

Ph.D. Qualifying Exam – Spring 2004

First problem

Suppose $f \in L^1(\mathbb{R})$. Let

$$\forall x \in \mathbb{R} \quad g(x) = \int_{\mathbb{R}} e^{-ixy^2} f(y) \, dy$$

1. Prove that the integral exists for every x .
2. Prove that g is a continuous function.
3. Prove that there is a dense subset S of $L^1(\mathbb{R})$ such that if $f \in S$, then

$$\lim_{|x| \rightarrow +\infty} g(x) = 0 \tag{*}$$

4. Prove that $(*)$ holds for any $f \in L^1(\mathbb{R})$.

Second problem

Suppose (X, \mathcal{A}, μ) is a measure space and that $(f_n)_{n \in \mathbb{N}}$ is a sequence in $L^1(\mu)$ converging weakly. Prove that

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall A \in \mathcal{A} \quad (\mu(A) < \delta \implies \forall n \in \mathbb{N} \quad \int_A |f_n| d\mu \leq \epsilon)$$

Third problem

Let $f : (a, b) \rightarrow \mathbb{R}$ be an arbitrary function. Let

$$S = \{x \in (a, b) \mid f'(x) \text{ exists and equals } 0\}$$

Prove that the image $f(S)$ of S under f has measure 0.

Fourth problem

Let X be a metric space.

1. Prove that if X is countably compact, then X is compact.
2. Prove that if every continuous function on X is bounded, then X is compact.

Fifth problem

Prove that there exists an orthonormal basis \mathcal{B} of $L^2([0, 1])$ such that

$$\forall f \in \mathcal{B} \quad \int_0^1 |f(x)| \frac{dx}{x} < \infty \quad \text{and} \quad \int_0^1 f(x) \frac{dx}{x} = 0$$

Sixth problem

Let (X, \mathcal{B}, μ) be a finite measure space. Suppose that $(f_n)_{n \in \mathbb{N}}$ is a sequence of functions in $L^1(\mu)$, converging almost everywhere to an $L^1(\mu)$ function f . Suppose also that

$$\|f_n\|_1 \xrightarrow{n \rightarrow \infty} \|f\|_1$$

1. Prove that for every measurable set A , $\int_A |f_n| d\mu \xrightarrow{n \rightarrow \infty} \int_A |f|$.
2. Prove that $\|f_n - f\|_1 \xrightarrow{n \rightarrow \infty} 0$.

Seventh problem

Suppose that $(f_n)_{n \in \mathbb{N}}$ is a decreasing sequence of continuous functions on $[0, 1]$, converging pointwise to a continuous function f . Prove that the convergence is uniform.

Eighth problem

Let $(c_n)_{n \in \mathbb{N}}$ be a sequence of complex numbers. Assume there are positive integers $N_k \xrightarrow[k \rightarrow \infty]{} \infty$ such that the trigonometric polynomials

$$P_k(t) = \sum_{j=-N_k}^{N_k} \left(1 - \frac{|j|}{N_k + 1}\right) c_j e^{ijt}$$

are nonnegative for all t . Prove that there exists a positive measure μ on \mathbb{T} such that

$$\forall n \in \mathbb{Z} \quad c_n = \int_{\mathbb{T}} e^{-int} d\mu(t)$$

Ninth problem

Let X be a vector space.

1. Prove that X has a Hamel basis (or algebraic basis).
2. If X is an infinite dimensional Banach space, prove that any Hamel basis must be uncountable.

Tenth problem

Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded function such that

$$\exists K > 0 \quad \forall x, h \in \mathbb{R} \quad |f(x+h) + f(x-h) - 2f(x)| \leq K|h|$$

Prove that

$$\forall x, y \in \mathbb{R} \quad |f(x) - f(y)| \leq c \left(1 + \ln^+ \frac{1}{|x-y|} \right) |x-y|$$

where c is a constant depending only on K and $\|f\|_\infty$.