

## Ph.D. Qualifying Exam – Spring 2005

## First problem

1. Let  $(B, \| \cdot \|)$  be a normed space and  $A : B \rightarrow B$  an invertible linear map such that  $\|A^n\| \leq c$  for all some constant  $c > 0$  and all  $n \in \mathbb{Z}$ . Prove that there exists an equivalent norm on  $B$  with respect to which  $A$  is an isometry.
2. Let  $\mu$  be a finite measure on  $[-1, 1]$  and assume that  $\int x^{kn} d\mu(x) = 0$  for some integer  $k$  and all non-negative integers  $n$ . Prove that if  $k$  is odd,  $\mu = 0$ . What can you say if  $k$  is even?
3. Prove that  $\mathcal{C}(0, 1)$  is not reflexive.
4. Prove that  $\mathcal{C}(0, 1)$  is not isomorphic – and in particular not isometric – to a uniformly convex Banach space.

**Solution**

**1** Let  $(B, \| \cdot \|)$  be a normed space and  $A$  be an invertible endomorphism of  $B$  such that

$$\exists c > 0 \quad \forall n \in \mathbb{Z} \quad \|A^n\| \leq c$$

Because of that, we know that

$$\forall x \in B \quad \forall n \in \mathbb{Z} \quad \|A^n x\| \leq \|A^n\| \|x\| \leq c \|x\| < \infty \quad (1)$$

which allows us to define

$$\forall x \in B \quad N(x) = \sup_{n \in \mathbb{Z}} \|A^n x\|$$

We first check that  $N$  is a norm on  $B$ .

- Let  $\lambda$  be a scalar and  $x \in B$ . Then

$$N(\lambda x) = \sup_{n \in \mathbb{Z}} \|A^n(\lambda x)\| = \sup_{n \in \mathbb{Z}} |\lambda| \|A^n x\| = |\lambda| \sup_{n \in \mathbb{Z}} \|A^n x\| = |\lambda| N(x)$$

- Let  $x$  and  $y$  be in  $B$ . Then

$$\forall n \in \mathbb{Z} \quad \|A^n(x + y)\| = \|A^n x + A^n y\| \leq \|A^n x\| + \|A^n y\| \leq N(x) + N(y)$$

Hence

$$N(x + y) \leq N(x) + N(y)$$

- Finally, notice that

$$\forall x \in B \quad \|x\| \leq N(x) \quad (2)$$

Thus, if  $N(x) = 0$ , then  $\|x\| = 0$  as well which in turn implies that  $x = 0$ .

N is a norm on B

Next, let's check that  $A$  is an isometry for  $N$ . Notice that  $N$  was defined specifically for that matter and this part is thus not hard to see:

$$\forall x \in B \quad N(Ax) = \sup_{n \in \mathbb{Z}} \|A^{n+1}x\| = N(x)$$

A is an isometry for N

Finally,  $N$  and  $\|\cdot\|$  should be equivalent. A first step towards showing this was already done when mentioning (2). And by (1), we have

$$\forall x \in B \quad N(x) \leq c \|x\|$$

Hence

$\|\cdot\|$  and  $N$  are equivalent.

2 Let  $\mu$  be a finite measure on  $[-1, 1]$ , such that

$$\exists k \in \mathbb{Z} \quad \forall n \in \mathbb{N} \quad \int_{[-1,1]} x^{kn} d\mu(x) = 0 \quad (3)$$

Suppose first that  $k$  is odd and let  $\mathcal{X}$  be the algebra generated by the function  $x \mapsto x^k$ . In other terms,  $f$  is in  $\mathcal{X}$  if and only if  $f$  is of the form

$$x \mapsto \sum_{n=0}^N a_n x^{kn}$$

Then  $\mathcal{X}$  contains the constant function 1, and separates points, since if  $x_1 \neq x_2$  are in  $[-1, 1]$ , the function  $x \mapsto x^k$  takes different values at  $x_1$  and  $x_2$ . By Weierstrass' theorem,  $\mathcal{X}$  is dense in  $\mathcal{C}([-1, 1])$ . Thus (3) generalizes to every continuous function on  $[-1, 1]$ :

$$\forall f \in \mathcal{C}([-1, 1]) \quad \int_{[-1,1]} f d\mu = 0$$

Since

$$\|\mu\| = \sup_{\substack{f \in \mathcal{C}([-1,1]) \\ \|f\| \leq 1}} \int_{[-1,1]} f d\mu = 0$$

it follows that

$$\mu = 0$$

Now suppose that  $k$  is even. This time the algebra  $\mathcal{X}$  does not separate points in  $[-1, 1]$ , since it is formed of even functions. However,  $\mathcal{X}$  separates points in  $[0, 1]$  and we can thus extend **(3)** to all even functions on  $[-1, 1]$ :

$$\forall f \in \mathcal{C}([-1, 1]) \quad f \text{ even} \quad \int_{[-1, 1]} f \, d\mu = 0$$

And it looks like we cannot say much more than that.

**[3]** By Riesz's representation theorem, we know that the dual space of  $\mathcal{C}(0, 1)$  can be identified with the space  $\mathcal{M}(0, 1)$  of signed measures on  $[0, 1]$ . More precisely, for every linear functional  $\varphi$  on  $\mathcal{C}(0, 1)$ , there exists a unique signed measure  $\mu$  on  $[0, 1]$  such that

$$\forall f \in \mathcal{C}(0, 1) \quad \langle \varphi, f \rangle = \int_{(0, 1)} f \, d\mu$$

We want to show that the dual of  $\mathcal{M}(0, 1)$  contains strictly  $\mathcal{C}(0, 1)$  so we have to come up with a linear functional on  $\mathcal{M}(0, 1)$  that cannot be represented by a continuous function. There are probably various ways of cooking up such an example, and what follows is only one of them.

Remember that  $L^1(0, 1)$  sits inside  $\mathcal{M}(0, 1)$  as a strict closed subspace.

Here is how it works: if  $f$  is an  $L^1$  function, we can define  $\mu_f = f \, dx$ , *i.e.*  $\mu_f$  is the measure with density  $f$  relatively to the Lebesgue measure on  $[0, 1]$ . It should have been seen in class as well that  $\|\mu_f\|_{\mathcal{M}} = \|f\|_{L^1}$ , which shows at once that  $L^1$  is closed inside  $\mathcal{M}(0, 1)$ . And finally, one exhibits elements of  $\mathcal{M}(0, 1)$ , such as the Dirac mass at 0, for example, that are not elements of  $L^1(0, 1)$ .

So  $L^1(0, 1)$  is certainly not dense in  $\mathcal{M}(0, 1)$ . By the Hahn-Banach theorem, there is a linear functional  $\varphi$  on  $\mathcal{M}(0, 1)$ , not identically 0, but whose restriction to  $L^1(0, 1)$  is 0. Suppose that  $\mathcal{C}(0, 1)$  is reflexive. Then  $\varphi$  is represented by a function  $g \in \mathcal{C}(0, 1)$ , which means:

$$\forall \mu \in \mathcal{M}(0, 1) \quad (\varphi, \mu) = (\mu, g) = \int_{[0, 1]} g \, d\mu \quad \text{and} \quad \|\varphi\| = \|g\|_{\infty}$$

But from the Riesz representation theorem for  $L^p$  spaces, we know in particular that

$$\|g\|_{\infty} = \sup_{\substack{f \in L^1 \\ \|f\|=1}} \int_{[0, 1]} g(x)f(x) \, dx = \sup_{\substack{f \in L^1 \\ \|f\|=1}} (\varphi, \mu_f) = 0$$

Thus  $\|\varphi\| = 0$ , which contradicts the fact that  $\varphi$  is not trivial. Hence  $\mathcal{C}(0, 1)$  is not reflexive.

**[4]** The Milmann-Pettis theorem states that any uniformly convex Banach space is reflexive. Thus  $\mathcal{C}(0, 1)$  is not uniformly convex, for any choice of equivalent norm.

This theorem is sometimes proved in 205B, in lecture or in homeworks. Sometimes it is not, it all depends on who teaches the class. So whether or not you are authorized to use it during quals is unclear. No matter what, it is still useful to know that this result exists and this is why I quote it here.

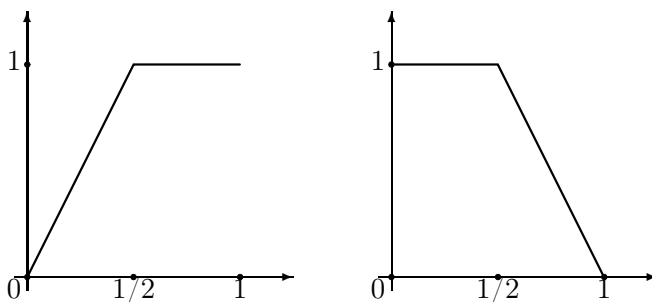
Its proof is hard, but not insanely hard and you should probably be able to do it by yourselves as a challenging exercise. It is done by using directly the definition of reflexivity (and not by use of, for example, Kakutani's theorem), requires good knowledge weak topologies, and the use of Banach-Alaoglu-Bourbaki's theorem.

Now back to the qual: Milmann-Pettis is, of course, a sledgehammer for this question, which can be treated directly.

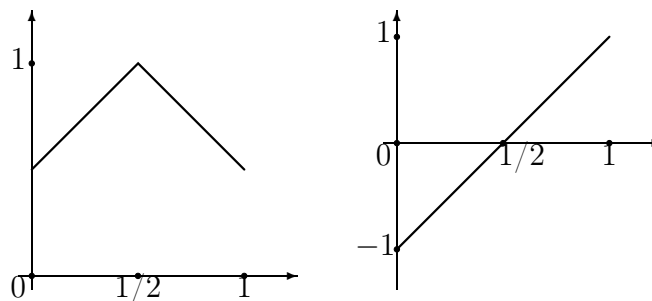
Here is an alternate, more direct, solution to this question. Remember that a Banach space  $B$  is said to be uniformly convex if and only if

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall f, g \in B \quad \left( \|f\| = \|g\| = 1 \quad \text{and} \quad \left\| \frac{f+g}{2} \right\| > 1 - \delta \right) \implies \|f - g\| < \epsilon$$

Just take  $f$  and  $g$  to be the following functions:



Then  $\frac{f+g}{2}$  and  $f - g$  respectively look like



and we have  $\|f\| = 1 \quad \|g\| = 1 \quad \left\| \frac{f+g}{2} \right\| = 1 \quad \text{and} \quad \|f - g\| = 1$

which contradicts the definition of uniform convexity. More precisely, if we take  $\epsilon = \frac{1}{2}$ , then for every positive  $\delta$ , we have at the same time  $\left\| \frac{f+g}{2} \right\| > 1 - \delta$  and  $\|f - g\| > \epsilon$ .

## Second problem

Let  $\mathcal{H}$  be a Hilbert space and let  $T$  be a bounded operator on  $\mathcal{H}$ . Show that  $T$  is compact if and only if it is the limit, in the operator norm, of a sequence of finite rank operators.

**Solution**

First, suppose that  $T$  is compact. Let  $\epsilon$  be a positive real number. The image of the unit ball of  $\mathcal{H}$  can be covered with finitely many (say  $n$ ) balls of radius  $\epsilon$ . Thus there exist points  $x_1, \dots, x_n$  in the unit ball of  $\mathcal{H}$ , such that

$$T(\mathcal{B}_{\mathcal{H}}) \subset \bigcup_{k=1}^n \mathcal{B}(Tx_k, \epsilon)$$

Let  $P$  be the orthogonal projection on the finite-dimensional subspace spanned by  $Tx_1, \dots, Tx_n$ . Then  $PT$  is a linear operator on  $\mathcal{H}$  of finite rank (its range is contained in the range of  $P$ ) and we proceed to show that  $\|PT - T\| \leq 2\epsilon$ , which will show that  $T$  is a limit of finite rank operators.

Let  $x \in \mathcal{H}$  be in the unit ball. Then  $Tx$  has to be in one of the balls  $\mathcal{B}(Tx_k, \epsilon)$  for some  $k \in \{1, \dots, n\}$ :

$$\|Tx - Tx_k\| \leq \epsilon$$

$$\text{Thus } \|PTx - Tx\| \leq \|PTx - \underbrace{Tx_k}_{=PTx_k}\| + \|Tx_k - Tx\| \leq \|P(Tx - Tx_k)\| + \epsilon \leq 2\epsilon$$

and

$$\|PT - T\| \leq 2\epsilon$$

Compact operators on a Hilbert space are limits of finite rank operators.

Conversely, suppose that there is a sequence  $(T_n)_{n \in \mathbb{N}}$  of finite rank operators converging to  $T$ . Let  $\epsilon$  be a positive real number. We know then that there exists an integer  $N$  such that  $\|T_N - T\| \leq \epsilon$ .

The operator  $T_N$  is of finite rank, hence compact. So  $T_N(\mathcal{B}_{\mathcal{H}})$  can be covered by finitely many balls  $B_1, \dots, B_m$  of radius  $\epsilon$ . Thus  $(T_N^{-1}(B_i) \cap \mathcal{B}_{\mathcal{H}})_{1 \leq i \leq m}$  is a cover of the unit ball of  $\mathcal{H}$  and  $(T(T_N^{-1}(B_i) \cap \mathcal{B}_{\mathcal{H}}))_{1 \leq i \leq m}$  is a finite cover of  $T(\mathcal{B}_{\mathcal{H}})$ . Let's evaluate the diameters of these sets.

Let  $i$  be an integer between 1 and  $m$ , and  $x$  and  $y$  be in  $T_N^{-1}(B_i) \cap \mathcal{B}_{\mathcal{H}}$ . Then

$$\|Tx - Ty\| \leq \|Tx - T_Nx\| + \|T_Nx - T_Ny\| + \|T_Ny - Ty\|$$

The first and last term are less than  $\epsilon$  because  $\|T - T_N\| \leq \epsilon$ . The middle term is less than  $2\epsilon$  since  $T_Nx$  and  $T_Ny$  are in the ball  $B_i$  of radius  $\epsilon$ . Hence

$$\|Tx - Ty\| \leq 4\epsilon$$

or in other words  $\forall i \in \{1, \dots, m\}$   $\text{diam}(T(T_N^{-1}(B_i) \cap \mathcal{B}_{\mathcal{H}})) \leq 4\epsilon$

Now, just choose a point  $x_i$  in  $T(T_N^{-1}(B_i) \cap \mathcal{B}_{\mathcal{H}})$  for each  $i$ . What we just showed ensures that

$$T(T_N^{-1}(B_i) \cap \mathcal{B}_{\mathcal{H}}) \subset \mathcal{B}(x_i, 4\epsilon)$$

Hence,  $T(\mathcal{B}_{\mathcal{H}})$  can be covered by finitely many balls of radius  $4\epsilon$ : this set is totally bounded, hence relatively compact since  $\mathcal{H}$  is complete.

A limit of compact operators is compact.

## Third problem

Suppose  $B$  is a Banach space and  $K \subset B$  is a subset. Recall that its convex hull  $\text{ch}(K)$  is the smallest convex subset containing  $K$ .

1. Prove that if  $K$  is compact, the closure  $\overline{\text{ch}}(K)$  is compact as well.
2. Show that the set of indicator functions  $\{\mathbb{1}_{[\tau, \tau + \frac{1}{5}]} \mid \tau \in \mathbb{T}\}$  is compact in  $L^1(\mathbb{T})$  and its convex hull is not.

**Solution**

Let  $B$  be a Banach space and  $K$  a subset of  $B$ . The convex hull of  $K$  is the smallest convex subset of  $B$  containing  $K$ . It is also the intersection of all convex sets containing  $K$ , or the set of all convex combinations of elements of  $K$ :

$$\text{ch}(K) = \left\{ \sum_{i=1}^n a_i x_i \mid n \in \mathbb{N}^* \quad x_1, \dots, x_n \in K \quad a_1, \dots, a_n \in [0, 1] \quad \sum_{i=1}^n a_i \leq 1 \right\}$$

**1** Suppose that  $K$  is compact. Because  $B$  is complete, to show that  $\overline{\text{ch}}(K)$  is compact, it is enough to show that  $\text{ch}(K)$  is totally bounded. Let  $\epsilon$  be a positive real number. Since  $K$  is compact, it can be covered by finitely many balls of radius  $\epsilon$ :

$$\exists x_1, \dots, x_n \in K \quad K \subset \bigcup_{i=1}^n \mathcal{B}(x_i, \epsilon) = \{x_1, \dots, x_n\} + \mathcal{B}(0, \epsilon)$$

The closed convex hull of the set  $X = \{x_1, \dots, x_n\}$  is

$$\text{ch}(X) = \overline{\text{ch}}(X) = \left\{ \sum_{i=1}^n a_i x_i \mid a_1, \dots, a_n \in [0, 1] \quad \sum_{i=1}^n a_i = 1 \right\}$$

which is included in the image of the compact set  $[0, 1]^n$  under the continuous map

$$\begin{aligned} [0, 1]^n &\longrightarrow B \\ (a_1, \dots, a_n) &\longmapsto \sum_{i=1}^n a_i x_i \end{aligned}$$

Hence  $\overline{\text{ch}}(X)$  is compact and can be covered by finitely many balls of radius  $\epsilon$ :

$$\exists y_1, \dots, y_m \in \text{ch}(X) \subset \text{ch}(K) \quad \text{ch}(X) \subset \bigcup_{i=1}^m \mathcal{B}(y_i, \epsilon)$$

But  $\text{ch}(X) + \mathcal{B}(0, \epsilon)$  is convex (easy to check) and contains  $X + \mathcal{B}(0, \epsilon)$ . Hence it contains  $\text{ch}(X + \mathcal{B}(0, \epsilon))$ . It follows that:

$$\text{ch}(K) \subset \text{ch}(X + \mathcal{B}(0, \epsilon)) \subset \text{ch}(X) + \mathcal{B}(0, \epsilon) \subset \bigcup_{i=1}^m \mathcal{B}(y_i, \epsilon) + \mathcal{B}(0, \epsilon) \subset \bigcup_{i=1}^m \mathcal{B}(y_i, 2\epsilon)$$

The convex hull of  $K$  can be covered by finitely many balls of radius  $2\epsilon$ , for any given  $\epsilon$ . Hence because  $B$  is complete,

If  $K$  is compact,  $\overline{\text{ch}(K)}$  is compact.

[2] We know from class that translations are continuous on  $L^1$ , which means precisely that

$$\forall f \in L^1(\mathbb{T}) \quad \lim_{\tau \rightarrow 0} \int_{\mathbb{T}} |f(t + \tau) - f(t)| dt = 0$$

In particular, taking  $f = \mathbf{1}_{[0, \frac{1}{5}]}$ , the function  $\varphi$  defined by

$$\forall \tau \in \mathbb{T} \quad \varphi(\tau) = \mathbf{1}_{[\tau, \tau + \frac{1}{5}]}$$

is continuous. The set of indicator functions  $\{\mathbf{1}_{[\tau, \tau + \frac{1}{5}]} \mid \tau \in \mathbb{T}\}$  is the image of  $\mathbb{T}$  (compact) by  $\varphi$  (continuous). Therefore,

$\{\mathbf{1}_{[\tau, \tau + \frac{1}{5}]} \mid \tau \in \mathbb{T}\}$  is compact.



## Fourth problem

Let  $(A_j)_{1 \leq j \leq N}$  be a collection of Lebesgue-measurable subsets of  $[0, 1]$  such that  $\mu(A_j) \geq \frac{1}{2}$  for each  $j$ . Let  $a \in (0, \frac{1}{2})$  and denote

$$E_a = \{x \in [0, 1] \mid x \in A_j \text{ for more than } aN \text{ values of } j\}$$

Prove that 
$$\mu(E_a) \geq \frac{1 - 2a}{2(1 - a)}$$

and that this estimate is the best possible (if it is to apply for all  $N$ ).

**Solution**

Following the hint, consider the function  $F = \sum_{k=1}^N \mathbb{1}_{A_k}$ . Then for every integer  $n$ ,

$$\{F = n\} = \{x \in [0, 1] \mid x \text{ is in exactly } n \text{ of the } A_j\text{'s}\}$$

Therefore 
$$\{F \leq n\} = \{x \in [0, 1] \mid x \text{ is in at most } n \text{ of the } A_j\text{'s}\}$$

and 
$$\{F \leq aN\} = \{x \in [0, 1] \mid x \text{ is in at most } aN \text{ of the } A_j\text{'s}\}$$

since  $F$  only takes integer values. It follows that

$$\{F > aN\} = \{x \in [0, 1] \mid x \text{ is in more than } aN \text{ of the } A_j\text{'s}\} = E_a$$

Thus  $E_a$  is measurable and we found a relationship between this set and  $F$ . Now, since all the  $A_j$ 's have measure greater than  $\frac{1}{2}$ , we have on the one hand

$$\frac{N}{2} = \sum_{k=1}^N \frac{1}{2} \leq \sum_{k=1}^N \mu(A_k) = \sum_{k=1}^N \int_{[0,1]} \mathbb{1}_{A_k} d\mu = \int_{[0,1]} F d\mu \quad (1)$$

On the other hand,

$$\int_{[0,1]} F d\mu = \int_{\{F > aN\}} F d\mu + \int_{\{F \leq aN\}} F d\mu \leq \int_{E_a} F d\mu + aN \int_{\{F \leq aN\}} d\mu \quad (2)$$

Let's look at each term separately:

$$\int_{E_a} F d\mu = \sum_{k=1}^N \int_{E_a} \mathbb{1}_{A_k} d\mu = \sum_{k=1}^N \int_{E_a \cap A_k} d\mu \leq N\mu(E_a)$$

while 
$$aN \int_{\{F \leq aN\}} d\mu = aN\mu(F \leq aN) = aN(1 - \mu(E_a))$$

Plugging this back in **(2)** yields

$$\int_{[0,1]} F d\mu \leq N\mu(E_a) + aN - aN\mu(E_a) = aN + N(1-a)\mu(E_a)$$

and then reinjecting in **(1)** gives us

$$\frac{N}{2} \leq aN + N(1-a)\mu(E_a)$$

Conclusion:

$$\boxed{\mu(E_a) \geq \frac{1-2a}{2(1-a)}}$$

## Fifth problem

The Hausdorff-Young inequality on the line states that if  $1 \leq p \leq 2$  and  $\frac{1}{p} + \frac{1}{q} = 1$ , then

$$f \in L^p(\mathbb{R}) \implies \widehat{f} \in L^q(\mathbb{R}) \quad \text{and} \quad \|\widehat{f}\|_q \leq \|f\|_p \quad (1)$$

Prove that the converse is true: (1) implies that  $\frac{1}{p} + \frac{1}{q} = 1$  and  $p \in [1, 2]$ .

**Solution**

Let  $p \geq 1$  be a real number and suppose that there exists a  $q \geq 1$  such that

$$\forall f \in L^p(\mathbb{R}) \quad \widehat{f} \in L^q(\mathbb{R}) \quad \text{and} \quad \|\widehat{f}\|_q \leq \|f\|_p$$

First let  $f \neq 0$  be in  $\mathcal{S}(\mathbb{R})$  (the Schwartz space) and  $\lambda$  be a positive real number. Then  $f$  is in  $L^p(\mathbb{R})$ ,  $\widehat{f}$  is in  $L^q(\mathbb{R})$  with  $\|\widehat{f}\|_q \neq 0$  by injectivity of the Fourier transform. Define

$$\forall x \in \mathbb{R} \quad f_\lambda(x) = f(\lambda x)$$

$$\text{Then} \quad \|f_\lambda\|_p = \left( \int_{\mathbb{R}} |f(\lambda x)|^p dx \right)^{1/p} = \left( \int_{\mathbb{R}} |f(x)|^p \frac{dx}{\lambda} \right)^{1/p} = \lambda^{-1/p} \|f\|_p$$

$$\forall \xi \in \mathbb{R} \quad \widehat{f}_\lambda(\xi) = \int_{\mathbb{R}} f(\lambda x) e^{-ix\xi} dx = \int_{\mathbb{R}} f(x) e^{-ix\xi/\lambda} \frac{dx}{\lambda} = \frac{1}{\lambda} \widehat{f}\left(\frac{\xi}{\lambda}\right)$$

$$\text{and} \quad \|\widehat{f}_\lambda\|_q = \frac{1}{\lambda} \times \lambda^{1/q} \|\widehat{f}\|_q$$

We chose here to take  $f$  in the Schwartz space, for a technical reason: we then know that  $\widehat{f}$  is also in  $\mathcal{S}(\mathbb{R})$ , which allows us to use the integral formula in order to relate  $\widehat{f}_\lambda$  and  $\widehat{f}$  like we did earlier.

If  $f$  is only taken in  $L^p(\mathbb{R})$ , then all we know is that  $\widehat{f}$  is defined in the sense of distributions and we have then to work in that framework. We would still have been able to find that  $\widehat{f}_\lambda = \frac{1}{\lambda} \widehat{f}_{1/\lambda}$ , but it would have been a bit more tedious.

$$\text{We know that} \quad \|\widehat{f}_\lambda\|_q \leq \|f_\lambda\|_p$$

$$\text{that is} \quad \lambda^{-1+1/q} \|\widehat{f}\|_q \leq \lambda^{-1/p} \|f\|_p$$

$$\text{or} \quad \forall \lambda > 0 \quad \|\widehat{f}\|_q \leq \lambda^{1-(1/p+1/q)} \|f\|_p$$

If  $1 - (\frac{1}{p} + \frac{1}{q}) > 0$ , letting  $\lambda$  tend to 0 yields  $\|\widehat{f}\|_q = 0$ , which is a contradiction. If  $1 - (\frac{1}{p} + \frac{1}{q}) < 0$ , let  $\lambda$  tend to  $+\infty$  in order to get the same contradiction. Hence

$$\frac{1}{p} + \frac{1}{q} = 1$$

Now to proving that  $p$  has to be between 1 and 2.

This part is extremely long and tedious, but I could not come up with anything better to solve it. There may be a trick to solve this question very easily, but I am starting to doubt it given the time I spent on it. It is my opinion that, whoever submitted that problem for the qualifying exam, seriously underestimated its difficulty. Anyway, you have been warned.

If  $z$  is a complex number in the closed upper half-plane  $\mathcal{H} = \{\text{Im } z \geq 0\}$ , define

$$\forall x \in \mathbb{R} \quad \varphi_z(x) = e^{izx}$$

Notice that if  $z \in \mathbb{R}_+^*$ , then  $\varphi_z$  is a continuous function of absolute value 1. Thus it does not have a Fourier transform (at least in the classical sense). If  $z \in i\mathbb{R}_+^*$ , then  $\varphi_z$  is a Gaussian function. And we know how to compute the Fourier transform of such functions:

$$\forall \alpha > 0 \quad \forall \xi \in \mathbb{R} \quad \widehat{\varphi_{i\alpha}}(\xi) = \int_{\mathbb{R}} e^{-\alpha x^2 - i\xi x} dx = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\xi^2}{4\alpha}} \quad (2)$$

Of course, no one is expected to remember this precise formula. But you should be able to recover it quickly. The easiest way, I believe, is by solving the differential equation satisfied by the Fourier transform. This equation is obtained by differentiating with respect to  $\xi$  under the integral and integrating by parts:

$$\begin{aligned} \widehat{\varphi_{i\alpha}}'(\xi) &= -i \int_{\mathbb{R}} x e^{-\alpha x^2} e^{-i\xi x} dx = \frac{i}{2\alpha} \int_{\mathbb{R}} (-2\alpha x) e^{-\alpha x^2} e^{-i\xi x} dx \\ &= \frac{i}{2\alpha} \left( [e^{-\alpha x^2} e^{-i\xi x}]_{-\infty}^{+\infty} + i\xi \int_{\mathbb{R}} e^{-\alpha x^2 - i\xi x} dx \right) = -\frac{\xi}{2\alpha} \widehat{\varphi_{i\alpha}}(\xi) \end{aligned}$$

Therefore  $\widehat{\varphi_{i\alpha}}(\xi) = K e^{-\frac{\xi^2}{4\alpha}}$  with  $K = \int_{\mathbb{R}} e^{-\alpha x^2} dx$

$K^2$  is then computed by using polar coordinates:

$$K^2 = \int_{\mathbb{R}^2} e^{-\alpha(x^2+y^2)} dx dy = \int_0^{2\pi} \int_0^{+\infty} r e^{-\alpha r^2} dr d\theta = -\frac{2\pi}{2\alpha} [e^{-\alpha r^2}]_0^{+\infty} = \frac{\pi}{\alpha}$$

Hence  $\widehat{\varphi_{i\alpha}}(\xi) = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\xi^2}{4\alpha}}$

In order to rewrite (2) in terms of  $i\alpha$  only, we choose the usual branch of the square root in  $\mathbb{C} \setminus \mathbb{R}_-$ , that is real on the positive real axis. We then get:

$$\widehat{\varphi_{i\alpha}}(\xi) = \sqrt{\frac{\pi}{-i(i\alpha)}} e^{-\frac{i\xi^2}{4i\alpha}} = e^{i\pi/4} \sqrt{\frac{\pi}{i\alpha}} \varphi_{\frac{-1}{4i\alpha}}(\xi)$$

which can be rewritten  $\forall z \in i\mathbb{R}_+^*$   $\widehat{\varphi}_z = e^{i\pi/4} \sqrt{\frac{\pi}{z}} \varphi_{\frac{-1}{4z}}$  (3)

Now, let  $\psi$  be a fixed  $\mathcal{C}^\infty$  function with compact support included in, say,  $[-1, 1]$ . Let  $\xi$  be a fixed real number. Define

$$\forall z \in \mathcal{H} \quad h(z) = \widehat{\varphi}_z \widehat{\psi}(\xi) = \int_{-1}^1 \varphi_z(x) \psi(x) e^{-i\xi x} dx$$

This function of a complex variable is perfectly well defined and continuous, since  $\varphi_z \psi$  is  $\mathcal{C}^\infty$  with support in  $[-1, 1]$ . Moreover,  $h$  is holomorphic in the interior of  $\mathcal{H}$ :

We used here a theorem from complex analysis:

**Theorem 1:** Let  $K$  be a compact set in  $\mathbb{R}^n$  and  $\Omega$  an open set in  $\mathbb{C}$ . Let  $(x, z) \mapsto f(x, z)$  be a function defined in  $K \times \Omega$ , measurable with respect to the first variable and holomorphic with respect to the second. Then

$$z \mapsto \int_K f(x, z) dx$$

defines a holomorphic function in  $\Omega$ .

There is another version of this theorem, which we should need later, for when the integration occurs on a non-compact set:

**Theorem 2:** Let  $\Omega$  be an open set in  $\mathbb{C}$ . Let  $(x, z) \mapsto f(x, z)$  be a function defined on  $\mathbb{R}^n \times \Omega$ , measurable in the first variable and holomorphic in the second. Suppose that there exists a function  $g$ , integrable over  $\mathbb{R}^n$ , such that

$$\forall x \in \mathbb{R}^n \quad \forall z \in \Omega \quad |f(x, z)| \leq g(x)$$

Then

$$z \mapsto \int_{\mathbb{R}^n} f(x, z) dx$$

defines a holomorphic function in  $\Omega$ .

Our goal is now to compute  $h$  for every  $z \in \mathcal{H} \setminus \mathbb{R}$ , using formula (3) and analytic continuation. We let  $\epsilon$  be a positive real number and

$$\Omega = \{z \in \mathcal{H} \setminus \mathbb{R} \mid \epsilon < |z| < M\}$$

Suppose that  $z$  is purely imaginary, with positive imaginary part ( $z \in i\mathbb{R}_+^*$ ). Then  $\varphi_z$  and  $\psi$  are Schwarz functions (all we really need is  $L^2$ , actually) and we know that

$$\begin{aligned} h(z) &= \widehat{\varphi}_z \star \widehat{\psi}(\xi) = \int_{\mathbb{R}} \widehat{\varphi}_z(u) \widehat{\psi}(\xi - u) du \\ &= e^{i\pi/4} \sqrt{\frac{\pi}{z}} \int_{\mathbb{R}} e^{-\frac{iu^2}{4z}} \widehat{\psi}(\xi - u) du \end{aligned}$$

But the expression on the righthandside makes sense for every  $z \in \Omega$  and defines a holomorphic function. Indeed, the part with the square root is no problem; as for the integral, it is dealt with using **Theorem 2**, since:

$$\forall z = |z| e^{i\theta} \in \Omega \quad \forall u \in \mathbb{R} \quad \left| e^{-\frac{iu^2}{4z}} \right| = \left| e^{-\frac{iu^2}{4|z|} e^{-i\theta}} \right| = e^{-\frac{u^2 \sin \theta}{4|z|}} \leq e^{-\frac{u^2 \sin \theta}{4\epsilon}} \leq 1$$

Since the two functions are holomorphic in  $\Omega$  and agree on the imaginary axis, they are equal:

$$\forall z \in \mathcal{H} \setminus \mathbb{R} \quad |z| > \epsilon \quad h(z) = \widehat{\phi_z \psi}(\xi) = e^{i\pi/4} \sqrt{\frac{\pi}{z}} \int_{\mathbb{R}} e^{-\frac{iu^2}{4z}} \widehat{\psi}(\xi - u) du$$

This is true for every positive  $\epsilon$ , thus

$$\forall z \in \mathcal{H} \setminus \mathbb{R} \quad \widehat{\phi_z \psi}(\xi) = e^{i\pi/4} \sqrt{\frac{\pi}{z}} \int_{\mathbb{R}} e^{-\frac{iu^2}{4z}} \widehat{\psi}(\xi - u) du$$

We are now able to estimate this Fourier transform:

$$\forall z \in \mathcal{H} \setminus \mathbb{R} \quad |\widehat{\phi_z \psi}(\xi)| \leq \sqrt{\frac{\pi}{|z|}} \int_{\mathbb{R}} |\widehat{\psi}(u)| du$$

But  $h$  is continuous on the closed upper half-plane, as stated earlier. Thus this estimate extends in particular to  $\mathbb{R}_+$ :

$$\forall \lambda > 0 \quad \forall \xi \in \mathbb{R} \quad |\widehat{\varphi_\lambda \psi}(\xi)| = \left| \int_{-1}^1 e^{i\lambda x^2} \psi(x) e^{-i\xi x} dx \right| \leq \frac{K}{\sqrt{\lambda}}$$

Conclusion:

$$\boxed{\lim_{\lambda \rightarrow +\infty} \|\widehat{\varphi_\lambda \psi}\|_\infty = 0}$$

Now we can go back to solving the problem. Suppose that  $p > 2$ . The Fourier transform  $\widehat{\varphi_\lambda \psi}$  is in the Schwartz space, since  $\varphi_\lambda \psi$  is in Schwartz. Thus it is in  $L^p$  and bounded, and we have:

$$\|\widehat{\varphi_\lambda \psi}\|_p^p = \int_{\mathbb{R}} |\widehat{\varphi_\lambda \psi}|^p = \int_{\mathbb{R}} |\widehat{\varphi_\lambda \psi}|^2 |\widehat{\varphi_\lambda \psi}|^{p-2} \leq \|\widehat{\varphi_\lambda \psi}\|_\infty^{p-2} \|\widehat{\varphi_\lambda \psi}\|_2^2$$

Note how we used the fact that  $p - 2 > 0$  in this computation. Now, using Parseval's formula and the fact that  $\varphi_\lambda$  has absolute value 1:

$$\|\widehat{\varphi_\lambda \psi}\|_2^2 = \|\varphi_\lambda \psi\|_2^2 = \|\psi\|_2^2$$

and therefore

$$\|\widehat{\varphi_\lambda \psi}\|_p^p \leq \|\widehat{\varphi_\lambda \psi}\|_\infty^{p-2} \|\psi\|_2^2 \xrightarrow{\lambda \rightarrow +\infty} 0$$

We know that the Fourier inversion formula holds for Schwartz functions, so the double Fourier transform of  $\varphi_\lambda \psi$  is  $x \mapsto \varphi_\lambda \psi(-x)$ , which has same  $L^q$  norm as  $\varphi_\lambda \psi$ . So by the inverted Hausdorff-Young inequality (**1**),

$$\|\varphi_\lambda \psi\|_q \leq \|\widehat{\varphi_\lambda \psi}\|_p \xrightarrow{\lambda \rightarrow +\infty} 0$$

And we have our contradiction, since  $\|\varphi_\lambda\psi\|_q$  does not depend on  $\lambda$ :

$$\|\varphi_\lambda\psi\|_q^q = \int_{\mathbb{R}} |\psi|^q$$

Conclusion:

(1) implies that $\frac{1}{p} + \frac{1}{q} = 1$ and $1 \leq p \leq 2$ .
--------------------------------------------------------------------------

## Sixth problem

1. A distribution  $\mu$  on  $\mathbb{T}$  is *positive* if  $\langle \mu, f \rangle \geq 0$  for every non-negative  $f \in \mathcal{C}^\infty(\mathbb{T})$ . Show that a positive distribution is a measure.
2. Assume  $f_n \in L^2(0, 1)$ ,  $n \in \mathbb{N}$  and  $\|f_n\| \leq 1$ . Prove that

$$\mu[|f_n| > n^{2/3}] < \frac{1}{n^{4/3}}$$

and conclude that for every positive  $\epsilon$ , there exists  $N \in \mathbb{N}$  such that

$$\mu[|f_n| \leq n^{2/3} \quad \forall n > N] \geq 1 - \epsilon$$

**Solution**

**1** Let  $\mu$  be a distribution on  $\mathbb{T}$ , that is a continuous linear functional on  $\mathcal{C}^\infty(\mathbb{T})$ , and suppose that  $\mu$  is positive. That is

$$\forall f \in \mathcal{C}^\infty(\mathbb{T}) \quad f \geq 0 \quad \langle \mu, f \rangle \geq 0$$

We will show that  $\mu$  extends to a bounded linear functional on  $\mathcal{C}(\mathbb{T})$ , which in turn implies that  $\mu$  is a measure, by Riesz' representation theorem.

Let  $f$  be a real-valued function in  $\mathcal{C}^\infty(\mathbb{T})$ . We write  $\mathbf{1}$  for the constant function equal to 1. Then  $\|f\|_\infty \mathbf{1} - f$  and  $\|f\|_\infty \mathbf{1} + f$  are both nonnegative on  $\mathbb{T}$ . Thus

$$\langle \mu, \|f\|_\infty \mathbf{1} - f \rangle \geq 0 \quad \text{and} \quad \langle \mu, \|f\|_\infty \mathbf{1} + f \rangle \geq 0$$

and it follows that

$$|\langle \mu, f \rangle| \leq \|f\|_\infty \langle \mu, \mathbf{1} \rangle$$

Now, if  $f$  a complex-valued  $\mathcal{C}^\infty$  function on  $\mathbb{T}$ , we write that  $f = \Re f + i\Im f$ . Then

$$|\langle \mu, f \rangle| \leq |\langle \mu, \Re f \rangle| + |\langle \mu, \Im f \rangle| \leq 2\langle \mu, \mathbf{1} \rangle \|f\|_\infty$$

This estimate, valid for every  $\mathcal{C}^\infty(\mathbb{T})$  function, combined with the fact that  $\mathcal{C}^\infty(\mathbb{T})$  is dense in  $\mathcal{C}(\mathbb{T})$  (for example, it contains trigonometric polynomials), tells us that  $\mu$  has a unique extension to  $\mathcal{C}(\mathbb{T})$ , that satisfies

$$\forall f \in \mathcal{C}(\mathbb{T}) \quad |\langle \mu, f \rangle| \leq 2\langle \mu, \mathbf{1} \rangle \|f\|_\infty$$

By Riesz' representation theorem, there exists a unique signed measure  $\nu$  on  $\mathbb{T}$  such that

$$\forall f \in \mathcal{C}(\mathbb{T}) \quad \langle \mu, f \rangle = \int_{\mathbb{T}} f \, d\nu$$

 $\mu$  a measure.



**2** Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of  $L^2(0, 1)$  functions such that  $\|f_n\|_2 \leq 1$  for every  $n$ . Let  $n$  be an integer. We have

$$\begin{aligned} \mu[|f_n| > n^{2/3}] &= \mu[|f_n|^2 > n^{4/3}] = \int_{[|f_n|^2 > n^{4/3}]} d\mu = n^{-4/3} \int_{[|f_n|^2 > n^{4/3}]} n^{4/3} d\mu \\ &< n^{-4/3} \int_{[|f_n|^2 > n^{4/3}]} |f_n|^2 d\mu \leq n^{-4/3} \|f_n\|_2^2 \end{aligned}$$

Hence

$$\boxed{\mu[|f_n| > n^{2/3}] < n^{-4/3}}$$

Now, let  $\epsilon$  be positive. Since the series  $\sum n^{-4/3}$  converges, there exists an integer  $N$  such that

$$\sum_{n > N} \frac{1}{n^{4/3}} < \epsilon$$

Then

$$\mu\left(\bigcup_{n > N} [|f_n| > n^{2/3}]\right) \leq \sum_{n > N} \frac{1}{n^{4/3}} < \epsilon$$

and

$$\boxed{\mu[|f_n| \leq n^{2/3} \mid n > N] = 1 - \mu\left(\bigcup_{n > N} [|f_n| > n^{2/3}]\right) > 1 - \epsilon}$$

## Problem 7

Let  $(f_n)_{n \in \mathbb{N}}$  be an orthonormal sequence in  $L^2(0, 1)$ . Let  $S_n = \frac{1}{n} \sum_{m=1}^n f_m$ .

1. Prove that  $\|S_n\|^2 = \frac{1}{n}$ . It follows that if  $\sum \frac{1}{\lambda_j} < \infty$ , and in particular if  $\lambda_j = [j \ln^2 j]$ , then  $\sum |S_{\lambda_j}|^2$  converges almost everywhere, and  $S_{\lambda_j}$  converges to 0 almost everywhere.
2. If  $N \in (\lambda_j, \lambda_{j+1})$ , then  $S_N = \frac{\lambda_j}{N} S_{\lambda_j} + \frac{1}{N} \sum_{m=\lambda_j+1}^N f_m$ . Use 6.2 to estimate the sum and deduce that  $(S_n)_{n \in \mathbb{N}}$  converges almost everywhere to 0.

**Solution**

1 Let  $n$  be an integer. Because  $f_1, \dots, f_n$  are orthonormal, we have

$$\|S_n\|^2 = \frac{1}{n^2} \sum_{m=1}^n \|f_m\|^2 = \frac{1}{n}$$

Let  $(\lambda_j)_{j \in \mathbb{N}}$  be a sequence of positive integers such that  $\sum \frac{1}{\lambda_j}$  converges and let  $C$  be its sum. For example, take  $\lambda_j = [j \ln^2 j]$ , as the problem suggests. Let

$$\forall k \in \mathbb{N} \quad A_k = \left[ \sum_{j=1}^{\infty} |S_{\lambda_j}|^2 > k \right] \quad \text{and} \quad A = \left[ \sum |S_{\lambda_j}|^2 \text{ diverges} \right]$$

so that  $A = \bigcap_{k=1}^{\infty} A_k$  and  $\mu(A) \leq \mu(A_k)$  for every  $k$

Using the same strategy as in **problem 6.b**, we show that

$$\mu(A_k) \leq \frac{1}{k} \int_{[0,1]} \sum_{j=1}^{\infty} |S_{\lambda_j}|^2 d\mu = \frac{1}{k} \sum_{j=1}^{\infty} \|S_{\lambda_j}\|^2 = \frac{C}{k}$$

and it follows that  $\forall k \in \mathbb{N}^* \quad \mu(A) \leq \frac{C}{k}$  so  $\mu(A) = 0$

Thus

The series  $\sum |S_{\lambda_j}|^2$  converges almost everywhere and  $(S_{\lambda_j})_{j \in \mathbb{N}}$  converges to 0 almost everywhere.

2

## Problem 8

Let  $B$  be the unit ball in  $\mathbb{R}^3$ ,  $G = \mathbb{1}_B$  its indicator function, and

$$g(x) = \int_{\mathbb{R}^2} G(x, y, z) \, dy \, dz$$

1. Compute  $g(x)$ .
2. Show that  $\widehat{g}(\xi) = O(\frac{1}{|\xi|^3})$ . (This is more than probably false)
3. Show that  $\widehat{G}(\xi, \eta, \zeta) = O(\frac{1}{(\xi^2 + \eta^2 + \zeta^2)^{3/2}})$ .
4. Let  $F_n(x, y, z) = \sin^2(nr)G(x, y, z)$  and  $f_n(x) = \int_{\mathbb{R}^2} F_n(x, y, z) \, dy \, dz$ . Prove that  $(f_n)_{n \in \mathbb{N}}$  converges uniformly to  $\frac{g}{2}$ .

**Solution**

**1** Computing  $g$  is not too much of a problem. Let  $x$  be a real number. If  $|x| > 1$ , there is no chance for  $(x, y, z)$  to be in the unit ball of  $\mathbb{R}^3$ , whatever  $y$  and  $z$  are. Thus  $g(x) = 0$ .

If  $|x| \leq 1$ , then  $(x, y, z)$  belongs to  $B$  if and only if

$$y^2 + z^2 \leq 1 - x^2$$

so  $\forall (y, z) \in \mathbb{R}^2 \quad G(x, y, z) = 1 \iff (y, z) \in \mathcal{D}(0, \sqrt{1 - x^2})$

Otherwise,  $G(x, y, z)$  is 0. So  $g(x)$  is simply the area of the disk centered at 0, with radius  $\sqrt{1 - x^2}$ :

$$\forall x \in \mathbb{R} \quad g(x) = \pi(1 - x^2)\mathbb{1}_{[-1,1]}(x)$$

**4** Let  $n$  be a positive integer and define

$$\forall (x, y, z) \in \mathbb{R}^3 \quad F_n(x, y, z) = \sin^2(nr)G(x, y, z)$$

and  $\forall x \in \mathbb{R} \quad f_n(x) = \int_{\mathbb{R}^2} F_n(x, y, z) \, dy \, dz$

Fix  $x \in \mathbb{R}$ . As explained in question **1**,  $(y, z) \mapsto G(x, y, z)$  is 0 if  $|x| > 1$ , and is the indicator function of the disk  $\mathcal{D}(0, \sqrt{1 - x^2})$  if  $|x| \leq 1$ . Thus

$$f_n(x) = \begin{cases} 0 & \text{if } |x| > 1 \\ \int_{\mathcal{D}(0, \sqrt{1-x^2})} \sin^2(nr) \, dy \, dz & \text{if } |x| \leq 1 \end{cases}$$

and  $f_n = 0 = \frac{g}{2}$  when  $|x| > 1$ , which of course gives us the uniform convergence for free. Now let's focus on the case where  $|x| \leq 1$  and switch to polar coordinates in order to compute the integral:

$$f_n(x) = \int_0^{2\pi} \int_0^{\sqrt{1-x^2}} \rho \sin^2(n\sqrt{x^2 + \rho^2}) \, d\rho d\theta = 2\pi \int_0^{\sqrt{1-x^2}} \rho \sin^2(n\sqrt{x^2 + \rho^2}) \, d\rho$$

Proceed next with the substitution

$$u = \sqrt{x^2 + \rho^2} \quad \rho^2 = u^2 - x^2 \quad \rho \, d\rho = u \, du$$

$$\begin{aligned} \text{so that } f_n(x) &= 2\pi \int_{|x|}^1 u \sin^2(nu) \, du = 2\pi \int_{|x|}^1 \frac{u}{2} (1 - \cos(2nu)) \, du \\ &= \pi \int_{|x|}^1 u \, du - \pi \int_{|x|}^1 u \cos(2nu) \, du = \frac{\pi}{2} \sqrt{1-x^2} - \pi \int_{|x|}^1 u \cos(2nu) \, du \\ f_n(x) &= \frac{g(x)}{2} - \pi \int_{|x|}^1 u \cos(2nu) \, du \end{aligned}$$

All that remains is to show that the integral converges to 0 uniformly in  $x$ . This is done, for example, by integration by parts:

$$\int_{|x|}^1 u \cos(2nu) \, du = \frac{1}{2n} [u \sin(2nu)]_{|x|}^1 - \frac{1}{2n} \int_{|x|}^1 \sin(2nu) \, du$$

The functions involved on the righthandside are less than 1 (remember that we are considering the case  $|x| < 1$ ), therefore

$$\left| \int_{|x|}^1 u \cos(2nu) \, du \right| \leq \frac{3}{2n}$$

This bound being independent of  $x$ , the claim is proved:

$$\left| \int_{|x|}^1 u \cos(2nu) \, du \right| \text{ tends to 0 uniformly in } x.$$

Therefore

$$\boxed{(f_n)_{n \in \mathbb{N}} \text{ converges uniformly to } \frac{g}{2} \text{ on } \mathbb{R}.}$$

## Problem 9

Let  $\forall x \in \mathbb{R} \quad f(x) = \sum_{n=0}^{\infty} \frac{\cos 10^{2n}x}{10^n}$

1. Prove that  $f$  is Hölder  $\frac{1}{2}$ .
2. Prove that  $f$  is nowhere differentiable.
3. Can a Lipschitz function be nowhere differentiable?

**Solution**

Of course,  $f$  is well defined and is continuous since the series converges normally:

$$\forall x \in \mathbb{R} \quad \forall n \in \mathbb{N} \quad \left| \frac{\cos 10^{2n}x}{10^n} \right| \leq \frac{1}{10^n}$$

1 Let  $x$  and  $h$  be real numbers. We have

$$f(x+h) - f(x) = \sum_{n=0}^{\infty} \frac{\cos 10^{2n}(x+h) - \cos 10^{2n}x}{10^n}$$

Since  $f$  is  $\frac{2\pi}{100}$ -periodic, we may assume that  $|h| \leq \frac{2\pi}{100}$ . We cut this sum in two parts, one between 0 and  $N$ , the other one between  $N+1$  and  $\infty$ , then estimate each part separately, and finally choose  $N$  cleverly in order to get the Hölder  $\frac{1}{2}$  inequality required:

$$|f(x+h) - f(x)| \leq \sum_{n=0}^N \frac{|\cos 10^{2n}(x+h) - \cos 10^{2n}x|}{10^n} + \sum_{n=N+1}^{\infty} \frac{|\cos 10^{2n}(x+h) - \cos 10^{2n}x|}{10^n}$$

Each term of the first sum is estimated through the mean value theorem:

$$|\cos 10^{2n}(x+h) - \cos 10^{2n}x| \leq 10^{2n}|h|$$

and each term of the second sum is roughly estimated

$$|\cos 10^{2n}(x+h) - \cos 10^{2n}x| \leq 2$$

so that

$$|f(x+h) - f(x)| \leq |h| \sum_{n=0}^N 10^n + 2 \sum_{n=N+1}^{\infty} \frac{1}{10^n}$$

We have

$$\sum_{n=0}^N 10^n = 10^N \sum_{n=0}^N \frac{1}{10^n} \leq 10^N \sum_{n=0}^{\infty} \frac{1}{10^n} = \frac{10^{N+1}}{9}$$

and 
$$\sum_{n=N+1}^{\infty} \frac{1}{10^n} = \frac{1}{10^{N+1}} \sum_{n=0}^{\infty} \frac{1}{10^n} = \frac{1}{9 \times 10^N}$$

Hence  $|f(x+h) - f(x)| \leq \frac{10^{N+1}|h|}{9} + \frac{2}{9 \times 10^N} = \frac{\sqrt{|h|}}{9} \left( 10^{N+1} \sqrt{|h|} + \frac{2}{10^N \sqrt{|h|}} \right)$

This computation is valid for every integer  $N$ . Now is time to choose it in order to optimize this inequality. Let's take  $N$  to be the only integer such that

$$\frac{1}{10^{N+1}} \leq \sqrt{|h|} < \frac{1}{10^N}$$

Then  $10^{N+1} \sqrt{|h|} < 10$  and  $\frac{1}{10^N \sqrt{|h|}} \leq 10$

and we finally get 
$$\boxed{|f(x+h) - f(x)| \leq \frac{30}{9} \sqrt{|h|}}$$

**[2]** Let  $x$  be in  $[0, \frac{2\pi}{100})$  and suppose that  $f$  is differentiable at  $x$ . Then, if  $(y_k)_{k \in \mathbb{N}}$  and  $(z_k)_{k \in \mathbb{N}}$  are sequences converging to  $x$ , such that  $y_k \leq x < z_k$  or  $z_k \leq x < y_k$  for every  $k$ , we have

$$\lim_{k \rightarrow \infty} \frac{f(x) - f(y_k)}{x - y_k} - f'(x) = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{f(x) - f(z_k)}{x - z_k} - f'(x) = 0$$

So there exist sequences  $(\epsilon_k)_{k \in \mathbb{N}}$  and  $(\eta_k)_{k \in \mathbb{N}}$  tending to 0, such that

$$\forall k \in \mathbb{N} \quad f(x) - f(y_k) = (x - y_k)(f'(x) + \epsilon_k)$$

and  $\forall k \in \mathbb{N} \quad f(x) - f(z_k) = (x - z_k)(f'(x) + \eta_k)$

Thus  $f(z_k) - f(y_k) = (z_k - y_k)f'(x) + (x - y_k)\epsilon_k + (z_k - x)\eta_k$

and 
$$\frac{f(z_k) - f(y_k)}{z_k - y_k} - f'(x) = \frac{x - y_k}{z_k - y_k} \epsilon_k + \frac{z_k - x}{z_k - y_k} \eta_k$$

But  $y_k \leq x \leq z_k$  or  $z_k \leq x < y_k$  so that

$$0 \leq |x - y_k| \leq |z_k - y_k| \quad \text{and} \quad 0 \leq |z_k - x| \leq |z_k - y_k|$$

So 
$$\left| \frac{f(z_k) - f(y_k)}{z_k - y_k} - f'(x) \right| \leq \epsilon_k + \eta_k$$

which proves that 
$$\boxed{\lim_{k \rightarrow \infty} \frac{f(z_k) - f(y_k)}{z_k - y_k} = f'(x)}$$

In order to show that  $f$  is not differentiable at  $x$ , we will exhibit two sequences  $(y_k)_{k \in \mathbb{N}}$  and  $(z_k)_{k \in \mathbb{N}}$  that contradict this result. Let's do this.

Let  $k$  be an integer. The collection of intervals  $\left\{ \left[ \frac{2p\pi}{10^{2k}}, \frac{2(p+1)\pi}{10^{2k}} \right) \mid 0 \leq p < 10^{2k-2} \right\}$  covers the interval  $\left[ 0, \frac{2\pi}{100} \right)$  so there exists (a unique)  $p$  such that

$$\frac{2p\pi}{10^{2k}} \leq x < \frac{2(p+1)\pi}{10^{2k}}$$

We then define

$$y_k = \frac{(2p+1)\pi}{10^{2k}} \quad \text{and} \quad z_k = \begin{cases} \frac{2p\pi}{10^{2k}} & \text{if } x < y_k \\ \frac{2(p+1)\pi}{10^{2k}} & \text{if } x \geq y_k \end{cases}$$

so that either  $z_k \leq x < y_k$  or  $y_k \leq x < z_k$

Assume, for example, that we are in the first case. The second case would be studied exactly the same. We want to look at every ratio  $\frac{\cos 10^{2n} z_k - \cos 10^{2n} y_k}{10^n(z_k - y_k)}$  and see how each of them behaves. Notice that

$$z_k - y_k = -\frac{\pi}{10^{2k}}$$

There are three cases to look at.

- **$n > k$ :** We have

$$10^{2n} z_k = 2p\pi 10^{2(n-k)} \quad \text{and} \quad 10^{2n} y_k = (2p+1)\pi 10^{2(n-k)}$$

which are even multiples of  $\pi$ . Therefore these numbers have same cosine and

$$\forall n > k \quad \frac{\cos 10^{2n} z_k - \cos 10^{2n} y_k}{10^n(z_k - y_k)} = 0$$

- **$n = k$ :** We have

$$10^{2k} z_k = 2p\pi \quad \text{and} \quad 10^{2k} y_k = (2p+1)\pi$$

Hence 
$$\frac{\cos 10^{2k} z_k - \cos 10^{2k} y_k}{10^k(z_k - y_k)} = \frac{-2 \cdot 10^k}{\pi}$$

- **$n < k$ :** We simply use the mean value estimate:

$$\left| \frac{\cos 10^{2n} z_k - \cos 10^{2n} y_k}{10^n(z_k - y_k)} \right| \leq \frac{10^{2n}(z_k - y_k)}{10^n(z_k - y_k)} = 10^n$$

Thus

$$\sum_{n=0}^{k-1} \left| \frac{\cos 10^{2n} z_k - \cos 10^{2n} y_k}{10^n (z_k - y_k)} \right| \leq \sum_{n=0}^{k-1} 10^n \leq \frac{10^k}{9}$$

From what precedes, we see that

$$\left| \frac{f(z_k) - f(y_k)}{z_k - y_k} \right| = \left| \sum_{n=0}^{k-1} \frac{\cos 10^{2n} z_k - \cos 10^{2n} y_k}{10^n (z_k - y_k)} - \frac{2 \cdot 10^k}{\pi} \right|$$

Using the triangle inequality:

$$\left| \frac{f(z_k) - f(y_k)}{z_k - y_k} \right| \geq \frac{2 \cdot 10^k}{\pi} - \left| \sum_{n=0}^{k-1} \frac{\cos 10^{2n} z_k - \cos 10^{2n} y_k}{10^n (z_k - y_k)} \right| \geq 10^k \times \left( \frac{2}{\pi} - \frac{1}{9} \right)$$

We would have the same estimate if we had made the other choice for  $z_k$ . So this lower bound is valid for every  $k$ , and since  $\frac{2}{\pi} - \frac{1}{9}$  is positive,

and

$$\lim_{k \rightarrow \infty} \left| \frac{f(z_k) - f(y_k)}{z_k - y_k} \right| = +\infty$$

This contradicts the existence of  $f'(x)$ . So

$f$  is nowhere differentiable.

**[3]** A Lipschitz function is absolutely continuous, hence differentiable almost everywhere. Thus it cannot be nowhere differentiable.



**Problem 10**

Let  $B$  be a Banach space and  $S : B \rightarrow \mathcal{C}([0, 1])$  a linear map such that, for every sequence  $(v_n)_{n \in \mathbb{N}}$  of elements of  $B$  converging to 0,  $(Sv_n)_{n \in \mathbb{N}}$  converges pointwise to 0. Prove that  $S$  is bounded.

**Solution**

$B$  and  $\mathcal{C}([0, 1])$  are Banach spaces, so the tool to show that  $S$  is bounded is of course the closed graph theorem. Let  $(v_n)_{n \in \mathbb{N}}$  be a sequence in  $B$ , converging to some  $v \in B$  and such that  $(Sv_n)_{n \in \mathbb{N}}$  converges (uniformly) to some  $f \in \mathcal{C}([0, 1])$ .

The sequence  $(v_n - v)_{n \in \mathbb{N}}$  converges in  $B$  to 0. Therefore,  $(Sv_n - Sv)_{n \in \mathbb{N}}$  converges pointwise to 0. In other terms,  $(Sv_n)_{n \in \mathbb{N}}$  converges pointwise to  $Sv$ . But we also know that it converges uniformly (hence pointwise) to  $f$ . Thus  $f = Sv$  and the convergence towards  $Sv$  is uniform.

This achieves showing that  $S$  has closed graph;

S is a bounded operator.