)\in\mathcal{P}} \frac{1}{\big|\mathbb{N}(\gamma_j \mathrm{i} + \delta_j)\big|^{2s}} \leqslant \frac{c_2^{s_1} d^{2s_1} d_0^{2s_1-2}}{c_3^{s_0} c(s_0-1)\mathcal{V}}$$

The Hilbert modular Eisenstein series converges uniformly in K. The bound on the righthandside, independent of s, shows that the convergence is also uniform in s in the strip $\{s_1 > \operatorname{Re} s > s_0 > 1\}$.

Now, on to checking that the definition does not depend on the choices for $\mathfrak{a} \in \mathrm{A}^{-1}$ and for the non-associated pairs of generators for $\mathfrak{a}$. The second claim is an immediate consequence of the product formula for units:

$$\forall \varepsilon \in \mathfrak{o}^\times \qquad \big|\mathbb{N}(\varepsilon)\big| = \prod_{i=1}^{n} \big|\varepsilon^{(i)}\big| = 1$$

so the product in the Eisenstein series is unchanged when replacing any pair (γ, δ) by an associated one $(\varepsilon\gamma, \varepsilon\delta)$.

As for the first claim, suppose that $\mathfrak{a}_1$ is another ideal in A^{-1}. Then, for some $a \in \mathrm{K}$, we have

$$\mathfrak{a} = a\mathfrak{a}_1$$

Thus $$\mathbb{N}\mathfrak{a} = \big|\mathbb{N}(a)\big|\mathbb{N}\mathfrak{a}_1$$

Also, $$\big\{(\gamma,\delta) \in \mathfrak{a}^2/\mathfrak{o}^\times \mid \langle\gamma,\delta\rangle = \mathfrak{a}\big\} = \big\{(a\gamma, a\delta) \mid (\gamma,\delta) \in \mathfrak{a}_1/\mathfrak{o}^\times \quad \langle\gamma,\delta\rangle = \mathfrak{a}_1\big\}$$

Hence $$\mathbb{N}(\mathfrak{a})^{2s} \sum_{\substack{(\gamma,\delta)\in\mathfrak{a}^2/\mathfrak{o}^\times \\ \langle\gamma,\delta\rangle=\mathfrak{a}}} \frac{\mathbb{N}(\mathbf{y})^s}{\big|\mathbb{N}(\gamma\mathbf{z}+\delta)\big|^{2s}} = \big|\mathbb{N}(a)\mathbb{N}(\mathfrak{a}_1)\big|^{2s} \sum_{\substack{(\gamma,\delta)\in\mathfrak{a}_1^2/\mathfrak{o}^\times \\ \langle\gamma,\delta\rangle=\mathfrak{a}_1}} \frac{\mathbb{N}(\mathbf{y})^s}{\big|\mathbb{N}(a\gamma\mathbf{z}+a\delta)\big|^{2s}}$$

$$= \mathbb{N}(\mathfrak{a}_1)^{2s} \sum_{\substack{(\gamma,\delta)\in\mathfrak{a}_1^2/\mathfrak{o}^\times \\ \langle\gamma,\delta\rangle=\mathfrak{a}_1}} \frac{\mathbb{N}(\mathbf{y})^s}{\big|\mathbb{N}(\gamma\mathbf{z}+\delta)\big|^{2s}}$$

The proof is now complete. $\square$

Now that the existence of the Eisenstein series has been shown, we can investigate its properties. In particular:

Theorem 23

The Hilbert modular Eisenstein series $\mathrm{E}_{\mathrm{K,A}}$ *is* $\mathrm{SL}_2(\mathfrak{o})$*-automorphic in the variable* $\boldsymbol{z}$*, for every given* s *such that* $\mathrm{Re}\, s > 1$*.*

Proof: In view of **Lemma 21**, we know that for every $\mathbf{z} \in \mathscr{H}^n$ and s with real part greater than 1,

$$\mathrm{E}_{\mathrm{K,A}}(\mathbf{z}, s) = \mathbb{N}(\mathfrak{a})^{2s} \sum_{\mathrm{M} \in \Gamma_\lambda \backslash \mathscr{M}} \mathrm{Im}\,(\mathrm{A}^{-1}\mathrm{M}\mathbf{z})^s$$

If N is in $\mathrm{SL}_2(\mathfrak{o})$, the operation

$$\begin{aligned} \Gamma_\lambda \backslash \mathscr{M} &\longrightarrow \Gamma_\lambda \backslash \mathscr{M} \\ \Gamma_\lambda \mathrm{M} &\longmapsto \Gamma_\lambda \mathrm{M}\mathrm{N}^{-1} \end{aligned}$$

simply reorganizes the cosets and is a bijection. Therefore,

$$\mathrm{E}_{\mathrm{K,A}}(\mathrm{N}\mathbf{z}, s) = \mathbb{N}(\mathfrak{a})^{2s} \sum_{\mathrm{M} \in \Gamma_\lambda \backslash \mathscr{M}} \mathrm{Im}\,(\mathrm{A}^{-1}\mathrm{M}\mathrm{N}\mathbf{z})^s = \mathrm{E}_{\mathrm{K,A}}(\mathbf{z}, s)$$

after replacing M by $\mathrm{M}\mathrm{N}^{-1}$. $\square$

In particular, $\mathrm{E}_{\mathrm{K,A}}\big((\mathbf{x} + o) + \mathrm{i}\mathbf{y}, s\big) = \mathrm{E}_{\mathrm{K,A}}(\mathbf{x} + \mathrm{i}\mathbf{y}, s)$ for every algebraic integer $o \in \mathfrak{o}$. In other words, $\mathrm{E}_{\mathrm{K,A}}$ is $\mathfrak{o}$-periodic in the variable $\mathbf{x}$ and it is natural to investigate its Fourier coefficients as a function on the group $\mathbb{R}^n/\mathfrak{o}$.

2.2 Fourier expansion and functional equation

In this section, we compute the Fourier expansion of the Eisenstein series. This will be useful in two ways: first, it will provide us with a meromorphic continuation in s to $\mathbb{C}$, as well as a functional equation; second, this expansion will be required in the next chapter, in order to express the renormalized Rankin-Selberg that is central to this thesis.

2.2.1 Normalization of the Eisenstein series

We wish to modify slightly the Hilbert modular Eisenstein series, in order to make its functional equation easier to express. To give a simple example where something similar is done, let's look at the usual Riemann zeta function. It is defined as

$$\forall s \in \mathbb{C} \quad \operatorname{Re} s > 1 \qquad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

The functional equation for ζ is

$$\zeta(1-s) = \pi^{s-\frac{1}{2}} \frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})} \zeta(s)$$

which means that, if we define the normalized ζ function

$$\zeta^{\star}(s) = \pi^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

then

$$\zeta^{\star}(1-s) = \zeta^{\star}(s) \tag{13}$$

Finding the right Γ factors and powers of π to put in front of the zeta function is called finding the right normalization. It turns out that the Hilbert modular Eisenstein series has to be normalized as well in order to get a nice functional such as (**13**). This normalization involves zeta functions of ideal classes for K.

We remind the reader of the definition of the normalized zeta function of an ideal class A of K:

$$\zeta^{\star}_{\mathrm{K,A}}(\mathbf{z}, s) = \mathrm{D}^{\frac{s}{2}} \pi^{-\frac{ns}{2}} \Gamma\left(\frac{s}{2}\right)^n \sum_{\mathfrak{b} \text{ integral } \in \mathrm{A}} \frac{1}{\mathbb{N}(\mathfrak{b})^s}$$

Its thorough study is done in Lang's *Algebraic Number Theory*, chapter 8 and 13 [6]. If $\mathfrak{a}$ is an ideal in A^{-1}, by definition of an ideal class, we have

$$\mathrm{A} = \left\{ \xi \mathfrak{a}^{-1} \mid \xi \in \mathrm{K}/\mathfrak{o}^{\times} \right\}$$

A fractional ideal in $\mathfrak{a}$ will be integral if and only if

$$\xi \mathfrak{a}^{-1} \subset \mathfrak{o} \iff \langle \xi \rangle \subset \mathfrak{a}$$

Therefore, we get all integral ideals $\mathfrak{b}$ in A as $\xi \mathfrak{a}^{-1}$, where $\xi \in \mathfrak{a}/\mathfrak{o}^{\times}$:

$$\zeta^{\star}_{\mathrm{K,A}}(\mathbf{z}, s) = \mathrm{D}^{\frac{s}{2}} \pi^{-\frac{ns}{2}} \Gamma\left(\frac{s}{2}\right)^n \sum_{\mathfrak{b} \text{ integral } \in \mathrm{A}} \frac{1}{\mathbb{N}(a^{-1}\xi)^s} = \mathrm{D}^{\frac{s}{2}} \pi^{-\frac{ns}{2}} \Gamma\left(\frac{s}{2}\right)^n \sum_{\xi \in \mathfrak{a}/\mathfrak{o}^{\times}} \frac{\mathbb{N}(\mathfrak{a})^s}{\left|\mathbb{N}(\xi)\right|^s}$$

We remind the reader that $\mathscr{C}$ denotes the ideal class group of K.

Definition 24 Let A be an ideal class of K. The normalized Hilbert modular Eisenstein series associated to A is defined as

$$\mathrm{E}^{\star}_{\mathrm{K,A}}(\mathsf{z},s)=\sum_{\mathrm{B}\in\mathscr{C}}\zeta^{\star}_{\mathrm{K,AB}^{-1}}(2s)\mathrm{E}_{\mathrm{K,B}}(\mathsf{z},s)\qquad \mathsf{z}\in\mathscr{H}^n\quad \mathrm{Re}\,s>1$$

We want to prove that $\mathrm{E}^{\star}_{\mathrm{K,A}}$ can actually be rewritten in a nicer fashion:

Proposition 25

Let A *be an ideal class of* K *and* $\mathfrak{a}$ *an ideal in* A^{-1}*. Then*

$$\mathrm{E}^{\star}_{\mathrm{K,A}}(\boldsymbol{z},s)=\mathbb{N}(\mathfrak{a})^{2s}\mathrm{D}^s\pi^{-ns}\Gamma(s)^n\sum_{\substack{(\alpha,\beta)\in\mathfrak{a}^2/\mathfrak{o}^\times\\(\alpha,\beta)\neq0}}\frac{\mathbb{N}(\boldsymbol{y})^s}{\left|\mathbb{N}(\alpha\boldsymbol{z}+\beta)\right|^{2s}}$$

This is merely a computation, except for one technicality taken care of by the following

Lemma 26

Let A *and* B *be two ideal classes in* K*, let* $\mathfrak{a},\mathfrak{b}$ *be ideals respectively in* A^{-1} *and* B^{-1}*. The map*

$$\begin{aligned}\varphi:\ \mathfrak{a}\mathfrak{b}^{-1}/\mathfrak{o}^\times\times\left\{(\gamma,\delta)\in\mathfrak{b}^2/\mathfrak{o}^\times\mid\langle\gamma,\delta\rangle=\mathfrak{b}\right\}&\longrightarrow\mathfrak{a}\times\mathfrak{a}\\(\xi,\gamma,\delta)&\longmapsto(\xi\gamma,\xi\delta)\end{aligned}$$

is a bijection onto a set of representatives for $\mathfrak{a}^2/\mathfrak{o}^\times$*, that generate an ideal of* B^{-1}*.*

Proof: Of course, if $\xi\in\mathfrak{a}\mathfrak{b}^{-1}$ and γ,δ are in $\mathfrak{b}$ and generate it as an ideal, then

$$\xi\gamma,\xi\delta\in\mathfrak{a}\mathfrak{b}^{-1}\,\mathfrak{b}=\mathfrak{a}$$

Furthermore $\langle\xi\gamma,\xi\delta\rangle=\xi\langle\gamma,\delta\rangle=\xi\mathfrak{b}\in\mathrm{B}^{-1}$

Now, let (α,β) be in $\mathfrak{a}^2/\mathfrak{o}^\times$, such that $\langle\alpha,\beta\rangle$ is in the ideal class B^{-1}. Thus there exists a $\xi\in\mathrm{K}$ such that $\langle\alpha,\beta\rangle=\xi\mathfrak{b}$. But

$$\langle\alpha,\beta\rangle\subset\mathfrak{a}$$

so $\xi\in\langle\alpha,\beta\rangle\mathfrak{b}^{-1}\subset\mathfrak{a}\mathfrak{b}^{-1}$ and $\xi^{-1}\in\mathfrak{a}^{-1}\mathfrak{b}$

Thus $$\gamma = \frac{\alpha}{\xi} \qquad \text{and} \qquad \delta = \frac{\beta}{\xi}$$
are both in $\mathfrak{b}$, and satisfy
$$\langle \gamma, \delta \rangle = \xi^{-1}\langle \alpha, \beta \rangle = \xi^{-1}\xi\mathfrak{b} = \mathfrak{b}$$
This shows that φ is surjective, which was the easier part.

Suppose now that we have two triples (ξ, γ, δ) and (ξ', γ', δ') in
$$\mathfrak{a}\mathfrak{b}^{-1}/\mathfrak{o}^{\times} \times \left\{(\gamma, \delta) \in \mathfrak{b}^2/\mathfrak{o}^{\times} \mid \langle \gamma, \delta \rangle = \mathfrak{b}\right\}$$
such that $(\xi\gamma, \xi\delta)$ and $(\xi'\gamma', \xi'\delta')$ are associated. There exists a unit ε such that
$$\xi\gamma = \varepsilon\xi'\gamma' \qquad \text{and} \qquad \xi\delta = \varepsilon\xi'\delta' \tag{\textbf{14}}$$
It follows that
$$\gamma\delta' = \gamma'\delta \tag{\textbf{15}}$$
and
$$\frac{\langle \gamma \rangle}{\mathfrak{b}} \frac{\langle \delta' \rangle}{\mathfrak{b}} = \frac{\langle \gamma' \rangle}{\mathfrak{b}} \frac{\langle \delta \rangle}{\mathfrak{b}}$$
But, for reasons already explained while proving **Theorem 3**, the ideals $\frac{\langle \gamma \rangle}{\mathfrak{b}}$ and $\frac{\langle \delta \rangle}{\mathfrak{b}}$ are relatively prime because $\langle \gamma, \delta \rangle = \mathfrak{b}$. The same holds for $\frac{\langle \gamma' \rangle}{\mathfrak{b}}$ and $\frac{\langle \delta' \rangle}{\mathfrak{b}}$. Thus
$$\frac{\langle \gamma \rangle}{\mathfrak{b}} = \frac{\langle \gamma' \rangle}{\mathfrak{b}} \qquad \text{and} \qquad \frac{\langle \delta \rangle}{\mathfrak{b}} = \frac{\langle \delta' \rangle}{\mathfrak{b}}$$
So there exist units ε_1 and ε_2 such that
$$\gamma' = \varepsilon_1\gamma \qquad \text{and} \qquad \delta' = \varepsilon_2\delta$$
But by (**15**), $$\varepsilon_2\gamma\delta = \varepsilon_1\gamma\delta$$
which implies $$\varepsilon_1 = \varepsilon_2 \qquad \text{and} \qquad (\gamma', \delta') = \varepsilon_1(\gamma, \delta)$$
Since (γ, δ) and (γ', δ') are in $\mathfrak{b}^2/\mathfrak{o}^{\times}$, they are equal. This in turn implies, because of (**14**), that ξ and ξ' are associated, thus equal. So φ is injective. □

Proof of Proposition 25: For every ideal class $\mathrm{B} \in \mathscr{C}$, we let $\mathfrak{b}_{\mathrm{B}}$ (or $\mathfrak{b}$ when there is no ambiguity) be an ideal in B^{-1}.

We let A be a particular one of those classes, and note $\mathfrak{a} = \mathfrak{b}_{\mathrm{A}}$. If $\mathbf{z} \in \mathscr{H}^n$ and s has real part bigger than 1, **Definition 24** tells us that

$$\begin{aligned}
\mathrm{E}^{\star}_{\mathrm{K,A}}(\mathbf{z}, s) &= \sum_{\mathrm{B} \in \mathscr{C}} \zeta_{\mathrm{K,AB}^{-1}}(2s) \mathrm{E}_{\mathrm{K,B}}(\mathbf{z}, s) \\
&= \mathrm{D}^s \pi^{-ns} \Gamma(s)^n \sum_{\mathrm{B} \in \mathscr{C}} \sum_{\xi \in \mathfrak{a}\mathfrak{b}^{-1}/\mathfrak{o}^{\times}} \frac{\mathbb{N}(\mathfrak{a}\mathfrak{b}^{-1})^{2s}}{\left|\mathbb{N}(\xi)\right|^{2s}} \times \mathbb{N}(\mathfrak{b})^{2s} \sum_{\substack{(\gamma,\delta) \in \mathfrak{b}^2/\mathfrak{o}^{\times} \\ \langle \gamma,\delta \rangle = \mathfrak{b}}} \frac{\mathbb{N}(\mathbf{y})^s}{\left|\mathbb{N}(\gamma \mathbf{z} + \delta)\right|^{2s}}
\end{aligned}$$

$$\mathrm{E}^{\star}_{\mathrm{K,A}}(\mathbf{z}, s) = \mathbb{N}(\mathfrak{a})^{2s} \mathrm{D}^s \pi^{-ns} \Gamma(s)^n \sum_{\mathrm{B} \in \mathscr{C}} \sum_{\xi \in \mathfrak{a}\mathfrak{b}^{-1}/\mathfrak{o}^{\times}} \sum_{\substack{(\gamma,\delta) \in \mathfrak{b}^2/\mathfrak{o}^{\times} \\ \langle \gamma,\delta \rangle = \mathfrak{b}}} \frac{\mathbb{N}(\mathbf{y})^s}{\left|\mathbb{N}(\xi\gamma \mathbf{z} + \xi\delta)\right|^{2s}}$$

We now invoke **Lemma 26** in order to collapse the two inner sums:

$$\mathrm{E}^{\star}_{\mathrm{K,A}}(\mathbf{z}, s) = \mathbb{N}(\mathfrak{a})^{2s} \mathrm{D}^s \pi^{-ns} \Gamma(s)^n \sum_{\mathrm{B} \in \mathscr{C}} \sum_{\substack{(\alpha,\beta) \in \mathfrak{a}^2/\mathfrak{o}^{\times} \\ \langle \alpha,\beta \rangle \in \mathrm{B}}} \frac{\mathbb{N}(\mathbf{y})^s}{\left|\mathbb{N}(\alpha \mathbf{z} + \beta)\right|^{2s}}$$

And since the ideal classes partition the collection of fractional ideals, the conclusion follows:

$$\mathrm{E}^{\star}_{\mathrm{K,A}}(\mathbf{z}, s) = \mathbb{N}(\mathfrak{a})^{2s} \mathrm{D}^s \pi^{-ns} \Gamma(s)^n \sum_{\substack{(\alpha,\beta) \in \mathfrak{a}^2/\mathfrak{o}^{\times} \\ (\alpha,\beta) \neq 0}} \frac{\mathbb{N}(\mathbf{y})^s}{\left|\mathbb{N}(\alpha \mathbf{z} + \beta)\right|^{2s}} \qquad \square$$

2.2.2 The Fourier expansion and its consequences

The inverse different $\mathfrak{D}^{-1}$ of K is defined as

$$\mathfrak{D}^{-1} = \{\xi \in \mathrm{K} \mid \mathrm{Tr}\,(\xi\mathfrak{o}) \subset \mathbb{Z}\}$$

Therefore, a complete set of characters for the compact group $\mathbb{R}^n/\mathfrak{o}$ is the collection of maps

$$\mathbf{x} \longmapsto \mathrm{e}^{2\pi \mathrm{i} \mathrm{Tr}\,(\mathbf{x}\xi)} \qquad \xi \in \mathfrak{D}^{-1}$$

We will need Bessel functions and divisor functions in order to express nicely the Fourier coefficients of the Eisenstein series:

$$\mathrm{K}_{\omega}(u) = \frac{1}{2} \int_0^{+\infty} \mathrm{e}^{-\frac{u}{2}(t+\frac{1}{t})} t^{\omega} \, \frac{\mathrm{d}t}{t} \qquad \omega \in \mathbb{C} \quad u > 0$$

$$\tau_{\omega}^{\mathrm{K,A}}(\mathfrak{c}) = \mathbb{N}(\mathfrak{c})^{-\omega} \sum_{\substack{\mathfrak{b} \text{ ideal in A} \\ \mathfrak{b}|\mathfrak{c}}} \mathbb{N}(\mathfrak{b})^{2\omega} \qquad \omega \in \mathbb{C} \quad \mathfrak{c} \text{ ideal in } \mathfrak{o}$$

All we need to know about the divisor function is that it is invariant under $\omega \longmapsto -\omega$, which is evident from the definition. As for the Bessel function, it has the same invariance; it has also rapid decay at $+\infty$:

$$\forall u > 2 \qquad \left|\mathrm{K}_{\omega}(u)\right| \leqslant \mathrm{K}_{\mathrm{Re}\,\omega}(2)\mathrm{e}^{-\frac{u}{2}}$$

And we will need the following integral formulas, valid for $y > 0$ and $\mathrm{Re}\, s > 1$:

$$y^s \pi^{-s}\Gamma(s) \int_{\mathbb{R}} \frac{\mathrm{e}^{-2\pi \mathrm{i} r x}}{(x^2+y^2)^s}\,\mathrm{d}x = \begin{cases} \pi^{-s+\frac{1}{2}}\Gamma\Big(s-\dfrac{1}{2}\Big)y^{1-s} & \text{if } r = 0 \\ 2|r|^{s-\frac{1}{2}}\sqrt{y}\mathrm{K}_{s-\frac{1}{2}}(2\pi y|r|) & \text{if } r = 0 \end{cases}$$

Those facts are rapidly proved in Bump's *Automorphic forms and Representations* [3], chapter 1.6. This last formula, in particular, implies that, for $y_1, \ldots, y_n > 0$ and $\xi \in \mathfrak{D}^{-1}$,

$$\mathbb{N}(\mathbf{y})^s \pi^{-ns}\Gamma(s)^n \int_{\mathbb{R}^n} \frac{\mathrm{e}^{-2\pi \mathrm{i} \mathrm{Tr}\,\mathbf{x}\xi}}{\mathbb{N}(\mathbf{x}^2+\mathbf{y}^2)^s}\,\mathrm{d}\mathbf{x} = \begin{cases} \pi^{-n(s-\frac{1}{2})}\Gamma\Big(s-\dfrac{1}{2}\Big)^n \mathbb{N}(\mathbf{y})^{1-s} \\ 2^n\left|\mathbb{N}(\xi)\right|^{s-\frac{1}{2}}\mathbb{N}(\mathbf{y})^{\frac{1}{2}} \prod\limits_{j=1}^{n} \mathrm{K}_{s-\frac{1}{2}}\big(2\pi y_j|\xi^{(j)}|\big) \end{cases} \qquad \textbf{(16)}$$

respectively when $\xi = 0$ and when $\xi \neq 0$.

Theorem 27

For every $\boldsymbol{z} \in \mathscr{H}^n$ and $s \in \mathbb{C}$ with real part greater than 1, we have

$$\begin{aligned} \mathrm{E}^{\star}_{\mathrm{K,A}}(\boldsymbol{z}, s) = {} & \mathbb{N}(\boldsymbol{y})^s\zeta^{\star}_{\mathrm{K,A}}(2s) + \mathbb{N}(\boldsymbol{y})^{1-s}\zeta^{\star}_{\mathrm{K,A}}(2-2s) \\ & + 2^n\mathbb{N}(\boldsymbol{y})^{\frac{1}{2}} \sum_{\xi \in \mathfrak{D}^{-1}} \tau^{\mathrm{K,A}}_{s-\frac{1}{2}}(\xi\mathfrak{D}) \prod_{j=1}^{n} \mathrm{K}_{s-\frac{1}{2}}\big(2\pi y_j|\xi^{(j)}|\big)\, e^{2\pi i Tr\, \boldsymbol{x}\xi} \end{aligned}$$

Proof: Let $\xi \in \mathfrak{D}^{-1}$ and let's compute the Fourier coefficient at ξ of the normalized Eisenstein series:

$$c(\xi) = \frac{1}{\mathrm{Vol}(\mathbb{R}^n/\mathfrak{o})} \int_{\mathbb{R}^n/\mathfrak{o}} \mathrm{E}^{\star}_{\mathrm{K,A}}(\mathbf{z}, s)\mathrm{e}^{-2\pi \mathrm{i}\mathrm{Tr}\,(\mathbf{x}\xi)}\,\mathrm{d}\mathbf{x}$$

We temporarily call $\Lambda(s)$ the factors that don't depend on $\mathbf{x}$ in the integrand:

$$\Lambda(s) = \frac{\mathbb{N}(\mathfrak{a})^{2s}\mathrm{D}^s\pi^{-ns}\Gamma(s)^n\mathbb{N}(\mathbf{y})^s}{\mathrm{Vol}(\mathbb{R}^n/\mathfrak{o})}$$

so that
$$c(\xi) = \Lambda(s) \sum_{\substack{(\alpha,\beta)\in\mathfrak{a}^2/\mathfrak{o}^\times \\ (\alpha,\beta)\neq 0}} \int_{\mathbb{R}^n/\mathfrak{o}} \frac{\mathrm{e}^{-2\pi\mathrm{i}\,\mathrm{Tr}\,(\mathbf{x}\xi)}}{\left|\mathbb{N}(\alpha\mathbf{z}+\beta)\right|^{2s}}\,\mathrm{d}\mathbf{x}$$

We get out of the way the terms where $\alpha = 0$, since the integrand is then just a character of $\mathbb{R}^n/\mathfrak{o}$, that integrates to 0 unless it is the trivial character:

$$\sum_{\substack{\beta\in\mathfrak{a}/\mathfrak{o}^\times \\ \beta\neq 0}} \frac{1}{\left|\mathbb{N}(\beta)\right|^{2s}} \int_{\mathbb{R}^n/\mathfrak{o}} \mathrm{e}^{-2\pi\mathrm{i}\,\mathrm{Tr}\,\mathbf{x}\xi}\,\mathrm{d}\mathbf{x} = \begin{cases} \mathrm{Vol}(\mathbb{R}^n/\mathfrak{o})\mathbb{N}(\mathfrak{a})^{-2s}\zeta_{\mathrm{K,A}}(2s) & \text{if } \xi = 0 \\ 0 & \text{if } \xi \neq 0 \end{cases}$$

and
$$\Lambda(s) \sum_{\substack{\beta\in\mathfrak{a}/\mathfrak{o}^\times \\ \beta\neq 0}} \frac{1}{\left|\mathbb{N}(\beta)\right|^{2s}} \int_{\mathbb{R}^n/\mathfrak{o}} \mathrm{e}^{-2\pi\mathrm{i}\,\mathrm{Tr}\,\mathbf{x}\xi}\,\mathrm{d}\mathbf{x} = \begin{cases} \mathbb{N}(\mathbf{y})^s\zeta^\star_{\mathrm{K,A}}(2s) & \text{if } \xi = 0 \\ 0 & \text{if } \xi \neq 0 \end{cases} \tag{17}$$

This done, we keep it aside until the computation of the other terms is completed. For convenience, call them $b(\xi)$:

$$b(\xi) = \Lambda(s) \sum_{\substack{\alpha\in\mathfrak{a}/\mathfrak{o}^\times \\ \alpha\neq 0}} \sum_{\beta\in\mathfrak{a}} \int_{\mathbb{R}^n/\mathfrak{o}} \frac{\mathrm{e}^{-2\pi\mathrm{i}\,\mathrm{Tr}\,(\mathbf{x}\xi)}}{\left|\mathbb{N}(\alpha\mathbf{z}+\beta)\right|^{2s}}\,\mathrm{d}\mathbf{x}$$

We can decompose $\mathfrak{a}$ into cosets modulo $\langle\alpha\rangle$ for each α:

$$b(\xi) = \Lambda(s) \sum_{\substack{\alpha\in\mathfrak{a}/\mathfrak{o}^\times \\ \alpha\neq 0}} \sum_{\beta\in\mathfrak{a}/\langle\alpha\rangle} \sum_{o\in\mathfrak{o}} \int_{\mathbb{R}^n/\mathfrak{o}} \frac{\mathrm{e}^{-2\pi\mathrm{i}\,\mathrm{Tr}\,(\mathbf{x}\xi)}}{\left|\mathbb{N}(\alpha\mathbf{z}+\beta+\alpha o)\right|^{2s}}\,\mathrm{d}\mathbf{x}$$

Replacing $\mathbf{x}$ by $\mathbf{x} - o$ allows us to collapse the integrals together. Remember that $\mathrm{Tr}\,(o\xi) \in \mathbb{Z}$ since $\xi \in \mathfrak{D}^{-1}$;

$$\begin{aligned} b(\xi) &= \Lambda(s) \sum_{\substack{\alpha\in\mathfrak{a}/\mathfrak{o}^\times \\ \alpha\neq 0}} \sum_{\beta\in\mathfrak{a}/\langle\alpha\rangle} \sum_{o\in\mathfrak{o}} \int_{-o+\mathbb{R}^n/\mathfrak{o}} \frac{\mathrm{e}^{-2\pi\mathrm{i}\,\mathrm{Tr}\,(\mathbf{x}\xi)}}{\left|\mathbb{N}(\alpha\mathbf{z}+\beta)\right|^{2s}}\,\mathrm{d}\mathbf{x} \\ &= \Lambda(s) \sum_{\substack{\alpha\in\mathfrak{a}/\mathfrak{o}^\times \\ \alpha\neq 0}} \sum_{\beta\in\mathfrak{a}/\langle\alpha\rangle} \int_{\mathbb{R}^n} \frac{\mathrm{e}^{-2\pi\mathrm{i}\,\mathrm{Tr}\,(\mathbf{x}\xi)}}{\left|\mathbb{N}(\alpha\mathbf{z}+\beta)\right|^{2s}}\,\mathrm{d}\mathbf{x} \\ b(\xi) &= \Lambda(s) \sum_{\substack{\alpha\in\mathfrak{a}/\mathfrak{o}^\times \\ \alpha\neq 0}} \sum_{\beta\in\mathfrak{a}/\langle\alpha\rangle} \int_{\mathbb{R}^n} \frac{\mathrm{e}^{-2\pi\mathrm{i}\,\mathrm{Tr}\,(\mathbf{x}\xi)}}{\left|\mathbb{N}(\alpha\mathbf{x}+\beta+\mathrm{i}\alpha\mathbf{y})\right|^{2s}}\,\mathrm{d}\mathbf{x} \end{aligned}$$

In each integral, factor out $|\mathbb{N}(\alpha)|^{2s}$ from the denominator, and replace $\mathbf{x}$ by $\mathbf{x} - \frac{\beta}{\alpha}$:

$$b(\xi) = \Lambda(s) \sum_{\substack{\alpha \in \mathfrak{a}/\mathfrak{o}^\times \\ \alpha \neq 0}} \frac{1}{\left|\mathbb{N}(\alpha)\right|^{2s}} \int\limits_{\mathbb{R}^n} \frac{\mathrm{e}^{-2\pi \mathrm{i} \operatorname{Tr} \mathbf{x}\xi}}{\left|\mathbb{N}(\mathbf{x} + \mathrm{i}\mathbf{y})\right|^{2s}} \,\mathrm{d}\mathbf{x} \sum_{\beta \in \mathfrak{a}/\langle\alpha\rangle} \mathrm{e}^{2\pi \mathrm{i} \operatorname{Tr} \frac{\beta\xi}{\alpha}}$$

The function $$\beta \longmapsto \mathrm{e}^{2\pi \mathrm{i} \operatorname{Tr} \frac{\beta\xi}{\alpha}}$$

is clearly a character of $\mathfrak{a}/\langle\alpha\rangle$. It will be trivial if and only if

$$\operatorname{Tr}\left(\tfrac{\xi}{\alpha}\,\mathfrak{a}\right) \subset \mathbb{Z} \iff \frac{\xi}{\alpha} \in \mathfrak{a}^{-1}\mathfrak{D}^{-1} \iff \xi\mathfrak{D} \subset \alpha\mathfrak{a}^{-1} \iff \alpha\mathfrak{a}^{-1} | \xi\mathfrak{D}$$

If this condition is satisfied, the sum on the right equals

$$\operatorname{Card}\big(\mathfrak{a}/\langle\alpha\rangle\big) = \frac{\left|\mathbb{N}(\alpha)\right|}{\mathbb{N}(\mathfrak{a})}$$

Otherwise, it is simply 0. Thus

$$b(\xi) = \frac{\Lambda(s)}{\mathbb{N}(\mathfrak{a})} \int\limits_{\mathbb{R}^n} \frac{\mathrm{e}^{-2\pi \mathrm{i} \operatorname{Tr} \mathbf{x}\xi}}{\left|\mathbb{N}(\mathbf{z})\right|^{2s}} \,\mathrm{d}\mathbf{x} \times \sum_{\substack{\alpha \in \mathfrak{a}/\mathfrak{o}^\times \\ \alpha\mathfrak{a}^{-1} | \xi\mathfrak{D}}} \frac{1}{\left|\mathbb{N}(\alpha)\right|^{2s-1}}$$

We explain already at the beginning of **section 2.2.1**, while presenting the zeta function of an ideal class, that the $(\alpha\mathfrak{a}^{-1})_{\alpha \in \mathfrak{a}/\mathfrak{o}^\times}$ run through all integral ideals in the class A. Thus, by letting $\mathfrak{b} = \alpha\mathfrak{a}^{-1}$ in the summation,

$$\begin{aligned} b(\xi) &= \frac{\Lambda(s)}{\mathbb{N}(\mathfrak{a})} \int\limits_{\mathbb{R}^n} \frac{\mathrm{e}^{-2\pi \mathrm{i} \operatorname{Tr} \mathbf{x}\xi}}{\left|\mathbb{N}(\mathbf{z})\right|^{2s}} \,\mathrm{d}\mathbf{x} \times \sum_{\substack{\mathfrak{b} \text{ ideal in A} \\ \mathfrak{b} | \xi\mathfrak{D}}} \frac{1}{\mathbb{N}(\mathfrak{a})^{2s-1}\mathbb{N}(\mathfrak{b})^{2s-1}} \\ &= \begin{cases} \dfrac{\Lambda(s)}{\mathbb{N}(\mathfrak{a})^{2s}} \times \dfrac{\tau^{\mathrm{K,A}}_{s-\frac{1}{2}}(\xi\mathfrak{D})}{\mathbb{N}(\xi\mathfrak{D})^{s-\frac{1}{2}}} \times \displaystyle\int\limits_{\mathbb{R}^n} \frac{\mathrm{e}^{-2\pi \mathrm{i} \operatorname{Tr} \mathbf{x}\xi}}{\left|\mathbb{N}(\mathbf{z})\right|^{2s}} \,\mathrm{d}\mathbf{x} & \text{if } \xi \neq 0 \\ \dfrac{\Lambda(s)}{\mathbb{N}(\mathfrak{a})^{2s}} \zeta_{\mathrm{K,A}}(\mathbf{z}, 2s-1) \displaystyle\int\limits_{\mathbb{R}^n} \frac{\mathrm{d}\mathbf{x}}{\left|\mathbb{N}(\mathbf{z})\right|^{2s}} & \text{if } \xi = 0 \end{cases} \end{aligned}$$

But remember that $$\Lambda(s) = \frac{\mathbb{N}(\mathfrak{a})^{2s} \mathrm{D}^s \pi^{-ns} \Gamma(s)^n \mathbb{N}(\mathbf{y})^s}{\operatorname{Vol}(\mathbb{R}^n/\mathfrak{o})}$$

Reinjecting this in our last expression for $b(\xi)$, and using the fact that $\mathrm{Vol}(\mathbb{R}^n/\mathfrak{o})$ is $\sqrt{\mathbb{N}(\mathfrak{D})} = \sqrt{\mathrm{D}}$, we get

$$b(\xi) = \begin{cases} \tau^{\mathrm{K,A}}_{s-\frac{1}{2}}(\xi\mathfrak{D})\, \dfrac{\pi^{-ns}\Gamma(s)^n\mathbb{N}(\mathbf{y})^s}{\left|\mathbb{N}(\xi)\right|^{s-\frac{1}{2}}} \displaystyle\int_{\mathbb{R}^n} \frac{\mathrm{e}^{-2\pi\mathrm{i}\,\mathrm{Tr}\,\mathbf{x}\xi}}{\left|\mathbb{N}(\mathbf{z})\right|^{2s}}\,\mathrm{d}\mathbf{x} & \text{if } \xi \neq 0 \\ \mathrm{D}^{s-\frac{1}{2}}\zeta_{\mathrm{K,A}}(2s-1)\mathbb{N}(\mathbf{y})^s\pi^{-ns}\Gamma(s)^n \displaystyle\int_{\mathbb{R}^n} \frac{\mathrm{d}\mathbf{x}}{\left|\mathbb{N}(\mathbf{z})\right|^{2s}} & \text{if } \xi = 0 \end{cases}$$

Next, according to formula (**16**),

$$b(\xi) = \begin{cases} 2^n\mathbb{N}(\mathbf{y})^{\frac{1}{2}}\tau^{\mathrm{K,A}}_{s-\frac{1}{2}}(\xi\mathfrak{D}) \displaystyle\prod_{j=1}^{n} \mathrm{K}_{s-\frac{1}{2}}\big(2\pi y_j|\xi^{(j)}|\big) & \text{if } \xi \neq 0 \\ \underbrace{\mathrm{D}^{s-\frac{1}{2}}\zeta_{\mathrm{K,A}}(2s-1)\pi^{-n(s-\frac{1}{2})}\Gamma\Big(s-\frac{1}{2}\Big)}_{=\zeta^{\star}_{\mathrm{K,A}}(2s-1)}\mathbb{N}(\mathbf{y})^{1-s} & \text{if } \xi = 0 \end{cases}$$

Putting this together with formula (**17**),

$$c(\xi) = \begin{cases} \mathbb{N}(\mathbf{y})^s\zeta^{\star}_{\mathrm{K,A}}(2s) + \mathbb{N}(\mathbf{y})^{1-s}\zeta^{\star}_{\mathrm{K,A}}(2s-1) & \text{if } \xi = 0 \\ 2^n\mathbb{N}(\mathbf{y})^{\frac{1}{2}}\tau^{\mathrm{K,A}}_{s-\frac{1}{2}}(\xi\mathfrak{D}) \displaystyle\prod_{j=1}^{n} \mathrm{K}_{s-\frac{1}{2}}\big(2\pi y|\xi^{(j)}|\big) & \text{if } \xi \neq 0 \end{cases}$$

Because Bessel functions have rapid decay, the $\big(c(\xi)\big)_{\xi\in\mathfrak{D}^{-1}}$ form an absolutely summable family; therefore, $\mathrm{E}^{\star}_{\mathrm{K,A}}(\mathbf{z}, s)$ is the sum of its Fourier series. □

Corollary 28

The Hilbert modular Eisenstein series can be analytically continued to $\mathbb{C}\setminus\{0,1\}$ *and satisfies the functional equation*

$$\forall \boldsymbol{z} \in \mathscr{H}^n \quad \forall s \in \mathbb{C}\setminus\{0,1\} \qquad \mathrm{E}^{\star}_{\mathrm{K,A}}(\boldsymbol{z}, 1-s) = \mathrm{E}^{\star}_{\mathrm{K,A}}(\boldsymbol{z}, s)$$

Proof: Those properties are clear, and inherited from the Bessel functions, the divisor functions and $\zeta^{\star}_{\mathrm{K,A}}$.

Chapter 3

Hidden functional equations of Rankin-Selberg transforms : New results

This final chapter constitutes the original mathematical work on my part. After a short introduction (**Section 3.1**) reminding the reader of the objects we will be dealing with and stating our main theorem, we proceed to building the proof (**Section 3.3**) by laying down, in **Section 3.2**, some facts that will be required all along.

3.1 Introduction and first notations

3.1.1 The $\mathrm{SL}_2(\mathbb{Z})$ Eisenstein series

The classical Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ is a function of two variables $z = x + \mathrm{i}y$, in the complex upper half plane $\mathscr{H}$, and $s \in \mathbb{C} \setminus \{0, 1\}$. It has the following "explicit" expression in the region $\{z \in \mathscr{H} \quad s \in \mathbb{C} \mid \operatorname{Re} s > 1\}$:

$$\mathrm{E}^\star(z, s) = \zeta^\star(2s)\mathrm{E}(z, s)$$

where

$$\mathrm{E}(z, s) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ \gcd(c,d)=1}} \frac{y^s}{|cz + d|^{2s}} \qquad (\mathbf{1})$$

and $$\zeta^\star(s) = \pi^{-s/2}\Gamma\left(\tfrac{s}{2}\right)\zeta(s).$$

This function is, for fixed s, automorphic in the variable z. When one actually fixes z, the function $s \longmapsto \mathrm{E}^\star(z,s)$ is analytic in the half plane $\{\operatorname{Re} s > 1\}$, has analytic continuation to $\mathbb{C}\setminus\{0,1\}$ with simple poles at 0 and 1, and is invariant under the transformation $s \longmapsto 1-s$.

3.1.2 The Hilbert modular Eisenstein series

We are given a finite field extension $\mathbb{Q} \hookrightarrow \mathrm{K}$ of degree N with discriminant D, that we suppose totally real. By definition, this means that the N distinct embeddings $\sigma_1,\ldots,\sigma_\mathrm{N}$ of K into $\mathbb{C}$ have actually real range. If x is an element of K, we define

$$\forall i \in \{1,\ldots,\mathrm{N}\} \qquad x^{(i)} = \sigma_i(x).$$

We write $\mathfrak{o}$ to denote the ring of integers of K; we know that $\mathfrak{o}$ is a free $\mathbb{Z}$-module of rank N. The Hilbert modular group is the group $\mathrm{SL}_2(\mathfrak{o})$ of 2×2 matrices with coefficients in $\mathfrak{o}$ and with determinant 1. It acts naturally on $\mathscr{H}^\mathrm{N}$ as follows:

$$\forall \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathfrak{o}) \quad \forall \mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_\mathrm{N} \end{bmatrix} \in \mathscr{H}^\mathrm{N} \qquad \gamma(\mathbf{z}) = \begin{bmatrix} \dfrac{a^{(1)}z_1 + b^{(1)}}{c^{(1)}z_1 + d^{(1)}} \\ \vdots \\ \dfrac{a^{(\mathrm{N})}z_\mathrm{N} + b^{(\mathrm{N})}}{c^{(\mathrm{N})}z_\mathrm{N} + d^{(\mathrm{N})}} \end{bmatrix}.$$

If A is an ideal class for K represented by a fractional ideal $\mathfrak{a}^{-1}$, the Hilbert modular Eisenstein series associated to A is the function of $\mathrm{N}+1$ variables $\mathbf{z} = (z_1,\ldots,z_n)$ in $\mathscr{H}^\mathrm{N}$ and $s \in \mathbb{C}\setminus\{0,1\}$, defined explicitely when $\operatorname{Re} s > 1$ by

$$\mathrm{E}^\star_{\mathrm{K,A}}(\mathbf{z},s) = \mathbb{N}(\mathfrak{a})^{2s}\pi^{-\mathrm{N}s}\Gamma(s)^\mathrm{N}\,\mathrm{D}^s \sum_{(\alpha,\beta)\in\mathfrak{a}^2/\mathfrak{o}^\times} \prod_{j=1}^{\mathrm{N}} \frac{(\operatorname{Im} z_j)^s}{\left|\alpha^{(j)}z_j + \beta^{(j)}\right|^{2s}}. \tag{2}$$

The action of $\mathfrak{o}^\times$ on $\mathfrak{a}$ is simply

$$\begin{aligned} \mathfrak{o}^\times \times \mathfrak{a} &\longrightarrow \mathfrak{a}, \\ (\varepsilon,\alpha) &\longmapsto \varepsilon\alpha. \end{aligned}$$

This series, especially its convergence, has been the extensive subject of **Chapter 2**. And one can check easily that the Hilbert modular Eisenstein series associated to A is well defined in $\{\mathrm{Re}\, s > 1\}$: its value does not depend on the choices for $\mathfrak{a}$ or for representatives in the quotient $\mathfrak{a}^2/\mathfrak{o}^\times$. Finally, just like the classical Eisenstein series, $\mathrm{E}^\star_{\mathrm{K,A}}$ is $\mathrm{SL}_2(\mathfrak{o})$-invariant in the variable $\mathbf{z}$, has analytic continuation in s to $\mathbb{C} \setminus \{0,1\}$ and is invariant under the transformation $s \longmapsto 1-s$.

After specializing $\mathrm{E}^\star_{\mathrm{K,A}}(\cdot, s)$ to the diagonal $\big\{(z,\dots,z) \bigm| z \in \mathscr{H}\big\}$, we obtain an $\mathrm{SL}_2(\mathbb{Z})$-automorphic function which is denoted $\mathrm{E}^\star_{\mathrm{K,A}}(z,s)$.

3.1.3 Zagier's renormalization

If F is an $\mathrm{SL}_2(\mathbb{Z})$-automorphic function that decays faster than any polynomial for large values of y, the following integral converges absolutely for $\mathrm{Re}\, s > 1$:

$$\mathrm{R}(\mathrm{F}, s) = \int\limits_{\mathrm{SL}_2(\mathbb{Z})/\mathscr{H}} \mathrm{E}^\star(z,s)\mathrm{F}(z)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}. \tag{3}$$

It inherits, from the Eisenstein series, analytic continuation to $\mathbb{C} \setminus \{0,1\}$ and an invariance under $s \longmapsto 1-s$. It is called the Rankin-Selberg transform of F, due to the fact that Rankin and Selberg noticed that this integral can be unfolded in order to yield

$$\mathrm{R}(\mathrm{F}, s) = \zeta^\star(2s)\int\limits_0^\infty a_0(y) y^{s-2}\,\mathrm{d}y, \tag{4}$$

where $a_0(y)$ is the constant term in the Fourier expansion of F.

In [9], Zagier lifts the restriction about the rapid decay of F for large values of y, defines the Rankin-Selberg transform for a much larger class of automorphic functions and analyzes its properties. We recall here part of his main result.

Let F be an automorphic function and suppose there exists a function φ of the form

$$\varphi(y) = \sum_{i=1}^{\ell} \frac{c_i}{n_i!} y^{\alpha_i} \log^{n_i} y \qquad c_i, \alpha_i \in \mathbb{C} \quad \ell, n_i \in \mathbb{N},$$

such that, for every positive integer n,

$$\mathrm{F}(z) \underset{y\to\infty}{=} \varphi(y) + \mathrm{O}\big(y^{-n}\big).$$

The renormalized Rankin-Selberg tranform of F is then defined as

$$\mathrm{RN}\int\limits_{\mathrm{SL}_2(\mathbb{Z})/\mathscr{H}} \mathrm{E}^\star(z,s)\mathrm{F}(z)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \mathrm{R}(\mathrm{F},s) \underset{\mathrm{def}}{=} \zeta^\star(2s)\int\limits_0^\infty \big(a_0(y)-\varphi(y)\big)y^{s-2}\,\mathrm{d}y.$$

Zagier shows that this function of s is meromorphic, with poles at 0, 1, the α_i's, and the $(1-\alpha_i)$'s; furthermore, it inherits the invariance $s \longmapsto 1-s$ from the Eisenstein series.

3.1.4 Previous results and present goal

In [9], Zagier presents a few examples to illustrate his renormalization technique. Among them, one finds

$$\mathrm{RN}\int\limits_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathscr{H}} \mathrm{E}^\star(z,s_0)\mathrm{E}^\star(z,s_1)\mathrm{E}^\star(z,s_2)\mathrm{E}^\star(z,s_3)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}, \tag{\textbf{5}}$$

$$\mathrm{RN}\int\limits_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathscr{H}} \mathrm{E}^\star(z,s_0)\mathrm{E}^\star_{\mathrm{K,A}}(z,s_1)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}, \tag{\textbf{6}}$$

where K is a totally real cubic field, and

$$\mathrm{RN}\int\limits_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathscr{H}} \mathrm{E}^\star(z,s_0)\mathrm{E}^\star(z,s_1)\mathrm{E}^\star_{\mathrm{K,A}}(z,s_2)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}, \tag{\textbf{7}}$$

where K is a (totally) real quadratic field and A an ideal class.

Zagier mentions that all those expressions now make sense and inherit obvious functional equations, under the transformations $s_i \longmapsto 1-s_i$, from the Eisenstein series involved. However, these integrals cannot be computed in closed form (*i.e.* in terms of zeta functions or Eisenstein series).

In [4], Bump and Goldfeld study (**6**) in order to obtain a Kronecker limit formula for totally real cubic fields. As part of their work, the full group of functional equations for this integral is identified and, surprisingly, is bigger than expected as it has order 12 while there are only 4 (trivial) functional equations.

The article [1] is entirely devoted to showing that the same phenomenon occurs again for the integral (**5**). The full group of functional equations has, this time, order 1152 while the trivial functional equations form a group of order 384. It has to be noted that (**5**) is dramatically harder to study than (**6**).

Finally, our goal here is to show similar results for the integral (**7**). More precisely, we will show

Theorem 1

Let K *be a real quadratic field and* A *be an ideal class in* K*. The renormalized integral*

$$\mathrm{R_A}(s_0, s_1, s_2) = \mathrm{RN} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathscr{H}} \mathrm{E}^\star(z, s_0)\mathrm{E}^\star(z, s_1)\mathrm{E}^\star_{\mathrm{K,A}}(z, s_2)\frac{dx\, dy}{y^2}$$

has a group of functional equations of order 48. It is generated by the transformations

$$s_0 \longmapsto 1 - s_0 \qquad s_1 \longmapsto 1 - s_1 \qquad s_2 \longmapsto 1 - s_2 \qquad s_0 \longmapsto s_1$$

and

$$w : \begin{bmatrix} s_0 \\ s_1 \\ s_2 \end{bmatrix} \longmapsto \begin{bmatrix} -\dfrac{s_0}{2} + \dfrac{s_1}{2} + s_2 \\ 1 - \dfrac{s_0}{2} + \dfrac{s_1}{2} - s_2 \\ 1 - \dfrac{s_0}{2} - \dfrac{s_1}{2} \end{bmatrix}.$$

Here is our strategy. The first step is to look at the polar divisor Π, provided by Zagier's theorem, for $\mathrm{R_A}$: if a transformation of $\mathbb{C}^3$ leaves $\mathrm{R_A}$ invariant, the same must happen with Π. We compute the order of the group of invariants for Π and find 48. Therefore, the group of functional equations for $\mathrm{R_A}$ has order at most 48.

Conversely, it is sufficient to show that $\mathrm{R_A} \circ w = \mathrm{R_A}$. Indeed, there are 16 trivial functional equations for $\mathrm{R_A}$, and w is a transformation of order 3. Hence, the trivial functional equations together with w generate a group of order 48, which is the full group of functional equations.

Showing that w leaves $\mathrm{R_A}$ invariant is the main part of the present paper. This is obtained by showing that $\mathrm{R_A}$ miraculously relates very closely to a period for the $\mathrm{SL}_3(\mathbb{Z})$-Eisenstein series; at least closely enough so that the (known) functional equations for the latter series transfer to $\mathrm{R_A}$.

3.2 Notations and definitions

Proving **Theorem 1** requires a lot of computation and involves many notations. This section regroups all the symbols used throughout the paper.

3.2.1 Ideals, bases

We elaborate here on the first three chapters of [6], in order to get results, about ideals in number fields, of interest for our problem.

In the remainder of the paper, K is a (totally) real quadratic field with discriminant D and ring of integers $\mathfrak{o}$. A is an ideal class in K and $\mathfrak{a}$ an integral ideal in A^{-1}. $\mathscr{A}$ will denote a set of representatives of the nonzero principal ideals in $\mathfrak{o}$ generated by elements of $\mathfrak{a}$. Finally, $\mathfrak{D}$ is the different of $\mathfrak{o}$.

Lemma 2
The map

$$\begin{aligned} \psi : \mathscr{A} \times \big((\mathfrak{a}\mathfrak{D})^{-1} \setminus \{0\}\big) &\longrightarrow \mathrm{K}^2 \\ (\alpha, \beta) &\longrightarrow (\alpha\beta, \alpha) \end{aligned}$$

is a bijection onto $\big\{(\xi, \alpha) \in \mathfrak{D}^{-1} \times \mathscr{A} \mid \xi \neq 0 \quad \alpha\mathfrak{a}^{-1} | \xi\mathfrak{D}\big\}$.

Proof: The injectivity of ψ is clear since elements of $\mathscr{A}$ cannot be 0.

Remember that if $\mathfrak{b}$ is a fractional ideal in K, we have

$$\mathfrak{b}^{-1} = \{\beta \in \mathrm{K} \mid \beta\mathfrak{b} \in \mathfrak{o}\}.$$

So if $\alpha \in \mathscr{A}$ and $\beta \in (\mathfrak{a}\mathfrak{D})^{-1}$, denoting $\alpha\beta$ by ξ, we have

$$\xi\mathfrak{D} = (\alpha\beta)\mathfrak{D} = \beta(\alpha\mathfrak{D}) \subset \beta(\mathfrak{a}\mathfrak{D}) \subset \mathfrak{o},$$

thus $$\xi \in \mathfrak{D}^{-1}$$

Further, $\frac{\xi}{\alpha} = \beta \in (\mathfrak{a}\mathfrak{D})^{-1} \iff (\mathfrak{a}\mathfrak{D})^{-1} | \big(\frac{\xi}{\alpha}\big) \iff \alpha\mathfrak{a}^{-1} | \xi\mathfrak{D}$.

We just showed that the range of ψ is included in

$$\big\{(\xi, \alpha) \in \mathfrak{D}^{-1} \times \mathscr{A} \mid \xi \neq 0 \quad \alpha\mathfrak{a}^{-1} | \xi\mathfrak{D}\big\}.$$

That everything in this set is attained is clear. $\square$

Lemma 3

$$\{\alpha\mathfrak{a}^{-1} \mid \alpha \in \mathscr{A}\} = \{\mathfrak{b} \textit{ integral ideal} \mid \mathfrak{b} \in \mathrm{A}\}.$$

More precisely, for every integral ideal $\mathfrak{b} \in \mathrm{A}$*, there is a unique* $\alpha \in \mathscr{A}$ *such that* $\mathfrak{b} = \alpha\mathfrak{a}^{-1}$.

Proof: Suppose that α is in $\mathscr{A}$. Then α is in $\mathfrak{a}$, which is $\left(\mathfrak{a}^{-1}\right)^{-1}$. Hence $\alpha\mathfrak{a}^{-1} \subset \mathfrak{o}$: $\alpha\mathfrak{a}^{-1}$ is an integral ideal, in the same ideal class as $\mathfrak{a}^{-1}$, that is A.

Conversely, given an ideal $\mathfrak{b}$ in A, it is the product of $\mathfrak{a}^{-1}$ by a principal ideal $(\tilde{\alpha})$:

$$\mathfrak{b} = \tilde{\alpha}\mathfrak{a}^{-1} \qquad \tilde{\alpha} \in \mathfrak{o}.$$

But then we have
$$(\tilde{\alpha}) = \mathfrak{a}\mathfrak{b},$$
which implies in particular that $\tilde{\alpha}$ belongs to $\mathfrak{a}$. Thus, by definition of $\mathscr{A}$, there exists a (unique) $\alpha \in \mathscr{A}$ such that $(\alpha) = (\tilde{\alpha})$. □

Since $\mathfrak{a}$ is a free $\mathbb{Z}$-module of rank 2, it has a $\mathbb{Z}$-basis (α_1, α_2) and we let (β_1, β_2) be the dual basis for the trace bilinear form. It is thus a $\mathbb{Z}$-basis for the ideal $(\mathfrak{a}\mathfrak{D})^{-1}$.

Up to renaming the α's and β's, we can assume that the determinant

$$\begin{vmatrix} {\alpha_1}^{(1)} & {\alpha_1}^{(2)} \\ {\alpha_2}^{(1)} & {\alpha_2}^{(2)} \end{vmatrix}$$

is positive. Its value is therefore $\mathbb{N}(\mathfrak{a})\sqrt{\mathrm{D}}$. Then we have a simple relationship between the ${\alpha_i}^{(j)}$'s and the ${\beta_i}^{(j)}$'s. Indeed, saying that (β_1, β_2) is the dual basis of (α_1, α_2) for the trace form means precisely that the matrices

$$\begin{bmatrix} {\alpha_1}^{(1)} & {\alpha_1}^{(2)} \\ {\alpha_2}^{(1)} & {\alpha_2}^{(2)} \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} {\beta_1}^{(1)} & {\beta_2}^{(1)} \\ {\beta_1}^{(2)} & {\beta_2}^{(2)} \end{bmatrix}$$

are inverse of each other. Hence the relations between their coefficients:

$$\begin{aligned} {\beta_1}^{(1)} &= \frac{{\alpha_2}^{(2)}}{\mathbb{N}(\mathfrak{a})\sqrt{\mathrm{D}}} & \qquad {\beta_1}^{(2)} &= -\frac{{\alpha_2}^{(1)}}{\mathbb{N}(\mathfrak{a})\sqrt{\mathrm{D}}} \\ {\beta_2}^{(1)} &= -\frac{{\alpha_1}^{(2)}}{\mathbb{N}(\mathfrak{a})\sqrt{\mathrm{D}}} & \qquad {\beta_2}^{(2)} &= \frac{{\alpha_1}^{(1)}}{\mathbb{N}(\mathfrak{a})\sqrt{\mathrm{D}}}. \end{aligned} \tag{8}$$

3.2.2 Complex parameters

We are given two triples of complex numbers (s_0, s_1, s_2) and (ν_1, ν_2, s) that are related to each other in the following way:

$$\begin{cases} 3\nu_1 = s_0 + s_1 + 2s_2 - 1 \\ 3\nu_2 = s_0 - s_1 - 2s_2 + 2 \\ \quad s = -2s_1 + 2s_2 \end{cases} \iff \begin{cases} \dfrac{3\nu_1 + 3\nu_2}{2} = s_0 + \dfrac{1}{2} \\ \dfrac{3\nu_1 - 3\nu_2 - 2s}{6} = s_1 - \dfrac{1}{2} \\ \dfrac{3\nu_1 - 3\nu_2 + s}{6} = s_2 - \dfrac{1}{2} \end{cases}. \qquad (\mathbf{9})$$

These numbers will be fixed most of the time and we require that $\operatorname{Re}\nu_1$ and $\operatorname{Re}\nu_2$ be large, and that $\operatorname{Re}(\nu_1 - \nu_2)$ be large compared to $\operatorname{Re} s$. The reasons for these restrictions, as well as the relations (**9**), will be made obvious later. But one can already notice that the transformation w, of (s_0, s_1, s_2), presented in **Theorem 1**, corresponds to $(\nu_1, \nu_2, s) \longmapsto (1 - \nu_1 - \nu_2, \nu_1, s)$.

3.2.3 Special functions

Special functions, such as Bessel functions and divisor sums, will be all over the place and therefore we remind the reader of what they are:

$$\mathrm{K}_\omega(u) = \frac{1}{2} \int_0^{+\infty} \mathrm{e}^{-\frac{u}{2}(t+\frac{1}{t})} t^\omega \frac{\mathrm{d}t}{t} \qquad \omega \in \mathbb{C} \quad u > 0,$$

$$\tau_\omega(n) = n^{-\omega} \sum_{\substack{d \mid n \\ d>0}} d^{2\omega} \qquad \omega \in \mathbb{C} \quad n \in \mathbb{N}^\star,$$

and
$$\tau_\omega^{\mathrm{K,A}}(\mathfrak{c}) = \mathbb{N}(\mathfrak{c})^{-\omega} \sum_{\substack{\mathfrak{b} \text{ ideal in A} \\ \mathfrak{b} \mid \mathfrak{c}}} \mathbb{N}(\mathfrak{b})^{2\omega} \qquad \omega \in \mathbb{C} \quad \mathfrak{c} \text{ ideal in } \mathfrak{o}.$$

In particular, **Lemma 3** implies that

$$\begin{aligned} \tau_\omega^{\mathrm{K,A}}(\mathfrak{c}) &= \mathbb{N}(\mathfrak{c})^{-\omega} \sum_{\substack{\mathfrak{b} \text{ ideal in A} \\ \mathfrak{b} \mid \mathfrak{c}}} \mathbb{N}(\mathfrak{b})^{2\omega} = \mathbb{N}(\mathfrak{c})^{-\omega} \sum_{\substack{\alpha \in \mathscr{A} \\ \alpha\mathfrak{a}^{-1} \mid \mathfrak{c}}} \mathbb{N}(\alpha\mathfrak{a}^{-1})^{2\omega} \\ &= \sum_{\substack{\alpha \in \mathscr{A} \\ \alpha\mathfrak{a}^{-1} \mid \mathfrak{c}}} \left(\frac{\mathbb{N}(\mathfrak{c})}{\mathbb{N}(\alpha\mathfrak{a}^{-1})^2} \right)^{-\omega}. \end{aligned} \qquad (\mathbf{10})$$

Notice that all these functions are unchanged when one replaces ω by $-\omega$.

As explained above, we will be studying the renormalized integral (**7**) and therefore it will be convenient to define for $z \in \mathscr{H}$:

$$\mathrm{F}(z) = \mathrm{E}^\star(z, s_1)\mathrm{E}^\star_{\mathrm{K,A}}(z, s_2).$$

3.2.4 The $\mathrm{SL}_3(\mathbb{Z})$-Eisenstein series

If t_1 and t_2 are positive real numbers, we define

$$\mathrm{M}(t_1,t_2) = \begin{bmatrix} c{\alpha_1}^{(1)} & c{\alpha_1}^{(2)} & 0 \\ c{\alpha_2}^{(1)} & c{\alpha_2}^{(2)} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & (t_1t_2)^{-1} \end{bmatrix} = \begin{bmatrix} ct_1{\alpha_1}^{(1)} & ct_2{\alpha_1}^{(2)} & 0 \\ ct_1{\alpha_2}^{(1)} & ct_2{\alpha_2}^{(2)} & 0 \\ 0 & 0 & (t_1t_2)^{-1} \end{bmatrix},$$

where c is chosen so that $\det \mathrm{M}(t_1, t_2) = 1$, namely $c = \mathbb{N}(\mathfrak{a})^{-\frac{1}{2}}\mathrm{D}^{-\frac{1}{4}}$.

Because of the dual properties of the α's and β's, the inverse of $\mathrm{M}(t_1, t_2)$ can be easily computed:

$$\begin{aligned} \mathrm{M}(t_1,t_2)^{-1} &= \begin{bmatrix} {t_1}^{-1} & 0 & 0 \\ 0 & {t_2}^{-1} & 0 \\ 0 & 0 & t_1t_2 \end{bmatrix} \begin{bmatrix} c^{-1}{\beta_1}^{(1)} & c^{-1}{\beta_2}^{(1)} & 0 \\ c^{-1}{\beta_1}^{(2)} & c^{-1}{\beta_2}^{(2)} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (ct_1)^{-1}{\beta_1}^{(1)} & (ct_1)^{-1}{\beta_2}^{(1)} & 0 \\ (ct_2)^{-1}{\beta_1}^{(2)} & (ct_2)^{-1}{\beta_2}^{(2)} & 0 \\ 0 & 0 & t_1t_2 \end{bmatrix}. \end{aligned}$$

Since ν_1 and ν_2 are two complex numbers with real part strictly bigger that $2/3$, Bump showed in [2] that the $\mathrm{SL}_3(\mathbb{Z})$-Eisenstein series with weight (ν_1, ν_2) evaluated at $\mathrm{M}(t_1,t_2)\,{}^t\mathrm{M}(t_1,t_2)$ is given by

$$\begin{aligned} \mathrm{G}_{\nu_1,\nu_2}(t_1,t_2) = {} & \frac{1}{4}\pi^{-\frac{3\nu_1}{2}}\Gamma\Big(\frac{3\nu_1}{2}\Big)\pi^{-\frac{3\nu_2}{2}}\Gamma\Big(\frac{3\nu_2}{2}\Big)\zeta^\star(3\nu_1+3\nu_2-1) \\ & \times \sum_{\substack{\mathbf{x},\mathbf{y}\in\mathbb{Z}^3 \\ \mathbf{x},\mathbf{y}\neq 0 \\ \mathbf{x}\cdot\mathbf{y}=0}} \left\|{}^t\mathrm{M}(t_1,t_2)\mathbf{x}\right\|^{-3\nu_1} \left\|{}^t\mathrm{M}(t_1,t_2)^{-1}\mathbf{y}\right\|^{-3\nu_2}, \end{aligned}$$

where $\|\ \|$ is the euclidean norm in $\mathbb{R}^3$. This function can be analytically continued to $\mathbb{C}^2$ and is invariant under the transformation $(\nu_1, \nu_2) \longmapsto (1 - \nu_1 - \nu_2, \nu_1)$.

Now, given two triples of integers

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \text{and} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

such that $\mathbf{x} \cdot \mathbf{y} = 0$, we define

$$\alpha = x_1\alpha_1 + x_2\alpha_2 \in \mathfrak{a} \qquad \text{and} \qquad \beta = y_1\beta_1 + y_2\beta_2 \in (\mathfrak{a}\mathfrak{D})^{-1}.$$

Because (α_1, α_2) is a basis for $\mathfrak{a}$ over $\mathbb{Z}$, this notation gives a bijective correspondance between α's in $\mathfrak{a}$ and couples of integers (x_1, x_2). Similarly, we have a bijective correspondance between β's in $(\mathfrak{a}\mathfrak{D})^{-1}$ and couples of integers (y_1, y_2). Notice also that

$$\mathrm{Tr}\,(\alpha\beta) = x_1y_1 + x_2y_2 \qquad \text{since} \qquad \begin{cases} \mathrm{Tr}\,(\alpha_1\beta_1) = \mathrm{Tr}\,(\alpha_2\beta_2) = 1 \\ \mathrm{Tr}\,(\alpha_1\beta_2) = \mathrm{Tr}\,(\alpha_2\beta_1) = 0 \end{cases}.$$

Thus the condition $\mathbf{x}\cdot\mathbf{y} = 0$ can be rewritten as $\mathrm{Tr}\,(\alpha\beta) + x_3y_3 = 0$. On the other hand,

$$\left\| {}^t\mathrm{M}(t_1,t_2)\mathbf{x} \right\|^2 = \left(ct_1\alpha^{(1)}\right)^2 + \left(ct_2\alpha^{(2)}\right)^2 + \left({t_1}^{-1}{t_2}^{-1}x_3\right)^2$$

and
$$\left\| {}^t\mathrm{M}(t_1,t_2)^{-1}\mathbf{y} \right\|^2 = \left((ct_1)^{-1}\beta^{(1)}\right)^2 + \left((ct_2)^{-1}\beta^{(2)}\right)^2 + (t_1t_2y_3)^2.$$

Therefore, we can rewrite our $\mathrm{SL}_3(\mathbb{Z})$ Eisenstein series as

$$\begin{aligned} \mathrm{G}_{\nu_1,\nu_2}(t_1,t_2) = {} & \frac{1}{4}\pi^{-\frac{3\nu_1}{2}}\Gamma\left(\frac{3\nu_1}{2}\right)\pi^{-\frac{3\nu_2}{2}}\Gamma\left(\frac{3\nu_2}{2}\right)\zeta^\star(3\nu_1+3\nu_2-1) \\ & \times \sum_{(\alpha,\beta,\xi,\eta)\in\Omega} \left[\left(ct_1\alpha^{(1)}\right)^2 + \left(ct_2\alpha^{(2)}\right)^2 + \left({t_1}^{-1}{t_2}^{-1}\xi\right)^2\right]^{-\frac{3\nu_1}{2}} \\ & \left[\left((ct_1)^{-1}\beta^{(1)}\right)^2 + \left((ct_2)^{-1}\beta^{(2)}\right)^2 + (t_1t_2\eta)^2\right]^{-\frac{3\nu_2}{2}}, \end{aligned}$$

with

$$\Omega = \left\{(\alpha,\beta,\xi,\eta) \in \mathfrak{a} \times (\mathfrak{a}\mathfrak{D})^{-1} \times \mathbb{Z} \times \mathbb{Z} \;\middle|\; (\alpha,\xi) \neq 0 \quad (\beta,\eta) \neq 0 \quad \mathrm{Tr}\,(\alpha\beta) + \xi\eta = 0\right\}.$$

Whenever we need to restrict the summation to a subset Φ of Ω, which will happen a lot, we will note $\mathrm{G}^{\Phi}_{\nu_1,\nu_2}$ the corresponding part of the Eisenstein series.

3.2.5 Action of $\mathfrak{o}^\times$ on $(\mathbb{R}_+^\star)^2$

One can define an action of the units in $\mathfrak{o}$ on the multiplicative group $(\mathbb{R}_+^\star)^2$ as follows:

$$\epsilon(t_1, t_2) = \big(|\epsilon^{(1)}|t_1, |\epsilon^{(2)}|t_2\big).$$

Proposition 4

The kernel for this action is $\{\pm 1\}$. If $\epsilon \in \mathfrak{o}^\times$ and $(t_1, t_2) \in (\mathbb{R}_+^\star)^2$, then

$$\epsilon(t_1, t_2) = \big(|\epsilon|t_1, |\epsilon|^{-1}t_2\big).$$

Moreover, if $\varepsilon > 1$ is a fundamental unit for $\mathfrak{o}^\times$, the set

$$\mathrm{W} = \Big\{(t_1, t_2) \in (\mathbb{R}_+^\star)^2 \;\Big|\; \varepsilon^{-1} \leqslant \tfrac{t_2}{t_1} \leqslant \varepsilon\Big\}$$

is a complete set of representatives for $(\mathbb{R}_+^\star)^2/\mathfrak{o}^\times$.

Proof: If ϵ is in the kernel, then $|\epsilon| = 1$ and therefore $\epsilon = \pm 1$ since K is totally real.

Now, let $\epsilon \in \mathfrak{o}^\times$. The norm of ϵ is $\epsilon^{(1)}\,\epsilon^{(2)}$ and is a unit in $\mathbb{Z}$. It follows that

$$\left|\epsilon^{(1)}\,\epsilon^{(2)}\right| = 1.$$

Thus
$$\left|\epsilon^{(2)}\right| = |\epsilon|^{-1}$$

and the second claim follows.

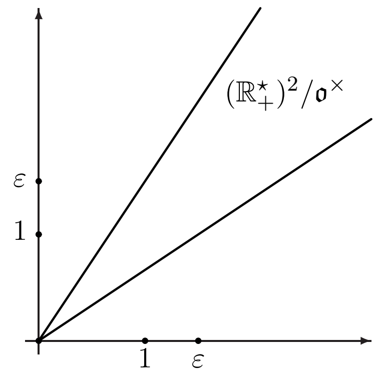
We identify now a set of representatives for $(\mathbb{R}_+^\star)^2/\mathfrak{o}^\times$. Because K is a real quadratic field, we know from the unit theorem that there exists $\varepsilon \in \mathfrak{o}^\times$ such that $\mathfrak{o}^\times = \big\{\pm\,\varepsilon^n \mid n \in \mathbb{Z}\big\}$. Up to taking $\pm\varepsilon^{-1}$ instead of ε, one can suppose that $\varepsilon > 1$. This is what we call a fundamental unit for $\mathfrak{o}^\times$.

Let (t_1, t_2) be a couple of positive real numbers. The sequence $\big(\varepsilon^{2k+1}\big)_{k\in\mathbb{Z}}$ is increasing, tends to 0 at $-\infty$, and to $+\infty$ at $+\infty$. Hence there exists a unique integer n such that

$$\varepsilon^{2n-1} \leqslant \frac{t_2}{t_1} < \varepsilon^{2n+1}.$$

Then
$$\varepsilon^{-1} \leqslant \frac{\varepsilon^{-n}t_2}{\varepsilon^n t_1} < \varepsilon$$

which shows that $\varepsilon^n(t_1, t_2) \in \mathrm{W}$. Thus every (t_1, t_2) is in the orbit of W. $\square$



3.3 Polar divisor considerations

3.3.1 Fourier expansions

According to Zagier's theory, studying $\mathrm{R_A}(s_0, s_1, s_2)$ requires computing a_0, the constant term in the Fourier expansion of F and the part φ of F that is not of rapid decay.

The Fourier expansion for $\mathrm{E}^\star$ is computed, for example, in Bump [3]. The expansion of $\mathrm{E}^\star_{\mathrm{K,A}}$ was computed in **Chapter 2**, **Theorem 27**:

$$\mathrm{E}^\star(z, s_1) = y^{s_1}\zeta^\star(2s_1) + y^{1-s_1}\zeta^\star(2-2s_1) \qquad (\mathbf{11})$$
$$+2\sqrt{y}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \tau_{s_1-\frac{1}{2}}(n)\mathrm{K}_{s_1-\frac{1}{2}}(2\pi y|n|)\,\mathrm{e}^{2\pi i n x}$$

and
$$\mathrm{E}^\star_{\mathrm{K,A}}(z, s_2) = y^{2s_2}\zeta^\star_{\mathrm{A}}(2s_2) + y^{2(1-s_2)}\zeta^\star_{\mathrm{A}}(2-2s_2) \qquad (\mathbf{12})$$
$$+4y\sum_{\substack{\xi\in\mathfrak{D}^{-1}\\ \xi\neq 0}} \tau^{\mathrm{K,A}}_{s_2-\frac{1}{2}}(\xi\mathfrak{D})\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\xi^{(1)}|\big)\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\xi^{(2)}|\big)\mathrm{e}^{2\pi \mathrm{i} x \operatorname{Tr}\xi}.$$

In each of these two expressions, the sums have rapid decay due to the presence of Bessel functions, while the first two terms have polynomial growth in y. Hence φ is immediately seen to be the product of these terms

$$\varphi(y) = \zeta^\star(2s_1)\zeta^\star_{\mathrm{A}}(2s_2)y^{s_1+2s_2} + \zeta^\star(2s_1)\zeta^\star_{\mathrm{A}}(2-2s_2)y^{2+s_1-2s_2} \qquad (\mathbf{13})$$
$$+\zeta^\star(2-2s_1)\zeta^\star_{\mathrm{A}}(2s_2)y^{1-s_1+2s_2} + \zeta^\star(2-2s_1)\zeta^\star_{\mathrm{A}}(2-2s_2)y^{3-s_1-2s_2}.$$

Finally, a_0 is obtained by multiplying (**11**) by (**12**) and discarding what depends on x:

$$a_0(y) = \varphi(y) \tag{14}$$
$$+4\zeta^\star(2s_1)y^{1+s_1} \sum_{\substack{\xi\in\mathfrak{D}^{-1}\\ \xi\neq 0, \operatorname{Tr}\xi=0}} \tau^{\mathrm{K,A}}_{s_2-\frac{1}{2}}(\xi\mathfrak{D})\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\xi^{(1)}|\big)\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\xi^{(2)}|\big)$$
$$+4\zeta^\star(2-2s_1)y^{2-s_1} \sum_{\substack{\xi\in\mathfrak{D}^{-1}\\ \xi\neq 0, \operatorname{Tr}\xi=0}} \tau^{\mathrm{K,A}}_{s_2-\frac{1}{2}}(\xi\mathfrak{D})\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\xi^{(1)}|\big)\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\xi^{(2)}|\big)$$
$$+8y^{\frac{3}{2}} \sum_{\substack{\xi\in\mathfrak{D}^{-1}, n\in\mathbb{Z}\\ \xi, n\neq 0\\ \operatorname{Tr}\xi+n=0}} \tau_{s_1-\frac{1}{2}}(n)\tau_{s_2-\frac{1}{2}}(\xi\mathfrak{D})\mathrm{K}_{s_1-\frac{1}{2}}(2\pi|n|y)\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\xi^{(1)}|\big)\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\xi^{(2)}|\big).$$

3.3.2 Investigating the polar divisor of $\mathrm{R_A}$

Expression (**13**) is all we need to compute the polar divisor of $\mathrm{R_A}$. Indeed, according to Zagier's theorem recalled in **Section 3.1.3**, the latter is given by the following collection of 14 hyperplanes in $\mathbb{C}^3$:

$$s_0 = s_1 + 2s_2 \qquad s_0 = 2 + s_1 - 2s_2 \qquad s_0 = 1 - s_1 + 2s_2 \qquad s_0 = 3 - s_1 - 2s_2$$

$$s_0 = 1 - s_1 - 2s_2 \qquad s_0 = -1 - s_1 + 2s_2 \qquad s_0 = s_1 - 2s_2 \qquad s_0 = -2 + s_1 + 2s_2$$

$$s_0, s_1, s_2 = 0 \qquad \text{and} \qquad s_0, s_1, s_2 = 1.$$

They cut out a polytope Π, a rhombic dodecahedron, displayed and described at

`http://en.wikipedia.org/wiki/Rhombic_dodecahedron` .

The order of the group of symetries Γ of Π can be computed in various ways. For example, Γ acts transitively on the set of vertices of Π that are connected to four other vertices. We count six such vertices, each of them being fixed by eight transformations in Γ: four rotations and four reflexions. Hence $|\Gamma| = 6 \times 8 = 48$ and the group of functional equations for $\mathrm{R_A}$ has order at most 48.

3.4 Proof of Theorem 1

As explained in section 1.4, now that we know that the group of functional equations for $\mathrm{R_A}$ has order less than 48, we are left with showing that w leaves $\mathrm{R_A}$ invariant. In order to do that, the $\mathrm{SL}_3(\mathbb{Z})$-Eisenstein series (section 2.4) is integrated over $(\mathbb{R}_+^\star)^2/\mathfrak{o}^\times$ (described in section 2.5).

As seen in formula (**8**), the Eisenstein series G_{ν_1,ν_2} consists in a sum over the set

$$\Omega = \Big\{(\alpha,\beta,\xi,\eta) \in \mathfrak{a} \times (\mathfrak{a}\mathfrak{D})^{-1} \times \mathbb{Z} \times \mathbb{Z} \;\Big|\; (\alpha,\xi) \neq 0 \quad (\beta,\eta) \neq 0 \quad \mathrm{Tr}\,(\alpha\beta) + \xi\eta = 0\Big\},$$

which we break up into smaller subsets that come up naturally when one investigates what indices, among α, β, ξ and η, can be simultaneously zero. This is all summarized in the following table:

	$\alpha = 0 \quad \xi = 0$	$\alpha = 0 \quad \xi \neq 0$	$\alpha \neq 0 \quad \xi = 0$	$\alpha \neq 0 \quad \xi \neq 0$
$\beta = 0 \quad \eta = 0$	Excluded	Excluded	Excluded	Excluded
$\beta = 0 \quad \eta \neq 0$	Excluded	Excluded	Ω_6	Excluded
$\beta \neq 0 \quad \eta = 0$	Excluded	Ω_5	Ω_4	Ω_2
$\beta \neq 0 \quad \eta \neq 0$	Excluded	Excluded	Ω_3	Ω_1

Some terms in our integral will be computed without trouble; some others will be modified through Poisson's summation formula. And some will be discarded because they are not integrable. This will give us precisely $\mathrm{R_A}$ but at this point, the functional equation w won't be immediately apparent because terms from the Eisenstein series have been removed: more manipulations will be required to exhibit the symmetry w for $\mathrm{R_A}$.

3.4.1 First computations

We begin by defining an action of $\mathfrak{o}^\times$ on the set Ω_1 defined in the table above:

$$\epsilon(\alpha,\beta,\xi,\eta) = \big(\epsilon\alpha, \epsilon^{-1}\beta, \xi, \eta\big).$$

And the following proposition identifies a fundamental domain:

Proposition 5

Let $\mathscr{A}$ be a set of representatives of the principal ideals generated by elements of $\mathfrak{a}$. Then a complete, nonredundant, set of representatives of $\Omega_1/\mathfrak{o}^\times$ is given by

$$\tilde{\Omega}_1 = \left\{(\alpha, \beta, \xi, \eta) \in \mathscr{A} \times (\mathfrak{a}\mathfrak{D})^{-1} \times \mathbb{Z} \times \mathbb{Z} \mid Tr(\alpha\beta) + \xi\eta = 0 \quad \alpha, \beta, \xi, \eta \neq 0\right\}.$$

Proof: Let $(\alpha, \beta, \xi, \eta)$ be an element of Ω_1. This means that

$$\alpha \in \mathfrak{a} \qquad \beta \in (\mathfrak{a}\mathfrak{D})^{-1} \qquad \xi, \eta \in \mathbb{Z} \qquad \alpha, \beta, \xi, \eta \neq 0$$

and
$$\mathrm{Tr}\,(\alpha\beta) + \xi\eta = 0.$$

Then there exists an $\tilde{\alpha}$ in $\mathscr{A}$ such that $(\alpha) = (\tilde{\alpha})$ and therefore, α and $\tilde{\alpha}$ differ by a unit:

$$\exists \epsilon \in \mathfrak{o}^\times \qquad \alpha = \epsilon\tilde{\alpha}.$$

From the definition of the action of $\mathfrak{o}^\times$, we have

$$(\alpha, \beta, \xi, \eta) = \epsilon(\tilde{\alpha}, \epsilon\beta, \xi, \eta).$$

But $(\tilde{\alpha}, \epsilon\beta, \xi, \eta)$ is in $\tilde{\Omega}_1$ since

$$\mathrm{Tr}\,(\tilde{\alpha}\epsilon\beta) + \xi\eta = \mathrm{Tr}\,(\underbrace{\epsilon\tilde{\alpha}}_{=\alpha}\beta) + \xi\eta = 0.$$

So any coset in $\Omega_1/\mathfrak{o}^\times$ can be represented by an element of $\tilde{\Omega}_1$.

To ensure there is no redundancy, suppose that $(\alpha_1, \beta_1, \xi_1, \eta_1)$ and $(\alpha_2, \beta_2, \xi_2, \eta_2)$ are in $\tilde{\Omega}_1$ and represent the same coset of Ω_1. Then

$$\exists \epsilon \in \mathfrak{o}^\times \qquad (\alpha_2, \beta_2, \xi_2, \eta_2) = \epsilon(\alpha_1, \beta_1, \xi_1, \eta_1) = (\epsilon\alpha_1, \epsilon^{-1}\beta_1, \xi_1, \eta_1).$$

So α_1 and α_2 generate the same principal ideal of $\mathfrak{o}$ and by definition of $\mathscr{A}$, they are equal. It follows that ϵ is actually equal to 1 and therefore

$$(\alpha_1, \beta_1, \xi_1, \eta_1) = (\alpha_2, \beta_2, \xi_2, \eta_2). \qquad \square$$

Proposition 6

*Let s_0, s_1, s_2, ν_1, ν_2 and s be complex numbers defined as in **Section 3.2.2**. Then*

$$\iint\limits_{(\mathbb{R}_+^\star)^2/\mathfrak{o}^\times} \mathrm{G}^{\Omega_1}_{(\nu_1,\nu_2)}(t_1,t_2)\,(t_1t_2)^s\,\frac{dt_1\,dt_2}{t_1\,t_2} = \frac{4}{3}\zeta^\star(2s_0)\sum_{\substack{\xi\in\mathfrak{D}^{-1},n\in\mathbb{Z}\\ \xi,n\neq 0\\ Tr\xi+n=0}}\tau_{s_1-\frac{1}{2}}(|n|)\tau^{\mathrm{K},\mathrm{A}}_{s_2-\frac{1}{2}}(\xi\mathfrak{D})$$

$$\times\int_0^\infty y^{\frac{3}{2}}\mathrm{K}_{s_1-\frac{1}{2}}(2\pi|n|y)\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\xi^{(1)}|\big)\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\xi^{(2)}|\big)y^{s_0-1}\,\frac{dy}{y}\ .$$

Proof: Given $(\alpha,\beta,\xi,\eta)\in\Omega_1$ and $t_1,t_2>0$, define

$$\mathrm{H}(\alpha,\beta,\xi,\eta,t_1,t_2) = (t_1t_2)^s\Big[\big(ct_1\alpha^{(1)}\big)^2+\big(ct_2\alpha^{(2)}\big)^2+\big(t_1{}^{-1}t_2{}^{-1}\xi\big)^2\Big]^{-\frac{3\nu_1}{2}}$$
$$\times\Big[\big((ct_1)^{-1}\beta^{(1)}\big)^2+\big((ct_2)^{-1}\beta^{(2)}\big)^2+(t_1t_2\eta)^2\Big]^{-\frac{3\nu_2}{2}}$$

and
$$\Lambda = \frac{1}{4}\pi^{-\frac{3\nu_1}{2}}\Gamma\Big(\frac{3\nu_1}{2}\Big)\pi^{-\frac{3\nu_2}{2}}\Gamma\Big(\frac{3\nu_2}{2}\Big)\zeta^\star(3\nu_1+3\nu_2-1),$$

so that
$$\mathrm{G}^{\Omega_1}_{\nu_1,\nu_2}(t_1,t_2)(t_1t_2)^s = \Lambda\sum_{(\alpha,\beta,\xi,\eta)\in\Omega_1}\mathrm{H}(\alpha,\beta,\xi,\eta,t_1,t_2).$$

The function H is the object that ties together the two different actions of $\mathfrak{o}^\times$, respectively on $(\mathbb{R}_+^\star)^2$ and Ω_1. Indeed, if ϵ is a unit in $\mathfrak{o}$, one checks easily that

$$\mathrm{H}\big(\epsilon(\alpha,\beta,\xi,\eta),t_1,t_2\big) = \mathrm{H}\big(\alpha,\beta,\xi,\eta,\epsilon(t_1,t_2)\big).$$

This allows us to unfold and collapse the following integral:

$$\iint\limits_{(\mathbb{R}_+^\times)^2/\mathfrak{o}^\times}\mathrm{G}^{\Omega_1}_{\nu_1,\nu_2}(t_1,t_2)(t_1t_2)^s\,\frac{\mathrm{d}t_1\mathrm{d}t_2}{t_1t_2} = \Lambda\sum_{(\alpha,\beta,\xi,\eta)\in\Omega_1}\ \iint\limits_{(\mathbb{R}_+^\times)^2/\mathfrak{o}^\times}\mathrm{H}(\alpha,\beta,\xi,\eta,t_1,t_2)\,\frac{\mathrm{d}t_1\mathrm{d}t_2}{t_1t_2}$$
$$= \Lambda\sum_{(\alpha,\beta,\xi,\eta)\in\tilde{\Omega}_1}\sum_{\varepsilon\in\mathfrak{o}^\times}\ \iint\limits_{(\mathbb{R}_+^\star)^2/\mathfrak{o}^\times}\mathrm{H}\big(\varepsilon(\alpha,\beta,\xi,\eta),t_1,t_2\big)\,\frac{\mathrm{d}t_1\mathrm{d}t_2}{t_1t_2}$$
$$= \Lambda\sum_{(\alpha,\beta,\xi,\eta)\in\tilde{\Omega}_1}\sum_{\varepsilon\in\mathfrak{o}^\times}\ \iint\limits_{\varepsilon^{-1}(\mathbb{R}_+^\times)^2/\mathfrak{o}^\times}\mathrm{H}(\alpha,\beta,\xi,\eta,t_1,t_2)\,\frac{\mathrm{d}t_1\mathrm{d}t_2}{t_1t_2}$$

$$\iint\limits_{(\mathbb{R}_+^\times)^2/\mathfrak{o}^\times}\mathrm{G}^{\Omega_1}_{\nu_1,\nu_2}(t_1,t_2)(t_1t_2)^s\,\frac{\mathrm{d}t_1\mathrm{d}t_2}{t_1t_2} = 2\Lambda\sum_{(\alpha,\beta,\xi,\eta)\in\tilde{\Omega}_1}\ \iint\limits_{(\mathbb{R}_+^\star)^2}\mathrm{H}(\alpha,\beta,\xi,\eta,t_1,t_2)\,\frac{\mathrm{d}t_1\mathrm{d}t_2}{t_1t_2}.$$

The factor of 2 comes out in the end because, as explained in **Proposition 2**, the action of $\mathfrak{o}^\times$ on $(\mathbb{R}_+^\star)^2$ has a kernel of cardinality 2. Moreover, this computation is made possible because of our assumption on the sizes of $\operatorname{Re}\nu_1$, $\operatorname{Re}\nu_2$ and $\operatorname{Re}(\nu_1-\nu_2)$ since any courageous person can check that

$$\iint\limits_{(\mathbb{R}_+^\star)^2} \left|\mathrm{H}(\alpha,\beta,\xi,\eta,t_1,t_2)\right| \frac{\mathrm{d}t_1\mathrm{d}t_2}{t_1t_2} < \infty.$$

We now attack the computation of the integral $\Lambda \iint\limits_{(\mathbb{R}_+^\star)^2} \mathrm{H}(\alpha,\beta,\xi,\eta,t_1,t_2)\, \frac{\mathrm{d}t_1\mathrm{d}t_2}{t_1t_2}$.

The parameters (α,β,ξ,η) will be fixed in $\tilde{\Omega}_1$ and we let

$$\mathrm{H}^\star(t_1,t_2) = \frac{1}{4}\pi^{-\frac{3\nu_1}{2}}\Gamma\Big(\frac{3\nu_1}{2}\Big)\pi^{-\frac{3\nu_2}{2}}\Gamma\Big(\frac{3\nu_2}{2}\Big) \times \mathrm{H}(\alpha,\beta,\xi,\eta,t_1,t_2).$$

After replacing the Gamma factors in Λ by their integral value, we get

$$\begin{aligned}\mathrm{H}^\star(t_1,t_2) = \frac{(t_1t_2)^s}{4}\int_0^\infty \mathrm{e}^{-r_1}\left(\frac{r_1}{\pi\big[\big(ct_1\alpha^{(1)}\big)^2 + \big(ct_2\alpha^{(2)}\big)^2 + \big({t_1}^{-1}{t_2}^{-1}\xi\big)^2\big]}\right)^{\frac{3\nu_1}{2}} \frac{\mathrm{d}r_1}{r_1}\\ \times \int_0^\infty \mathrm{e}^{-r_2}\left(\frac{r_2}{\pi\big[\big((ct_1)^{-1}\beta^{(1)}\big)^2 + \big((ct_2)^{-1}\beta^{(2)}\big)^2 + (t_1t_2\eta)^2\big]}\right)^{\frac{3\nu_2}{2}} \frac{\mathrm{d}r_2}{r_2}.\end{aligned}$$

It is then natural to perform the substitutions

$$\begin{cases} r_1 \longmapsto r_1\pi\big[\big(t_1\alpha^{(1)}\big)^2 + \big(t_2\alpha^{(2)}\big)^2 + \big({t_1}^{-1}{t_2}^{-1}\xi\big)^2\big] \\ r_2 \longmapsto r_2\pi\big[\big({t_1}^{-1}\beta^{(1)}\big)^2 + \big({t_2}^{-1}\beta^{(2)}\big)^2 + (t_1t_2\eta)^2\big], \end{cases}$$

respectively in the first and second integral to obtain

$$\begin{aligned}\mathrm{H}^\star(t_1,t_2) = \frac{(t_1t_2)^s}{4}\int_0^\infty\!\!\int_0^\infty \mathrm{e}^{-r_1\pi\left[(ct_1\alpha^{(1)})^2+(ct_2\alpha^{(2)})^2+\left(\frac{\xi}{t_1t_2}\right)^2\right]-r_2\pi\left[\left(\frac{\beta^{(1)}}{ct_1}\right)^2+\left(\frac{\beta^{(2)}}{ct_2}\right)^2+(t_1t_2\eta)^2\right]}\\ \times {r_1}^{\frac{3\nu_1}{2}}{r_2}^{\frac{3\nu_2}{2}}\,\frac{\mathrm{d}r_1\,\mathrm{d}r_2}{r_1\,r_2}.\end{aligned}$$

Thus

$$\begin{aligned}\iint\limits_{(\mathbb{R}_+^\star)^2}\mathrm{H}^\star(t_1,t_2)\,\frac{\mathrm{d}t_1\mathrm{d}t_2}{t_1t_2} = \frac{1}{4}\int\limits_{(\mathbb{R}_+^\star)^4} \mathrm{e}^{-r_1\pi\left[(ct_1\alpha^{(1)})^2+(ct_2\alpha^{(2)})^2+\left(\frac{\xi}{t_1t_2}\right)^2\right]}\\ \times \mathrm{e}^{-r_2\pi\left[\left(\frac{\beta^{(1)}}{ct_1}\right)^2+\left(\frac{\beta^{(2)}}{ct_2}\right)^2+(t_1t_2\eta)^2\right]}{r_1}^{\frac{3\nu_1}{2}}{r_2}^{\frac{3\nu_2}{2}}(t_1t_2)^s\,\frac{\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}t_1\,\mathrm{d}t_2}{r_1\,r_2\,t_1\,t_2}.\end{aligned}$$

We make then the variable change

$$u = \sqrt{r_1 r_2} \qquad v = {t_1}^2 \sqrt{\frac{r_1}{r_2}} \qquad w = {t_2}^2 \sqrt{\frac{r_1}{r_2}} \qquad x = {t_1}^{-2} {t_2}^{-2} \sqrt{\frac{r_1}{r_2}}.$$

This transformation is $\mathscr{C}^1$ on $(\mathbb{R}_+^\star)^4$, and invertible since we can recover r_1, r_2, t_1 and t_2 from u, v, w and x in the following way:

$$r_1 = u(vwx)^{\frac{1}{3}} \qquad r_2 = \frac{u}{(vwx)^{\frac{1}{3}}} \qquad t_1 = \frac{v^{\frac{1}{3}}}{(wx)^{\frac{1}{6}}} \qquad t_2 = \frac{w^{\frac{1}{3}}}{(vx)^{\frac{1}{6}}}.$$

This is again obviously a $\mathscr{C}^1$ transformation on $\mathbb{R}^4$: we have, therefore, defined a $\mathscr{C}^1$-diffeomorphism from $(\mathbb{R}_+^\star)^4$ onto itself. The Jacobian is tedious to compute, but in the end we obtain the very simple

$$\frac{\mathrm{d}r_1 \,\mathrm{d}r_2 \,\mathrm{d}t_1 \,\mathrm{d}t_2}{r_1\, r_2\, t_1\, t_2} = \frac{1}{6}\, \frac{\mathrm{d}u \,\mathrm{d}v \,\mathrm{d}w \,\mathrm{d}x}{u\, v\, w\, x},$$

so that after rearranging everything, we get

$$\iint\limits_{(\mathbb{R}_+^\star)^2} \mathrm{H}^\star(t_1, t_2)\, \frac{\mathrm{d}t_1 \,\mathrm{d}t_2}{t_1\, t_2} = \frac{1}{24} \int\limits_{(\mathbb{R}_+^\star)^4} \mathrm{e}^{-\pi u \left[vc^2 {\alpha^{(1)}}^2 + wc^2 {\alpha^{(2)}}^2 + x\xi^2 + v^{-1} \frac{{\beta^{(1)}}^2}{c^2} + w^{-1} \frac{{\beta^{(2)}}^2}{c^2} + x^{-1}\eta^2 \right]}$$
$$\times u^{\frac{3\nu_1+3\nu_2}{2}} v^{\frac{3\nu_1-3\nu_2+s}{6}} w^{\frac{3\nu_1-3\nu_2+s}{6}} x^{\frac{3\nu_1-3\nu_2-2s}{6}} \,\frac{\mathrm{d}u \,\mathrm{d}v \,\mathrm{d}w \,\mathrm{d}x}{u\, v\, w\, x}$$

We finally see Bessel functions appearing. Indeed, we know that for positive numbers a and b,

$$\int\limits_0^\infty \mathrm{e}^{-at-bt^{-1}} t^\nu \,\frac{\mathrm{d}t}{t} = 2\left(\frac{b}{a}\right)^{\frac{\nu}{2}} \mathrm{K}_\nu\big(2\sqrt{ab}\big),$$

so that, for example,

$$\int\limits_0^\infty \mathrm{e}^{-\pi u v c^2 {\alpha^{(1)}}^2 - \pi u v^{-1} \frac{{\beta^{(1)}}^2}{c^2}} v^{\frac{3\nu_1-3\nu_2+s}{6}} \,\frac{\mathrm{d}v}{v} = 2\left| c^{-2} \frac{\beta^{(1)}}{\alpha^{(1)}} \right|^{\frac{3\nu_1-3\nu_2+s}{6}} \mathrm{K}_{\frac{3\nu_1-3\nu_2+s}{6}}\big(2\pi u \big|\alpha^{(1)}\beta^{(1)}\big|\big).$$

Hence,

$$\iint\limits_{(\mathbb{R}_+^\star)^2} \mathrm{H}^\star(t_1, t_2)\, \frac{\mathrm{d}t_1 \,\mathrm{d}t_2}{t_1\, t_2} = \frac{1}{3}\left(c^{-4} \left| \mathrm{N}\frac{\beta}{\alpha} \right| \right)^{\frac{3\nu_1-3\nu_2+s}{6}} \times \left| \frac{\eta}{\xi} \right|^{\frac{3\nu_1-3\nu_2-2s}{6}} \int\limits_0^\infty \mathrm{K}_{\frac{3\nu_1-3\nu_2-2s}{6}}\big(2\pi u |\xi\eta|\big)$$
$$\times \mathrm{K}_{\frac{3\nu_1-3\nu_2+s}{6}}\big(2\pi u \big|\alpha^{(1)}\beta^{(1)}\big|\big) \mathrm{K}_{\frac{3\nu_1-3\nu_2+s}{6}}\big(2\pi u \big|\alpha^{(2)}\beta^{(2)}\big|\big) u^{\frac{3\nu_1+3\nu_2}{2}} \,\frac{\mathrm{d}u}{u}.$$

Using the relationships (**9**) and the definition of c,

$$\iint\limits_{(\mathbb{R}_+^\star)^2} \mathrm{H}^\star(t_1,t_2)\,\frac{\mathrm{d}t_1\,\mathrm{d}t_2}{t_1\,t_2} = \frac{1}{3}\left(\mathbb{N}(\mathfrak{a})^2\mathrm{D}\left|\mathbb{N}\frac{\beta}{\alpha}\right|\right)^{s_2-\frac{1}{2}} \times \left|\frac{\eta}{\xi}\right|^{s_1-\frac{1}{2}} \int_0^\infty u^{\frac{3}{2}}\mathrm{K}_{s_1-\frac{1}{2}}\big(2\pi u|\xi\eta|\big)$$
$$\times\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi u\big|\alpha^{(1)}\beta^{(1)}\big|\big)\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi u\big|\alpha^{(2)}\beta^{(2)}\big|\big)u^{s_0-1}\,\frac{\mathrm{d}u}{u}.$$

Thus
$$\iint\limits_{(\mathbb{R}_+^\star)^2/\mathfrak{o}^\times} \mathrm{G}^{\Omega_1}_{(\nu_1,\nu_2)}(t_1,t_2)\,(t_1t_2)^s\,\frac{\mathrm{d}t_1\,\mathrm{d}t_2}{t_1\,t_2} = \frac{2}{3}\zeta^\star(2s_0)\sum_{\substack{\xi\in\mathbb{Z}\\ \xi\neq 0}}\sum_{\substack{\eta\in\mathbb{Z}\\ \eta\neq 0}}\sum_{\alpha\in\mathscr{A}}\sum_{\substack{\beta\in(\mathfrak{a}\mathfrak{D})^{-1}\\ \beta\neq 0\\ \mathrm{Tr}\,(\alpha\beta)+\xi\eta=0}}$$
$$\left(\mathbb{N}(\mathfrak{a})^2\mathrm{D}\left|\mathbb{N}\frac{\beta}{\alpha}\right|\right)^{s_2-\frac{1}{2}} \times \left|\frac{\eta}{\xi}\right|^{s_1-\frac{1}{2}} \int_0^\infty y^{\frac{3}{2}}\mathrm{K}_{s_1-\frac{1}{2}}(2\pi y|\xi\eta|)\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\alpha^{(1)}\beta^{(1)}|\big)$$
$$\times\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\alpha^{(2)}\beta^{(2)}|\big)y^{s_0-1}\,\frac{\mathrm{d}y}{y}.$$

Consider the map

$$\begin{aligned}(\mathbb{Z}^\star)^2 &\longrightarrow \big\{(n,m)\in\mathbb{Z}^2 \mid n\neq 0 \quad m>0 \quad m|n\big\}\\ (\xi,\eta) &\longmapsto (\xi\eta,|\eta|).\end{aligned}$$

It is surjective, by definition; and if (n,m) is any element in the set on the right, it has exactly two preimages, which are

$$\left(\frac{n}{m},m\right) \qquad \text{and} \qquad \left(-\frac{n}{m},-m\right).$$

Therefore, when we change indices in the quadruple sum above by putting $n=\xi\eta$ and $m=|\eta|$, the divisor sum $\tau_{s_1-\frac{1}{2}}$ appears and a factor 2 is pulled out:

$$\iint\limits_{(\mathbb{R}_+^\star)^2/\mathfrak{o}^\times} \mathrm{G}^{\Omega_1}_{(\nu_1,\nu_2)}(t_1,t_2)\,(t_1t_2)^s\,\frac{\mathrm{d}t_1\,\mathrm{d}t_2}{t_1\,t_2} = \frac{4}{3}\zeta^\star(2s_0)\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\tau_{s_1-\frac{1}{2}}(n)$$
$$\times\sum_{\alpha\in\mathscr{A}}\sum_{\substack{\beta\in(\mathfrak{a}\mathfrak{D})^{-1}\\ \beta\neq 0\\ \mathrm{Tr}\,(\alpha\beta)+n=0}}\left(\mathbb{N}(\mathfrak{a})^2\mathrm{D}\left|\mathbb{N}\frac{\beta}{\alpha}\right|\right)^{s_2-\frac{1}{2}}$$
$$\times\int_0^\infty y^{\frac{3}{2}}\mathrm{K}_{s_1-\frac{1}{2}}(2\pi y|n|)\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\alpha^{(1)}\beta^{(1)}|\big)\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\alpha^{(2)}\beta^{(2)}|\big)y^{s_0-1}\,\frac{\mathrm{d}y}{y}\ .$$

The next change of indices consists in defining $\xi = \alpha\beta$, leaving α alone and invoking **Lemma 2**:

$$\iint\limits_{(\mathbb{R}_+^\star)^2/\mathfrak{o}^\times} \mathrm{G}^{\Omega_1}_{(\nu_1,\nu_2)}(t_1,t_2)\,(t_1t_2)^s\,\frac{\mathrm{d}t_1\,\mathrm{d}t_2}{t_1\,t_2} = \frac{4}{3}\zeta^\star(2s_0)\sum_{\substack{n\in\mathbb{Z},\xi\in\mathfrak{D}^{-1}\\ n,\xi\neq 0\\ \operatorname{Tr}\xi+n=0}}\tau_{s_1-\frac{1}{2}}(n)$$
$$\times\sum_{\substack{\alpha\in\mathscr{A}\\ \alpha\mathfrak{a}^{-1}|\xi\mathfrak{D}}}\left(\mathbb{N}(\mathfrak{a})^2\mathrm{D}\left|\mathbb{N}\frac{\xi}{\alpha^2}\right|\right)^{s_2-\frac{1}{2}}$$
$$\int\limits_0^\infty y^{\frac{3}{2}}\mathrm{K}_{s_1-\frac{1}{2}}(2\pi y|n|)\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\xi^{(1)}|\big)\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\xi^{(2)}|\big)y^{s_0-1}\,\frac{\mathrm{d}y}{y}\ .$$

Remembering that the discriminant is the norm of the different and using the multiplicative properties of the norms of ideals, we get

$$\iint\limits_{(\mathbb{R}_+^\star)^2/\mathfrak{o}^\times} \mathrm{G}^{\Omega_1}_{(\nu_1,\nu_2)}(t_1,t_2)\,(t_1t_2)^s\,\frac{\mathrm{d}t_1\,\mathrm{d}t_2}{t_1\,t_2} = \frac{4}{3}\zeta^\star(2s_0)\sum_{\substack{n\in\mathbb{Z},\xi\in\mathfrak{D}^{-1}\\ \xi,n\neq 0\\ \operatorname{Tr}\xi+n=0}}\tau_{s_1-\frac{1}{2}}(n)$$
$$\times\sum_{\substack{\alpha\in\mathscr{A}\\ \alpha\mathfrak{a}^{-1}|\xi\mathfrak{D}}}\left(\frac{\mathbb{N}(\xi\mathfrak{D})}{\mathbb{N}(\alpha\mathfrak{a}^{-1})^2}\right)^{s_2-\frac{1}{2}}$$
$$\int\limits_0^\infty y^{\frac{3}{2}}\mathrm{K}_{s_1-\frac{1}{2}}(2\pi y|n|)\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\xi^{(1)}|\big)\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\xi^{(2)}|\big)y^{s_0-1}\,\frac{\mathrm{d}y}{y}\ .$$

Finally, applying formula (**10**) achieves the proof of **Proposition 6**. □

The same techniques would allow us to show

Proposition 7

*Let s_0, s_1, s_2, ν_1, ν_2 and s be complex numbers defined as in section **3.2.2**. Then*

$$\iint\limits_{(\mathbb{R}_+^\star)^2/\mathfrak{o}^\times} \mathrm{G}^{\Omega_2}_{(\nu_1,\nu_2)}(t_1,t_2)(t_1t_2)^s\,\frac{dt_1\,dt_2}{t_1t_2} = \frac{2}{3}\zeta^\star(2s_0)\zeta^\star(2s_1-1)\sum_{\substack{\xi\in\mathfrak{D}^{-1}\\ \xi\neq 0}}\tau^{\mathrm{K,A}}_{s_2-\frac{1}{2}}(\xi\mathfrak{D})$$
$$\int\limits_0^\infty y^{2-s_1}\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\xi^{(1)}|\big)\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\xi^{(2)}|\big)y^{s_0-1}\,\frac{dy}{y}.$$

3.4.2 Where Poisson's summation formula comes into play

Comparing the formulas proved in **Propositions 6** and **7** with the expression (**14**), we see that we have obtained already a good amount of terms of $\mathrm{R_A}(s_0, s_1, s_2)$. So far, the tediousness of the computation has been the only difficulty: the integrals considered were absolutely convergent given our choices for the parameters ν_1, ν_2 and s.

The natural thing would be to expect that the remaining terms of $\mathrm{R_A}$ come from

$$\iint\limits_{(\mathbb{R}_+^\star)^2/\mathfrak{o}^\times} \mathrm{G}^{\Omega_i}_{\nu_1,\nu_2}(t_1,t_2)(t_1t_2)^s \frac{\mathrm{d}t_1\,\mathrm{d}t_2}{t_1t_2} \qquad i = 3, 4, 5 \text{ or } 6.$$

However it is not quite the case because none of these integrals makes sense with our choices of ν_1, ν_2 and s. But Poisson's summation formula will allow us to transform $\mathrm{G}^{\Omega_3\cup\Omega_4}_{\nu_1,\nu_2}$ and separate the integrable part from the non-integrable one. Of course, the former, once integrated, will give us the last remaining terms of $\mathrm{R_A}(s_0, s_1, s_2)$.

Proposition 8

*Let s_0, s_1, s_2, ν_1, ν_2 and s be complex numbers defined as in section **3.2.2**. Then*

$$\iint\limits_{(\mathbb{R}_+^\star)^2/\mathfrak{o}^\times} \big(\mathrm{G}^{\Omega_3\cup\Omega_4}_{(\nu_1,\nu_2)}(t_1,t_2) - \mathrm{V}_{\nu_1,\nu_2}(t_1,t_2)\big)(t_1t_2)^s\, \frac{dt_1\,dt_2}{t_1t_2} = \frac{2}{3}\zeta^\star(2s_0)\zeta^\star(2s_1)$$

$$\times \sum_{\substack{\xi\in\mathfrak{D}^{-1}\\ \xi\neq 0}} \tau^{\mathrm{K,A}}_{s_2-\frac{1}{2}}(\xi\mathfrak{D}) \int\limits_0^\infty y^{s_1+1}\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\xi^{(1)}|\big)\mathrm{K}_{s_2-\frac{1}{2}}\big(2\pi y|\xi^{(2)}|\big)y^{s_0-1}\,\frac{dy}{y}\ ,$$

where

$$\mathrm{V}_{\nu_1,\nu_2}(t_1,t_2) = \zeta^\star(3\nu_1)\zeta^\star(3\nu_2-1)(t_1t_2)^{-\frac{3\nu_1-3\nu_2+3}{2}}\mathrm{E}^\star\left(\frac{t_1{\alpha_1}^{(1)} - it_2{\alpha_1}^{(2)}}{t_1{\alpha_2}^{(1)} - it_2{\alpha_2}^{(2)}}, 3\nu_1+3\nu_2-1\right).$$

Proof: Remember that

$$\Omega_3\cup\Omega_4 = \big\{(\alpha,\beta,0,\eta)\in\mathfrak{a}\times(\mathfrak{a}\mathfrak{D})^{-1}\times\mathbb{Z}\times\mathbb{Z}\ \big|\ \alpha\neq 0 \quad \beta\neq 0 \quad \mathrm{Tr}\,(\alpha\beta)+\xi\eta = 0\big\}$$

and

$$\mathrm{G}^{\Omega_3\cup\Omega_4}_{\nu_1,\nu_2}(t_1,t_2) = \frac{1}{4}\pi^{-\frac{3\nu_1}{2}}\Gamma\Big(\frac{3\nu_1}{2}\Big)\pi^{-\frac{3\nu_2}{2}}\Gamma\Big(\frac{3\nu_2}{2}\Big)\zeta^\star(3\nu_1+3\nu_2-1)$$
$$\times \sum_{\substack{0\neq\alpha\in\mathfrak{a}\\ 0\neq\beta\in(\mathfrak{a}\mathfrak{D})^{-1}\\ \mathrm{Tr}\,(\alpha\beta)=0}}\sum_{\eta\in\mathbb{Z}}\Big[\big(ct_1\alpha^{(1)}\big)^2+\big(ct_2\alpha^{(2)}\big)^2\Big]^{-\frac{3\nu_1}{2}}\Big[\Big(\frac{\beta^{(1)}}{ct_1}\Big)^2+\Big(\frac{\beta^{(2)}}{ct_2}\Big)^2+(\eta t_1t_2)^2\Big]^{-\frac{3\nu_2}{2}}.$$

Let us fix α and β and focus on the sum over η. More precisely, define

$$\varphi(t_1,t_2) = \pi^{-\frac{3\nu_2}{2}}\Gamma\Big(\frac{3\nu_2}{2}\Big)\sum_{\eta\in\mathbb{Z}}\Big[\Big(\frac{\beta^{(1)}}{ct_1}\Big)^2+\Big(\frac{\beta^{(2)}}{ct_2}\Big)^2+(\eta t_1t_2)^2\Big]^{-\frac{3\nu_2}{2}}.$$

Using the integral expression of the Gamma function, we obtain

$$\varphi(t_1,t_2) = \int_0^\infty\sum_{\eta\in\mathbb{Z}}\mathrm{e}^{-r_2\pi\Big[\big(\frac{\beta^{(1)}}{ct_1}\big)^2+\big(\frac{\beta^{(2)}}{ct_2}\big)^2+\eta^2{t_1}^2{t_2}^2\Big]}\,{r_2}^{\frac{3\nu_2}{2}}\,\frac{\mathrm{d}r_2}{r_2}.$$

Poisson's summation formula tells us that

$$\sum_{\eta\in\mathbb{Z}}\mathrm{e}^{-\pi r_2\eta^2{t_1}^2{t_2}^2} = \frac{1}{t_1t_2\sqrt{r_2}}\sum_{\eta\in\mathbb{Z}}\mathrm{e}^{-\frac{\pi\eta^2}{r_2{t_1}^2{t_2}^2}},$$

so that $$\varphi(t_1,t_2) = \frac{1}{t_1t_2}\int_0^\infty\sum_{\eta\in\mathbb{Z}}\mathrm{e}^{-r_2\pi\Big[\big(\frac{\beta^{(1)}}{ct_1}\big)^2+\big(\frac{\beta^{(2)}}{ct_2}\big)^2\Big]-\frac{\pi\eta^2}{r_2{t_1}^2{t_2}^2}}\,{r_2}^{\frac{3\nu_2-1}{2}}\,\frac{\mathrm{d}r_2}{r_2}.$$

The term corresponding to $\eta=0$ in the sum can be rewritten as

$$\frac{1}{t_1t_2}\pi^{-\frac{3\nu_2-1}{2}}\Gamma\Big(\frac{3\nu_2-1}{2}\Big)\Big[\Big(\frac{\beta^{(1)}}{ct_1}\Big)^2+\Big(\frac{\beta^{(2)}}{ct_2}\Big)^2\Big]^{-\frac{3\nu_2-1}{2}}.$$

Its contribution to the expression of $\mathrm{G}^{\Omega_3\cup\Omega_4}_{\nu_1,\nu_2}$ is denoted by V:

$$\mathrm{V}_{\nu_1,\nu_2}(t_1,t_2) = \frac{1}{4t_1t_2}\pi^{-\frac{3\nu_1}{2}}\Gamma\Big(\frac{3\nu_1}{2}\Big)\pi^{-\frac{3\nu_2-1}{2}}\Gamma\Big(\frac{3\nu_2-1}{2}\Big)\zeta^\star(3\nu_1+3\nu_2-1) \qquad \textbf{(15)}$$
$$\times\sum_{\substack{\alpha\in\mathfrak{a}\\ \alpha\neq0}}\sum_{\substack{\beta\in(\mathfrak{a}\mathfrak{D})^{-1}\\ \beta\neq0\\ \mathrm{Tr}\,(\alpha\beta)=0}}\Big[\big(t_1\alpha^{(1)}\big)^2+\big(t_2\alpha^{(2)}\big)^2\Big]^{-\frac{3\nu_1}{2}}\Big[\Big(\frac{\beta^{(1)}}{ct_1}\Big)^2+\Big(\frac{\beta^{(2)}}{ct_2}\Big)^2\Big]^{-\frac{3\nu_2-1}{2}}$$

and $$\mathrm{G}^{\Omega_3\cup\Omega_4}_{\nu_1,\nu_2}(t_1,t_2)-\mathrm{V}_{\nu_1,\nu_2}(t_1,t_2) = \frac{1}{4}\zeta^\star(3\nu_1+3\nu_2-1)\sum_{\substack{\alpha\in\mathfrak{a}\\ \alpha\neq0}}\sum_{\substack{\beta\in(\mathfrak{a}\mathfrak{D})^{-1}\\ \beta\neq0\\ \mathrm{Tr}\,(\alpha\beta)=0}}\sum_{\substack{\eta\in\mathbb{Z}\\ \eta\neq0}}$$
$$\frac{1}{t_1t_2}\int_0^\infty\!\!\int_0^\infty\mathrm{e}^{-r_1\pi\left[(t_1\alpha^{(1)})^2+(t_2\alpha^{(2)})^2\right]-r_2\pi\Big[\big(\frac{\beta^{(1)}}{ct_1}\big)^2+\big(\frac{\beta^{(2)}}{ct_2}\big)^2\Big]-\frac{\pi\eta^2}{r_2{t_1}^2{t_2}^2}}\,{r_1}^{\frac{3\nu_1}{2}}{r_2}^{\frac{3\nu_2-1}{2}}\,\frac{\mathrm{d}r_1\,\mathrm{d}r_2}{r_1\,r_2}.$$

Computing the integral of this expression over $(\mathbb{R}_+^\star)/\mathfrak{o}^\times$ is an almost exact copy of the proof of **Proposition 6** – only the details of the computation change. Ultimately, we get the formula displayed in the current proposition. What remains to show at this point is that the V defined in (**15**) can indeed be rewritten as a classical Eisenstein series.

We focus on the double sum in (**15**), that we call $\Sigma(t_1, t_2)$. Because (α_1, α_2) and (β_1, β_2) are $\mathbb{Z}$-bases respectively for $\mathfrak{a}$ and $(\mathfrak{a}\mathfrak{D})^{-1}$, we have

$$\Sigma(t_1,t_2) = \sum_{\substack{\mathbf{x}\in\mathbb{Z}^2\\ \mathbf{x}\neq 0}} \sum_{\substack{\mathbf{y}\in\mathbb{Z}^2\\ \mathbf{y}\neq 0\\ \mathbf{x}\cdot\mathbf{y}=0}} \left[c^2{t_1}^2\big(x_1{\alpha_1}^{(1)} + x_2{\alpha_2}^{(1)}\big)^2 + c^2{t_2}^2\big(x_1{\alpha_1}^{(2)} + x_2{\alpha_2}^{(2)}\big)^2\right]^{-\frac{3\nu_1}{2}}$$
$$\times\left[c^{-2}{t_1}^{-2}\big(y_1{\beta_1}^{(1)} + y_2{\beta_2}^{(1)}\big)^2 + c^{-2}{t_2}^{-2}\big(y_1{\beta_1}^{(2)} + y_2{\beta_2}^{(2)}\big)^2\right]^{-\frac{3\nu_2-1}{2}}.$$

We can transform this sum by factoring out the gcds of each (x_1, x_2) and (y_1, y_2) and summing over all the possible values of these gcds. This pulls out two zeta functions:

$$\Sigma(t_1,t_2) = \zeta(3\nu_1)\zeta(3\nu_2-1)$$
$$\times \sum_{\substack{\mathbf{x}\in\mathbb{Z}^2\\ \gcd(x_1,x_2)=1}} \sum_{\substack{\mathbf{y}\in\mathbb{Z}^2\\ \gcd(y_1,y_2)=1\\ x_1y_1+x_2y_2=0}} \left[c^2{t_1}^2\big(x_1{\alpha_1}^{(1)} + x_2{\alpha_2}^{(1)}\big)^2 + c^2{t_2}^2\big(x_1{\alpha_1}^{(2)} + x_2{\alpha_2}^{(2)}\big)^2\right]^{-\frac{3\nu_1}{2}}$$
$$\times\left[c^{-2}{t_1}^{-2}\big(y_1{\beta_1}^{(1)} + y_2{\beta_2}^{(1)}\big)^2 + c^{-2}{t_2}^{-2}\big(y_1{\beta_1}^{(2)} + y_2{\beta_2}^{(2)}\big)^2\right]^{-\frac{3\nu_2-1}{2}}.$$

Now, let (x_1, x_2) and (y_1, y_2) be two couples of integers, each with gcd equal to 1 and such that $x_1y_1 + x_2y_2 = 0$. Then

$$x_1y_1 = -x_2y_2.$$

Because x_1 and x_2 are relatively prime, we get on the one hand

$$x_1|y_2 \qquad \text{and} \qquad x_2|y_1.$$

On the other hand, because y_1 and y_2 are relatively prime, we get

$$y_2|x_1 \qquad \text{and} \qquad y_1|x_2.$$

Hence, either $$y_2 = x_1 \qquad \text{and} \qquad y_1 = -x_2$$

or $$y_2 = -x_1 \qquad \text{and} \qquad y_1 = x_2.$$

Either way, the values of the corresponding summands in $\Sigma(t_1,t_2)$ are the same. So

$$\begin{aligned}\Sigma(t_1,t_2) &= 2\zeta(3\nu_1)\zeta(3\nu_2-1)\\ &\times \sum_{\substack{\mathbf{x}\in\mathbb{Z}^2\\ \gcd(x_1,x_2)=1}} \left[c^2{t_1}^2\big(x_1{\alpha_1}^{(1)}+x_2{\alpha_2}^{(1)}\big)^2+c^2{t_2}^2\big(x_1{\alpha_1}^{(2)}+x_2{\alpha_2}^{(2)}\big)^2\right]^{-\frac{3\nu_1}{2}}\\ &\times\left[c^{-2}{t_1}^{-2}\big(x_2{\beta_1}^{(1)}-x_1{\beta_2}^{(1)}\big)^2+c^{-2}{t_2}^{-2}\big(x_2{\beta_1}^{(2)}-x_1{\beta_2}^{(2)}\big)^2\right]^{-\frac{3\nu_2-1}{2}}.\end{aligned}$$

It is now time to make use of the relations (**8**) between the α's and β's, replace c by its value $\mathbb{N}(\mathfrak{a})^{-\frac{1}{2}}\mathrm{D}^{-\frac{1}{4}}$, and factor out $(t_1t_2)^{-2}$ from the expression in brackets on the third line. We obtain:

$$\begin{aligned}\Sigma(t_1,t_2) &= 2\zeta(3\nu_1)\zeta(3\nu_2-1)(t_1t_2)^{3\nu_2-1}\\ &\times \sum_{\substack{\mathbf{x}\in\mathbb{Z}^2\\ \gcd(x_1,x_2)=1}} \left[c^2{t_1}^2\big(x_1{\alpha_1}^{(1)}+x_2{\alpha_2}^{(1)}\big)^2+c^2{t_2}^2\big(x_1{\alpha_1}^{(2)}+x_2{\alpha_2}^{(2)}\big)^2\right]^{-\frac{3\nu_1+3\nu_2-1}{2}}.\end{aligned}$$

We recognize here the zeta function of a quadratic form. Transforming these objects in particular values of an Eisenstein series is a well-known technique that yields ultimately

$$\Sigma(t_1,t_2) = 4\zeta(3\nu_1)\zeta(3\nu_2-1)(t_1t_2)^{-\frac{3\nu_1-3\nu_2+1}{2}}\mathrm{E}\left(\frac{t_1{\alpha_1}^{(1)}-\mathrm{i}t_2{\alpha_1}^{(2)}}{t_1{\alpha_2}^{(1)}-\mathrm{i}t_2{\alpha_2}^{(2)}},3\nu_1+3\nu_2-1\right).$$

Thus

$$\begin{aligned}\mathrm{V}_{\nu_1,\nu_2}(t_1,t_2) &= \frac{1}{4t_1t_2}\pi^{-\frac{3\nu_1}{2}}\Gamma\Big(\frac{3\nu_1}{2}\Big)\pi^{-\frac{3\nu_2-1}{2}}\Gamma\Big(\frac{3\nu_2-1}{2}\Big)\zeta^\star(3\nu_1+3\nu_2-1)\Sigma(t_1,t_2)\\ &= \zeta^\star(3\nu_1)\zeta^\star(3\nu_2-1)(t_1t_2)^{-\frac{3\nu_1-3\nu_2+3}{2}}\mathrm{E}^\star\left(\frac{t_1{\alpha_1}^{(1)}-\mathrm{i}t_2{\alpha_1}^{(2)}}{t_1{\alpha_2}^{(1)}-\mathrm{i}t_2{\alpha_2}^{(2)}},\frac{3\nu_1+3\nu_2-1}{2}\right).\square\end{aligned}$$

3.4.3 What have we got so far?

In this short paragraph, we do a quick recap of everything we've done so far. We are interested in the renormalized integral

$$\mathrm{R}_\mathrm{A}(s_0,s_1,s_2) = \mathrm{RN}\int\limits_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathscr{H}} \mathrm{E}^\star(z,s_0)\mathrm{E}^\star(z,s_1)\mathrm{E}_{\mathrm{K,A}}(z,s_2)\,\frac{\mathrm{d}z}{y^2}.$$

By Zagier's definition, $\mathrm{R}_\mathrm{A}(s_0,s_1,s_2) = \zeta^\star(2s_0)\int\limits_0^\infty\big(a_0(y)-\varphi(y)\big)y^{s_0-1}\,\frac{\mathrm{d}y}{y},$

where $a_0 - \varphi$ has been computed in formula (**14**). Thus **Propositions 6**, **7** and **8** can be summarized in the formula

$$\mathrm{R_A}(s_0, s_1, s_2) =$$

$$6 \iint\limits_{(\mathbb{R}_+^\star)^2/\mathfrak{o}^\times} \big(\mathrm{G}_{\nu_1,\nu_2}(t_1,t_2) - \mathrm{V}_{\nu_1,\nu_2}(t_1,t_2) - \mathrm{G}^{\Omega_5}_{\nu_1,\nu_2}(t_1,t_2) - \mathrm{G}^{\Omega_6}_{\nu_1,\nu_2}(t_1,t_2)\big)(t_1t_2)^s \,\frac{\mathrm{d}t_1\,\mathrm{d}t_2}{t_1t_2}.$$

From the definition of Ω_5, we have

$$\begin{aligned}\mathrm{G}^{\Omega_5}_{\nu_1,\nu_2}(t_1,t_2) &= \frac{1}{4}\pi^{-\frac{3\nu_1}{2}}\Gamma\Big(\frac{3\nu_1}{2}\Big)\pi^{-\frac{3\nu_2}{2}}\Gamma\Big(\frac{3\nu_2}{2}\Big)\zeta^\star(3\nu_1+3\nu_2-1)\\ &\qquad\times\sum_{\substack{\xi\in\mathbb{Z}\\ \xi\neq 0}}\ \sum_{\substack{\beta\in(\mathfrak{a}\mathfrak{D})^{-1}\\ \beta\neq 0}}\Big(\frac{t_1t_2}{|\xi|}\Big)^{3\nu_1}\Big[\Big(\frac{\beta^{(1)}}{ct_1}\Big)^2+\Big(\frac{\beta^{(2)}}{ct_2}\Big)^2\Big]^{-\frac{3\nu_2}{2}}\\ &= \frac{(t_1t_2)^{3\nu_1}}{2}\zeta^\star(3\nu_1)\zeta^\star(3\nu_1+3\nu_2-1)\pi^{-\frac{3\nu_2}{2}}\Gamma\Big(\frac{3\nu_2}{2}\Big)\\ &\qquad\times\sum_{\substack{\beta\in(\mathfrak{a}\mathfrak{D})^{-1}\\ \beta\neq 0}}\Big[\Big(\frac{\beta^{(1)}}{ct_1}\Big)^2+\Big(\frac{\beta^{(2)}}{ct_2}\Big)^2\Big]^{-\frac{3\nu_2}{2}}.\end{aligned}$$

This is again the zeta function of a quadratic form, which we rewrite as an Eisenstein series:

$$\mathrm{G}^{\Omega_5}_{\nu_1,\nu_2}(t_1,t_2) = (t_1t_2)^{3\nu_1+\frac{3\nu_2}{2}}\zeta^\star(3\nu_1)\zeta^\star(3\nu_1+3\nu_2-1)\mathrm{E}^\star\Big(\frac{t_1{\alpha_1}^{(1)}-\mathrm{i}t_2{\alpha_1}^{(2)}}{t_1{\alpha_2}^{(1)}-\mathrm{i}t_2{\alpha_2}^{(2)}},\frac{3\nu_2}{2}\Big).$$

Similarly,

$$\mathrm{G}^{\Omega_6}_{\nu_1,\nu_2}(t_1,t_2) = (t_1t_2)^{-3\nu_2-\frac{3\nu_1}{2}}\zeta^\star(3\nu_2)\zeta^\star(3\nu_1+3\nu_2-1)\mathrm{E}^\star\Big(\frac{t_1{\alpha_1}^{(1)}-\mathrm{i}t_2{\alpha_1}^{(2)}}{t_1{\alpha_2}^{(1)}-\mathrm{i}t_2{\alpha_2}^{(2)}},\frac{3\nu_1}{2}\Big).$$

Hence we can summarize what we have so far in the following

Proposition 9

*Let s_0, s_1, s_2, ν_1, ν_2 and s be complex numbers defined as in **Section 3.2.2**. Then*

$$\mathrm{R_A}(s_0,s_1,s_2) = 6\iint\limits_{(\mathbb{R}_+^\star)^2/\mathfrak{o}^\times}\big(\mathrm{G}_{\nu_1,\nu_2}(t_1,t_2) - f_{\nu_1,\nu_2}(t_1,t_2)\big)(t_1t_2)^s\,\frac{dt_1\,dt_2}{t_1t_2}, \qquad (\mathbf{16})$$

where

$$
\begin{aligned}
f_{\nu_1,\nu_2}(t_1,t_2) &= \mathrm{V}_{\nu_1,\nu_2}(t_1,t_2) + \mathrm{G}^{\Omega_5}_{\nu_1,\nu_2}(t_1,t_2) + \mathrm{G}^{\Omega_6}_{\nu_1,\nu_2}(t_1,t_2) \\
&= (t_1t_2)^{3\nu_1+\frac{3\nu_2}{2}}\zeta^\star(3\nu_1)\zeta^\star(3\nu_1+3\nu_2-1)\mathrm{E}^\star\left(\frac{t_1{\alpha_1}^{(1)}-it_2{\alpha_1}^{(2)}}{t_1{\alpha_2}^{(1)}-it_2{\alpha_2}^{(2)}}, \frac{3\nu_2}{2}\right) \\
&\quad +(t_1t_2)^{-3\nu_2-\frac{3\nu_1}{2}}\zeta^\star(3\nu_2)\zeta^\star(3\nu_1+3\nu_2-1)\mathrm{E}^\star\left(\frac{t_1{\alpha_1}^{(1)}-it_2{\alpha_1}^{(2)}}{t_1{\alpha_2}^{(1)}-it_2{\alpha_2}^{(2)}}, \frac{3\nu_1}{2}\right) \\
&\quad +(t_1t_2)^{-\frac{3\nu_1-3\nu_2+3}{2}}\zeta^\star(3\nu_1)\zeta^\star(3\nu_2-1)\mathrm{E}^\star\left(\frac{t_1{\alpha_1}^{(1)}-it_2{\alpha_1}^{(2)}}{t_1{\alpha_2}^{(1)}-it_2{\alpha_2}^{(2)}}, 3\nu_1+3\nu_2-1\right).
\end{aligned}
$$

The problem now is that, even though G is invariant under the transformation $(\nu_1,\nu_2) \longmapsto (1-\nu_1-\nu_2,\nu_1)$, this symmetry cannot be transfered to R_A because it is not apparent in f. Some more work is thus required.

3.4.4 End of the proof

In section 2.5, we gave a description of a set of representatives for $(\mathbb{R}_+^\star)^2/\mathfrak{o}^\times$. This set is the collection of all lines in the quarter plane $(\mathbb{R}_+^\star)^2$ that go through 0 and with slopes between ε^{-1} and ε. Thus the map

$$(t_1,t_2) \longmapsto \left(\frac{t_2}{t_1}, t_1t_2\right)$$

is a $\mathscr{C}^1$-difféomorphism from $(\mathbb{R}_+^\star)^2/\mathfrak{o}^\times$ onto $\left[\varepsilon^{-1},\varepsilon\right] \times \mathbb{R}_+^\star$. So we make the variable change

$$u = \frac{t_1}{t_2} \qquad v = t_1t_2 \qquad t_1 = \sqrt{\frac{v}{u}} \qquad t_2 = \sqrt{uv} \qquad \frac{\mathrm{d}t_1\,\mathrm{d}t_2}{t_1t_2} = \frac{1}{2}\frac{\mathrm{d}u\,\mathrm{d}v}{u\,v}$$

in the integral (**16**), factor the t_2's on the numerator and denominator in the argument of the Eisenstein series, and obtain

$$
\begin{aligned}
\mathrm{R}_\mathrm{A}(s_0,s_1,s_2) = 3\int_0^\infty\int_{\varepsilon^{-1}}^{\varepsilon}\Bigg(&\mathrm{G}_{\nu_1,\nu_2}\left(\sqrt{\frac{v}{u}},\sqrt{uv}\right) \\
&-v^{-\frac{3\nu_1-3\nu_2+3}{2}}\zeta^\star(3\nu_1)\zeta^\star(3\nu_2-1)\mathrm{E}^\star\left(\frac{{\alpha_1}^{(1)}-\mathrm{i}u{\alpha_1}^{(2)}}{{\alpha_2}^{(1)}-\mathrm{i}u{\alpha_2}^{(2)}}, \frac{3\nu_1+3\nu_2-1}{2}\right) \\
&-v^{3\nu_1+\frac{3\nu_2}{2}}\zeta^\star(3\nu_1)\zeta^\star(3\nu_1+3\nu_2-1)\mathrm{E}^\star\left(\frac{{\alpha_1}^{(1)}-\mathrm{i}u{\alpha_1}^{(2)}}{{\alpha_2}^{(1)}-\mathrm{i}u{\alpha_2}^{(2)}}, \frac{3\nu_2}{2}\right) \\
&-v^{-3\nu_2-\frac{3\nu_1}{2}}\zeta^\star(3\nu_2)\zeta^\star(3\nu_1+3\nu_2-1)\mathrm{E}^\star\left(\frac{{\alpha_1}^{(1)}-\mathrm{i}u{\alpha_1}^{(2)}}{{\alpha_2}^{(1)}-\mathrm{i}u{\alpha_2}^{(2)}}, \frac{3\nu_1}{2}\right)\Bigg)v^s\frac{\mathrm{d}u\,\mathrm{d}v}{u\,v}.
\end{aligned}
$$

The first integral with respect to u, between ε^{-1} and ε is not a problem and we define

$$\mathcal{G}_{\nu_1,\nu_2}(v) = \int_{\varepsilon^{-1}}^{\varepsilon} \mathrm{G}_{\nu_1,\nu_2}\left(\sqrt{\frac{v}{u}}, \sqrt{uv}\right) \frac{\mathrm{d}u}{u}$$

and
$$\mathrm{L}(\omega) = \int_{\varepsilon^{-1}}^{\varepsilon} \mathrm{E}^{\star}\left(\frac{{\alpha_1}^{(1)} - \mathrm{i}u{\alpha_1}^{(2)}}{{\alpha_2}^{(1)} - \mathrm{i}u{\alpha_2}^{(2)}}, \omega\right) \frac{\mathrm{d}u}{u}.$$

All that is important to know about these two functions is that they are invariant, respectively under $(\nu_1, \nu_2) \longmapsto (1 - \nu_1 - \nu_2, \nu_1)$ and $\omega \longmapsto 1 - \omega$.

> **Remark:** Actually, one can say a little more about L. Indeed, using the technique from Hecke, described by Zagier in [8], one can actually show that $2\mathrm{L}(s) = \zeta^{\star}_{\mathrm{A}}(s)$. This identity constitutes, therefore, a proof for functional equation $\zeta^{\star}_{\mathrm{A}}(s) = \zeta^{\star}_{\mathrm{A}}(1-s)$. Since we know already (see for example Lang [6]) that $\zeta^{\star}_{\mathrm{A}}(1-s) = \zeta^{\star}_{\mathrm{A}^{-1}\mathfrak{D}^{-1}}(s)$, we get the result that, in a real quadratic field
>
> $$\zeta^{\star}_{\mathrm{A}}(1-s) = \zeta^{\star}_{\mathrm{A}^{-1}\mathfrak{D}^{-1}}(s) = \zeta^{\star}_{\mathrm{A}}(s).$$

With these new notations, we get

$$\begin{aligned}\frac{\mathrm{R}_{\mathrm{A}}(s_0, s_1, s_2)}{3} = \int_0^{\infty} \Bigg(\mathcal{G}_{\nu_1,\nu_2}(v) - v^{-\frac{3\nu_1 - 3\nu_2 + 3}{2}} \mathrm{L}\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right)\zeta^{\star}(3\nu_1)\zeta^{\star}(3\nu_2 - 1) \\ - v^{3\nu_1 + \frac{3\nu_2}{2}} \mathrm{L}\left(\frac{3\nu_2}{2}\right)\zeta^{\star}(3\nu_1)\zeta^{\star}(3\nu_1 + 3\nu_2 - 1) \\ - v^{-3\nu_2 - \frac{3\nu_1}{2}} \mathrm{L}\left(\frac{3\nu_1}{2}\right)\zeta^{\star}(3\nu_2)\zeta^{\star}(3\nu_1 + 3\nu_2 - 1)\Bigg) v^s \frac{\mathrm{d}v}{v}.\end{aligned}$$

Now, we split this integral $\int_0^{\infty}$ as $\int_0^1 + \int_1^{\infty}$ and we perform the substitution $v \to v^{-1}$ in the integral on $(0, 1]$. We regroup everything under a same integral over $[1, \infty)$ and finally let $x = \sqrt{v}$, in order to get rid of these fractional powers:

$$\begin{aligned}\frac{\mathrm{R}_{\mathrm{A}}(s_0,s_1,s_2)}{6} = \int_1^\infty \Bigg(& \mathcal{G}_{\nu_1,\nu_2}(x^2)x^{2s} + \mathcal{G}_{\nu_1,\nu_2}\left(x^{-2}\right)x^{-2s} \\ & - x^{-(3\nu_1-3\nu_2+3-2s)}\mathrm{L}\left(\frac{3\nu_1+3\nu_2-1}{2}\right)\zeta^\star(3\nu_1)\zeta^\star(3\nu_2-1) \\ & - x^{6\nu_1+3\nu_2+2s}\mathrm{L}\left(\frac{3\nu_2}{2}\right)\zeta^\star(3\nu_1)\zeta^\star(3\nu_1+3\nu_2-1) \\ & - x^{-(3\nu_1+6\nu_2-2s)}\mathrm{L}\left(\frac{3\nu_1}{2}\right)\zeta^\star(3\nu_2)\zeta^\star(3\nu_1+3\nu_2-1) \\ & - x^{3\nu_1-3\nu_2+3-2s}\mathrm{L}\left(\frac{3\nu_1+3\nu_2-1}{2}\right)\zeta^\star(3\nu_1)\zeta^\star(3\nu_2-1) \\ & - x^{-(6\nu_1+3\nu_2+2s)}\mathrm{L}\left(\frac{3\nu_2}{2}\right)\zeta^\star(3\nu_1)\zeta^\star(3\nu_1+3\nu_2-1) \\ & - x^{3\nu_1+6\nu_2-2s}\mathrm{L}\left(\frac{3\nu_1}{2}\right)\zeta^\star(3\nu_2)\zeta^\star(3\nu_1+3\nu_2-1)\Bigg)\frac{\mathrm{d}x}{x}.\end{aligned}$$

Because of our choices for $\operatorname{Re}\nu_1$, $\operatorname{Re}\nu_2$ and $\operatorname{Re}(\nu_1-\nu_2)$, some terms are integrable (those on lines 2, 4 and 6) and can be taken out of the integral:

$$\begin{aligned}\frac{\mathrm{R}_{\mathrm{A}}(s_0,s_1,s_2)}{6} = & \int_1^\infty \Bigg(\mathcal{G}_{\nu_1,\nu_2}(x^2)x^{2s} + \mathcal{G}_{\nu_1,\nu_2}\left(x^{-2}\right)x^{-2s} \\ & - x^{6\nu_1+3\nu_2+2s}\mathrm{L}\left(\frac{3\nu_2}{2}\right)\zeta^\star(3\nu_1)\zeta^\star(3\nu_1+3\nu_2-1) \\ & - x^{3\nu_1-3\nu_2+3-2s}\mathrm{L}\left(\frac{3\nu_1+3\nu_2-1}{2}\right)\zeta^\star(3\nu_1)\zeta^\star(3\nu_2-1) \\ & - x^{3\nu_1+6\nu_2-2s}\mathrm{L}\left(\frac{3\nu_1}{2}\right)\zeta^\star(3\nu_2)\zeta^\star(3\nu_1+3\nu_2-1)\Bigg)\frac{\mathrm{d}x}{x} \qquad \textbf{(17)} \\ & - \frac{1}{3\nu_1-3\nu_2+3-2s}\mathrm{L}\left(\frac{3\nu_1+3\nu_2-1}{2}\right)\zeta^\star(3\nu_1)\zeta^\star(3\nu_2-1) \\ & - \frac{1}{3\nu_1+6\nu_2-2s}\mathrm{L}\left(\frac{3\nu_1}{2}\right)\zeta^\star(3\nu_2)\zeta^\star(3\nu_1+3\nu_2-1) \\ & - \frac{1}{6\nu_1+3\nu_2+2s}\mathrm{L}\left(\frac{3\nu_2}{2}\right)\zeta^\star(3\nu_1)\zeta^\star(3\nu_1+3\nu_2-1).\end{aligned}$$

Finally, the following formula holds:

$$
\begin{aligned}
\frac{\mathrm{R}_{\mathrm{A}}(s_0,s_1,s_2)}{3} = & \int_1^{\infty}\Bigg(2\mathcal{G}_{\nu_1,\nu_2}(x^2)x^{2s}+2\mathcal{G}_{\nu_1,\nu_2}\big(x^{-2}\big)x^{-2s} \\
& -x^{6\nu_1+3\nu_2+2s}\mathrm{L}\left(\frac{3\nu_2}{2}\right)\zeta^\star(3\nu_1)\zeta^\star(3\nu_1+3\nu_2-1) \\
& -x^{3\nu_1-3\nu_2+3-2s}\mathrm{L}\left(\frac{3\nu_1+3\nu_2-1}{2}\right)\zeta^\star(3\nu_1)\zeta^\star(3\nu_2-1) \\
& -x^{3\nu_1+6\nu_2-2s}\mathrm{L}\left(\frac{3\nu_1}{2}\right)\zeta^\star(3\nu_2)\zeta^\star(3\nu_1+3\nu_2-1) \\
& -x^{-(3\nu_1+6\nu_2-6-2s)}\mathrm{L}\left(\frac{3\nu_1}{2}\right)\zeta^\star(3\nu_2-1)\zeta^\star(3\nu_1+3\nu_2-2) \\
& -x^{-(3\nu_1-3\nu_2-3-2s)}\mathrm{L}\left(\frac{3\nu_1+3\nu_2-1}{2}\right)\zeta^\star(3\nu_1-1)\zeta^\star(3\nu_2) \\
& -x^{-(6\nu_1+3\nu_2-6+2s)}\mathrm{L}\left(\frac{3\nu_2}{2}\right)\zeta^\star(3\nu_1+3\nu_2-2)\zeta^\star(3\nu_1-1)\Bigg)\frac{\mathrm{d}x}{x} \quad (\mathbf{18}) \\
& -\frac{1}{3\nu_1-3\nu_2+3-2s}\mathrm{L}\left(\frac{3\nu_1+3\nu_2-1}{2}\right)\zeta^\star(3\nu_1)\zeta^\star(3\nu_2-1) \\
& -\frac{1}{3\nu_1+6\nu_2-2s}\mathrm{L}\left(\frac{3\nu_1}{2}\right)\zeta^\star(3\nu_2)\zeta^\star(3\nu_1+3\nu_2-1) \\
& -\frac{1}{6\nu_1+3\nu_2+2s}\mathrm{L}\left(\frac{3\nu_2}{2}\right)\zeta^\star(3\nu_1)\zeta^\star(3\nu_1+3\nu_2-1) \\
& +\frac{1}{3\nu_1+6\nu_2-6-2s}\mathrm{L}\left(\frac{3\nu_1}{2}\right)\zeta^\star(3\nu_2-1)\zeta^\star(3\nu_1+3\nu_2-2) \\
& +\frac{1}{3\nu_1-3\nu_2-3-2s}\mathrm{L}\left(\frac{3\nu_1+3\nu_2-1}{2}\right)\zeta^\star(3\nu_1-1)\zeta^\star(3\nu_2) \\
& +\frac{1}{6\nu_1+3\nu_2+2s-6}\mathrm{L}\left(\frac{3\nu_2}{2}\right)\zeta^\star(3\nu_1+3\nu_2-2)\zeta^\star(3\nu_1-1).
\end{aligned}
$$

From (**17**) to (**18**), we have added the terms on lines 5, 6, 7, 11, 12 and 13. It turns out that when one integrates the terms on lines 5, 6 and 7, they cancel the terms on lines 11, 12 and 13. So nothing really happened here.

Until now, we had restrictions on the real parts of our parameters ν_1, ν_2 and s. Those are now lifted by meromorphic continuation and formula (**18**) holds in fact for all values of our paramaters away from the polar divisor of $\mathrm{R_A}$.

Finally, as a courageous reader can check, everything that is inside and outside the integral is invariant when one replaces ν_1 by $1 - \nu_1 - \nu_2$ and ν_2 by ν_1. So this invariance transfers to $\mathrm{R_A}$. And this achieves showing that w is a functional equation for $\mathrm{R_A}$, since $(\nu_1, \nu_2) \longmapsto (1-\nu_1-\nu_2, \nu_2)$ translates into w for s_0, s_1 and s_2, considering the relations (**9**). $\square$

Appendix A

More computations

In this last chapter, we present two computations that were alluded to, but now explained, during the proof of our main theorem. We want to explain

- how one can transform the zeta function of a positive definite quadratic form into a particular value of the classical Eisenstein series. This will be done with the particular example of V_{ν_1,ν_2} (page 63) but the strategy can be applied to prove the formulas on the bottom of page 64.
- how the function $\mathrm{L}(\omega)$ introduced on page 66 relates to the zeta function of the ideal class A.

A.1 Quadratic forms and Eisenstein series

In the proof of **Proposition 8**, we encounter the series

$$\Sigma(t_1,t_2) = 2\zeta(3\nu_1)\zeta(3\nu_2-1)(t_1t_2)^{3\nu_2-1}$$
$$\times \sum_{\substack{\mathbf{x}\in\mathbb{Z}^2\\ \gcd(x_1,x_2)=1}} \left[c^2{t_1}^2\left(x_1{\alpha_1}^{(1)}+x_2{\alpha_2}^{(1)}\right)^2 + c^2{t_2}^2\left(x_1{\alpha_1}^{(2)}+x_2{\alpha_2}^{(2)}\right)^2\right]^{-\frac{3\nu_1+3\nu_2-1}{2}}$$

and we claim that it can be rewritten as

$$\Sigma(t_1,t_2) = 4\zeta(3\nu_1)\zeta(3\nu_2-1)(t_1t_2)^{-\frac{3\nu_1-3\nu_2+1}{2}}\mathrm{E}\left(\frac{t_1{\alpha_1}^{(1)}-\mathrm{i}t_2{\alpha_1}^{(2)}}{t_1{\alpha_2}^{(1)}-\mathrm{i}t_2{\alpha_2}^{(2)}}, \frac{3\nu_1+3\nu_2-1}{2}\right)$$

Define the quadratic form

$$\forall (m,n)\in\mathbb{Z}^2 \qquad \mathrm{Q}(m,n) = c^2{t_1}^2\big(m{\alpha_1}^{(1)} + n{\alpha_2}^{(1)}\big)^2 + c^2{t_2}^2\big(m{\alpha_1}^{(2)} + n{\alpha_2}^{(2)}\big)^2$$

so that $\quad \Sigma(t_1,t_2) = 2\zeta(3\nu_1)\zeta(3\nu_2-1)(t_1t_2)^{3\nu_2-1} \displaystyle\sum_{\substack{(m,n)\in\mathbb{Z}^2\\ \gcd(m,n)=1}} \frac{1}{\mathrm{Q}(m,n)^{\frac{3\nu_1+3\nu_2-1}{2}}}$

We have

$$\begin{aligned}\frac{\mathrm{Q}(m,n)}{c^2} &= {t_1}^2\big(m{\alpha_1}^{(1)} + n{\alpha_2}^{(1)}\big)^2 + {t_2}^2\big(m{\alpha_1}^{(2)} + n{\alpha_2}^{(2)}\big)^2\\ &= m^2\big((t_1{\alpha_1}^{(1)})^2 + (t_2{\alpha_1}^{(2)})^2\big) + n^2\big(\big(t_1{\alpha_2}^{(1)}\big)^2 + (t_2{\alpha_2}^{(2)})^2\big)\\ &\qquad\qquad\qquad + 2mn\big({t_1}^2{\alpha_1}^{(1)}{\alpha_2}^{(1)} + {t_2}^2{\alpha_1}^{(2)}{\alpha_2}^{(2)}\big)\end{aligned}$$

We want to put this under the form $|mz+nw|^2$, for some complex numbers z and w such that $\frac{z}{w}$ is in the upper half-plane. The natural choices are

$$z = t_1{\alpha_1}^{(1)} - \mathrm{i}t_2{\alpha_1}^{(2)}$$

and

$$w = t_1{\alpha_2}^{(1)} - \mathrm{i}t_2{\alpha_2}^{(2)}$$

We have

$$|z|^2 = (t_1{\alpha_1}^{(1)})^2 + (t_2{\alpha_1}^{(2)})^2$$

$$|w|^2 = \big(t_1{\alpha_2}^{(1)}\big)^2 + (t_2{\alpha_2}^{(2)})^2$$

$$\mathrm{Re}\,\big(z\overline{w}\big) = {t_1}^2{\alpha_1}^{(1)}{\alpha_2}^{(1)} + {t_2}^2{\alpha_1}^{(2)}{\alpha_2}^{(2)}$$

and $\quad \mathrm{Im}\left(\dfrac{z}{w}\right) = \mathrm{Im}\left(\dfrac{z\overline{w}}{|w|^2}\right) = \dfrac{t_1t_2\big({\alpha_1}^{(1)}{\alpha_2}^{(2)} - {\alpha_1}^{(2)}{\alpha_2}^{(1)}\big)}{|w|^2} = \dfrac{t_1t_2}{c^2|w|^2} > 0$

Therefore $\quad |mz+nw|^2 = m^2|z|^2 + n^2|w|^2 + 2mn\mathrm{Re}\,\big(z\overline{w}\big) = \dfrac{\mathrm{Q}(m,n)}{c^2}$

or

$$\frac{\mathrm{Q}(m,n)}{c^2} = |w|^2\left|m\frac{z}{w} + n\right|^2$$

If we define $\mathrm{Z}\in\mathscr{H}$ to be the ratio $\frac{z}{w}$:

$$\mathrm{Z} = \frac{t_1{\alpha_1}^{(1)} - \mathrm{i}t_2{\alpha_1}^{(2)}}{t_1{\alpha_2}^{(1)} - \mathrm{i}t_2{\alpha_2}^{(2)}} \qquad \text{and} \qquad \mathrm{Im}\,\mathrm{Z} = \frac{t_1t_2}{c^2|w|^2}$$

we get

$$\mathrm{Q}(m,n) = c^2|w|^2\,|m\mathrm{Z}+n|^2 = t_1t_2\times\frac{|m\mathrm{Z}+n|^2}{\mathrm{Im}\,\mathrm{Z}}$$

Hence $\Sigma(t_1,t_2) = 2\zeta(3\nu_1)\zeta(3\nu_2-1)(t_1t_2)^{3\nu_2-1} \sum_{\substack{(m,n)\in\mathbb{Z}^2\\ \gcd(m,n)=1}} \left(\dfrac{\operatorname{Im} \mathrm{Z}}{t_1t_2|m\mathrm{Z}+n|^2}\right)^{\frac{3\nu_1+3\nu_2-1}{2}}$

$$= 4\zeta(3\nu_1)\zeta(3\nu_2-1)(t_1t_2)^{-\frac{3\nu_1-3\nu_2+1}{2}} \mathrm{E}\left(\mathrm{Z}, \frac{3\nu_1+3\nu_2+1}{2}\right)$$

A.2 Eisenstein series and zeta functions of ideal classes

On page 66, we anounce in a remark that the function L, defined as

$$\forall \omega \in \mathbb{C}\setminus\{0,1\} \qquad \mathrm{L}(\omega) = \int_{\varepsilon^{-1}}^{\varepsilon} \mathrm{E}^\star\left(\frac{\alpha_1{}^{(1)} - \mathrm{i}u\alpha_1{}^{(2)}}{\alpha_2{}^{(1)} - \mathrm{i}u\alpha_2{}^{(2)}}, \omega\right)$$

is related to the normalized zeta function of the ideal class A:

$$\mathrm{L}(\omega) = \frac{\zeta^\star_{\mathrm{A}}(\omega)}{2} = \frac{\mathrm{D}^{\frac{\omega}{2}}\pi^{-\omega}}{2}\Gamma\left(\frac{\omega}{2}\right)^2 \sum_{\mathfrak{b} \text{ integral in A}} \frac{1}{\mathbb{N}(\mathfrak{b})^\omega}$$

Even though this identity is irrelevant to our result, it is interesting enough to mention and prove.

The computation done in the previous section shows that

$$\sum_{\substack{(m,n)\in\mathbb{Z}^2\\ \gcd(m,n)=1}} \frac{1}{\left[c^2{t_1}^2\left(m\alpha_1{}^{(1)}+n\alpha_2{}^{(1)}\right)^2 + c^2{t_2}^2\left(m\alpha_1{}^{(2)}+n\alpha_2{}^{(2)}\right)^2\right]^\omega} =$$

$$\frac{2}{(t_1t_2)^\omega}\mathrm{E}\left(\frac{t_1\alpha_1{}^{(1)} - \mathrm{i}t_2\alpha_2{}^{(1)}}{t_1\alpha_1{}^{(2)} - \mathrm{i}t_2\alpha_2{}^{(2)}}, \omega\right)$$

Multiply both sides by $\zeta^\star(2\omega)$, put $t_1 = 1$ and $t_2 = u$, to obtain:

$$\frac{2}{u^\omega}\mathrm{E}^\star\left(\frac{\alpha_1{}^{(1)} - \mathrm{i}u\alpha_2{}^{(1)}}{\alpha_1{}^{(2)} - \mathrm{i}u\alpha_2{}^{(2)}}, \omega\right) =$$

$$\pi^{-\omega}\Gamma(\omega) \sum_{\substack{(m,n)\in\mathbb{Z}^2\\ (m,n)\neq 0}} \frac{1}{\left[c^2\left(m\alpha_1{}^{(1)}+n\alpha_2{}^{(1)}\right)^2 + c^2u^2\left(m\alpha_1{}^{(2)}+n\alpha_2{}^{(2)}\right)^2\right]^\omega}$$

$$= \frac{\pi^{-\omega}\Gamma(\omega)}{c^{2\omega}} \sum_{\substack{\alpha\in\mathfrak{a}\\ \alpha\neq 0}} \frac{1}{\left({\alpha^{(1)}}^2 + u^2{\alpha^{(2)}}^2\right)^\omega}$$

$$\mathrm{E}^\star\left(\frac{\alpha_1{}^{(1)} - \mathrm{i}u\alpha_2{}^{(1)}}{\alpha_1{}^{(2)} - \mathrm{i}u\alpha_2{}^{(2)}}, \omega\right) = \frac{\pi^{-\omega}\Gamma(\omega)}{2c^{2\omega}} \sum_{\substack{\alpha\in\mathfrak{a}\\ \alpha\neq 0}} \frac{1}{\left(u^{-1}{\alpha^{(1)}}^2 + u{\alpha^{(2)}}^2\right)^\omega}$$

Then condition the sum on cosets of $\mathfrak{a}/\mathfrak{o}^\times$:

$$\mathrm{E}^\star\left(\frac{{\alpha_1}^{(1)} - \mathrm{i}u{\alpha_2}^{(1)}}{{\alpha_1}^{(2)} - \mathrm{i}u{\alpha_2}^{(2)}}, \omega\right) = \frac{\pi^{-\omega}\Gamma(\omega)}{2c^{2\omega}} \sum_{\substack{\alpha\in\mathfrak{a}/\mathfrak{o}^\times \\ \alpha\neq 0}} \sum_{\epsilon\in\mathfrak{o}^\times} \frac{1}{\left(u^{-1}{\epsilon^{(1)}}^2{\alpha^{(1)}}^2 + u{\epsilon^{(2)}}^2{\alpha^{(2)}}^2\right)^\omega}$$

As explained in **Proposition 4**, for every unit ε, we have

$$\left|\epsilon^{(2)}\right| = \frac{1}{\left|\epsilon^{(1)}\right|} = \frac{1}{|\epsilon|}$$

so after factoring $\left|\epsilon^{(1)}\epsilon^{(2)}\right|^\omega = 1$ from the denominator, we get

$$\mathrm{E}^\star\left(\frac{{\alpha_1}^{(1)} - \mathrm{i}u{\alpha_2}^{(1)}}{{\alpha_1}^{(2)} - \mathrm{i}u{\alpha_2}^{(2)}}, \omega\right) = \frac{\pi^{-\omega}\Gamma(\omega)}{2c^{2\omega}} \sum_{\substack{\alpha\in\mathfrak{a}/\mathfrak{o}^\times \\ \alpha\neq 0}} \sum_{\epsilon\in\mathfrak{o}^\times} \frac{1}{\left(u^{-1}\epsilon^2{\alpha^{(1)}}^2 + u\epsilon^{-2}{\alpha^{(2)}}^2\right)^\omega}$$

Next, by the unit theorem, there exists a fundamental unit $\varepsilon > 1$ such that

$$\mathfrak{o}^\times = \left\{ \pm\varepsilon^k \mid k \in \mathbb{Z}\right\}$$

So $$\mathrm{E}^\star\left(\frac{{\alpha_1}^{(1)} - \mathrm{i}u{\alpha_2}^{(1)}}{{\alpha_1}^{(2)} - \mathrm{i}u{\alpha_2}^{(2)}}, \omega\right) = \frac{\pi^{-\omega}\Gamma(\omega)}{c^{2\omega}} \sum_{\substack{\alpha\in\mathfrak{a}/\mathfrak{o}^\times \\ \alpha\neq 0}} \sum_{k\in\mathbb{Z}} \frac{1}{\left(u^{-1}\varepsilon^{2k}{\alpha^{(1)}}^2 + u\varepsilon^{-2k}{\alpha^{(2)}}^2\right)^\omega}$$

Everything is now set up to compute the integral $\mathrm{L}(\omega)$:

$$\mathrm{L}(\omega) = \frac{\pi^{-\omega}\Gamma(\omega)}{c^{2\omega}} \sum_{\substack{\alpha\in\mathfrak{a}/\mathfrak{o}^\times \\ \alpha\neq 0}} \sum_{k\in\mathbb{Z}} \int_{\varepsilon^{-1}}^{\varepsilon} \frac{1}{\left(u^{-1}\varepsilon^{2k}{\alpha^{(1)}}^2 + u\varepsilon^{-2k}{\alpha^{(2)}}^2\right)^\omega} \frac{\mathrm{d}u}{u}$$

Change variables and replace $u^{-1}\varepsilon^{2k}$ by v:

$$\mathrm{L}(\omega) = \frac{\pi^{-\omega}\Gamma(\omega)}{c^{2\omega}} \sum_{\substack{\alpha\in\mathfrak{a}/\mathfrak{o}^\times \\ \alpha\neq 0}} \sum_{k\in\mathbb{Z}} \int_{\varepsilon^{2k-1}}^{\varepsilon^{2k+1}} \frac{1}{\left(v{\alpha^{(1)}}^2 + v^{-1}{\alpha^{(2)}}^2\right)^\omega} \frac{\mathrm{d}v}{v}$$

The integrals and the sum over k collapse nicely and we obtain

$$\mathrm{L}(\omega) = \frac{\pi^{-\omega}\Gamma(\omega)}{c^{2\omega}} \sum_{\substack{\alpha\in\mathfrak{a}/\mathfrak{o}^\times \\ \alpha\neq 0}} \int_{0}^{\infty} \frac{1}{\left(v{\alpha^{(1)}}^2 + v^{-1}{\alpha^{(2)}}^2\right)^\omega} \frac{\mathrm{d}v}{v}$$

Factor $\left|\alpha^{(1)}\alpha^{(2)}\right|^\omega = \mathbb{N}(\alpha)^\omega$ out of the denominators:

$$\mathrm{L}(\omega) = \frac{\pi^{-\omega}\Gamma(\omega)}{c^{2\omega}} \sum_{\substack{\alpha\in\mathfrak{a}/\mathfrak{o}^\times \\ \alpha\neq 0}} \frac{1}{\mathbb{N}(\alpha)^\omega} \int_{0}^{\infty} \frac{1}{\left(v\,\frac{\alpha^{(1)}}{\alpha^{(2)}} + v^{-1}\,\frac{\alpha^{(2)}}{\alpha^{(1)}}\right)^\omega} \frac{\mathrm{d}v}{v}$$

Change variables again, replacing $v\,\frac{\alpha^{(1)}}{\alpha^{(2)}}$ by u. Also, remember that $c^{-2} = \mathbb{N}(\mathfrak{a})\mathrm{D}^{\frac{1}{2}}$:

$$\mathrm{L}(\omega) = \mathrm{D}^{\frac{\omega}{2}}\pi^{-\omega}\Gamma(\omega)\sum_{\substack{\alpha\in\mathfrak{a}/\mathfrak{o}^\times\\ \alpha\neq 0}}\frac{\mathbb{N}(\mathfrak{a})^\omega}{\mathbb{N}(\alpha)^\omega}\int_0^{+\infty}\frac{1}{(u+u^{-1})^\omega}\,\frac{\mathrm{d}u}{u}$$

Relating this to the facts about the zeta function of the ideal class A, presented in **Section 2.2.1** (page 32), it follows that

$$\mathrm{L}(\omega) = \mathrm{D}^{\frac{\omega}{2}}\pi^{-\omega}\,\zeta_{\mathrm{K,A}}(\omega)\Gamma(\omega)\int_0^{+\infty}\frac{1}{(u+u^{-1})^\omega}\,\frac{\mathrm{d}u}{u}$$

But
$$\begin{aligned}
\Gamma(\omega)\int_0^{+\infty}\frac{1}{(u+u^{-1})^\omega}\,\frac{\mathrm{d}u}{u} &= \int_0^{+\infty}\int_0^{+\infty}\mathrm{e}^{-v}\Big(\frac{v}{u+u^{-1}}\Big)^\omega\,\frac{\mathrm{d}v\mathrm{d}u}{vu}\\
&= \int_0^{+\infty}\int_0^{+\infty}\mathrm{e}^{-vu-vu^{-1}}v^\omega\,\frac{\mathrm{d}v\mathrm{d}u}{vu}\\
\Gamma(\omega)\int_0^{+\infty}\frac{1}{(u+u^{-1})^\omega}\,\frac{\mathrm{d}u}{u} &= \int_0^{+\infty}\int_0^{+\infty}\mathrm{e}^{-vu}(vu)^{\frac{\omega}{2}}\,\mathrm{e}^{-vu^{-1}}(vu^{-1})^{\frac{\omega}{2}}\,\frac{\mathrm{d}v\mathrm{d}u}{vu}
\end{aligned}$$

Just proceed to the substitution

$$\begin{cases} x = vu \\ y = vu^{-1}\end{cases}\qquad \frac{\mathrm{d}x\mathrm{d}y}{2xy} = \frac{\mathrm{d}v\mathrm{d}u}{uv}$$

to get
$$\Gamma(\omega)\int_0^{+\infty}\frac{1}{(u+u^{-1})^\omega}\,\frac{\mathrm{d}u}{u} = \frac{1}{2}\Gamma\Big(\frac{\omega}{2}\Big)^2$$

As anounced,
$$\mathrm{L}(\omega) = \frac{\zeta^\star_{\mathrm{K,A}}(\omega)}{2}$$

Bibliography

[1] J. Beineke and D. Bump, *Hidden symmetries for a renormalized integral of Eisenstein series*, Forum Math. **14** (2002), 457–475.

[2] D. Bump, *Automorphic Forms on* $\mathrm{GL}_3(\mathbb{R})$, Ph.D. thesis, The University of Chicago, 1982.

[3] ———, *Automorphic Forms and Representations*, vol. 55, Cambridge Studies in Advanced Mathematics, 1997.

[4] D. Bump and D. Goldfeld, *A Kronecker Limit Formula for Cubic Fields*, Modular forms, Symp. Durham/Engl. (1984), 43–49.

[5] A. Frohlich, *Algebraic Number Theory*, Cambridge University Press, 1991.

[6] S. Lang, *Algebraic Number Theory*, Springer-Verlag, Graduate Texts in Mathematics, 1994.

[7] C. Siegel, *Lecture Notes on Advanced Analytic Number Theory*, Tata Institute, Bombay (1961).

[8] D. Zagier, *A Kronecker Limit Formula for Real Quadratic Fields*, Math. Ann. **213** (1975), 153–184.

[9] ———, *The Rankin-Selberg method for Automorphic Functions which are not of Rapid Decay*, J. Fac. Sci. Univ. Tokyo (1981).