

Ph.D. Qualifying Exam – Spring 2004

First problem

Suppose $f \in L^1(\mathbb{R})$. Let

$$\forall x \in \mathbb{R} \quad g(x) = \int_{\mathbb{R}} e^{-ixy^2} f(y) \, dy$$

1. Prove that the integral exists for every x .
2. Prove that g is a continuous function.
3. Prove that there is a dense subset S of $L^1(\mathbb{R})$ such that if $f \in S$, then

$$\lim_{|x| \rightarrow +\infty} g(x) = 0 \tag{*}$$

4. Prove that $(*)$ holds for any $f \in L^1(\mathbb{R})$.

Solution

[1] We have $\forall x, y \in \mathbb{R} \quad |e^{-ixy^2} f(y)| = |f(y)|$ (1)

and since f is $L^1(\mathbb{R})$, the function $x \mapsto f(y)e^{-ixy^2}$ is also $L^1(\mathbb{R})$ for every real number x . Thus

The function g is well defined.

[2] Because of (1), and because the function $(x, y) \mapsto f(y)e^{-ixy^2}$ is continuous in x , we can apply the continuity theorem for integrals and deduce that

g is continuous on \mathbb{R} .

In the remainder of the solution, we will write Γf for g . That is

$$\forall f \in L^1(\mathbb{R}) \quad \forall x \in \mathbb{R} \quad \Gamma f(x) = \int_{\mathbb{R}} f(y) e^{-ixy^2} \, dy$$

[3] Let S_1 be the set of L^1 functions, that are 0 almost everywhere in some neighbourhood of 0. That is:

$$S_1 = \{f \in L^1(\mathbb{R}) \mid \exists \epsilon > 0 \quad f \mathbf{1}_{[-\epsilon, \epsilon]} = 0\}$$

We first show that S_1 is dense. Let f be in L^1 . For every positive integer n , define

$$f_n = f - f \mathbb{1}_{[-\frac{1}{n}, \frac{1}{n}]}$$

Then

$$f_n \mathbb{1}_{[-\frac{1}{n}, \frac{1}{n}]} = 0$$

so that

$$\forall n \in \mathbb{N}^* \quad f_n \in S$$

Furthermore $\forall n \in \mathbb{N} \quad \|f - f_n\|_1 = \int_{-\frac{1}{n}}^{\frac{1}{n}} |f| \xrightarrow{n \rightarrow \infty} 0$

which proves that S_1 is dense. We now let S be the elements of S_1 which are \mathcal{C}^∞ with compact support. The intersection of two dense sets is dense, hence

S is dense in $L^1(\mathbb{R})$.

If f belongs to S , then $h : y \mapsto \frac{f(y)}{y}$ is \mathcal{C}^∞ with compact support and 0 in a neighbourhood of 0. Thus h and h' are integrable. We have

$$\forall x \in \mathbb{R} \quad \mathbb{T}f(x) = \int_{\mathbb{R}} f(y) e^{-ixy^2} dy = \int_{\mathbb{R}} h(y) y e^{-ixy^2} dy$$

Integrate by parts:

$$\forall x \in \mathbb{R} \setminus \{0\} \quad \mathbb{T}f(x) = \underbrace{\left[\frac{h(y)e^{-ixy^2}}{-2ix} \right]_{-\infty}^{+\infty}}_{=0} + \frac{1}{2ix} \int_{\mathbb{R}} h'(y) e^{-ixy^2} dy$$

Thus

$$\forall x \in \mathbb{R} \setminus \{0\} \quad |\mathbb{T}f(x)| \leq \frac{1}{2|x|} \|h'\|_1 \xrightarrow{|x| \rightarrow +\infty} 0$$

There is a dense subset S of L^1 such that

$$\forall f \in S \quad \lim_{|x| \rightarrow +\infty} \mathbb{T}f(x) = 0$$

[4] Let f be in L^1 and g be in S . We can write that $\mathbb{T}f = \mathbb{T}g + \mathbb{T}(f - g)$, and notice that

$$\forall x \in \mathbb{R} \quad |\mathbb{T}(f - g)(x)| \leq \|f - g\|_1$$

Then

$$\limsup_{|x| \rightarrow +\infty} |\mathbb{T}f(x)| \leq \underbrace{\limsup_{|x| \rightarrow +\infty} |\mathbb{T}g(x)|}_{=0} + \|f - g\|_1 = \|f - g\|_1$$

This holds for every g in S , which is dense in L^1 . Thus $\limsup_{|x| \rightarrow +\infty} |\mathbb{T}f(x)|$ is 0:

$$\forall f \in L^1(\mathbb{R}) \quad \lim_{|x| \rightarrow +\infty} \mathbb{T}f(x) = 0$$

Second problem

Suppose (X, \mathcal{A}, μ) is a measure space and that $(f_n)_{n \in \mathbb{N}}$ is a sequence in $L^1(\mu)$ converging weakly. Prove that

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall A \in \mathcal{A} \quad (\mu(A) < \delta \implies \forall n \in \mathbb{N} \quad \int_A |f_n| d\mu \leq \epsilon)$$

Solution

The following result is a key ingredient for this problem:

Theorem 1: Let (X, \mathcal{A}, μ) be a measure space. Let f be a positive integrable function on X . Then

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall A \in \mathcal{A} \quad (\mu(A) \leq \delta \implies \int_A f d\mu \leq \epsilon)$$

I believe it is usually proved in 205A, so it can, of course, be used. Yet it took me so much time remembering why it is true that I want to recall the proof:

Proof: For every positive integer n , define

$$f_n = \text{Min}(f, n)$$

The sequence $(f_n)_{n \in \mathbb{N}}$ is increasing and converges pointwise to f . By the monotone convergence theorem, $f_n \xrightarrow[n \rightarrow \infty]{L^1(\mu)} f$. Let A be a measurable set. We have

$$\int_A f d\mu = \int_A (f - f_n) d\mu + \int_A f_n d\mu \leq \int_X (f - f_n) d\mu + n\mu(A)$$

Taking n big enough so that $\int_X (f - f_n) d\mu \leq \frac{\epsilon}{2}$, and letting A be any set of measure less than $\frac{\epsilon}{2n}$ yield

$$\int_A f d\mu \leq \epsilon \quad \square$$

Suppose the conclusion false:

$$\exists \epsilon > 0 \quad \forall \delta > 0 \quad \exists A \in \mathcal{A} \quad (\mu(A) < \delta \quad \text{and} \quad \int_A |f_n| d\mu > \epsilon \quad \text{for some } n) \quad (\star)$$

In other words,

$$\text{Inf} \left\{ \mu(A) \mid A \in \mathcal{A} \text{ such that } \int_A |f_n| d\mu > \epsilon \text{ for some } n \right\} = 0$$

Start by taking $\delta_1 = 1$ in (\star) . Then there exists an integer k_1 and a measurable set A_1 such that

$$\mu(A_1) \leq \delta_1 \quad \text{and} \quad \int_{A_1} |f_{k_1}| d\mu > \epsilon$$

Let's rename f_{k_1} to g_1 . This is the first step in the inductive construction of a subsequence $(g_n)_{n \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}}$, a sequence $(A_n)_{n \in \mathbb{N}}$ of measurable sets, and a sequence $(\delta_n)_{n \in \mathbb{N}}$ of positive real numbers.

Suppose that we have constructed the first terms g_1, \dots, g_n of the subsequence, sets A_1, \dots, A_n and positive real numbers $\delta_1, \dots, \delta_n$, with the following properties:

- For $m \leq n$, δ_m is such that

$$\forall A \in \mathcal{A} \quad (\mu(A) \leq \delta_m \implies \int_A |g_j| d\mu < \frac{\epsilon}{2^m} \text{ for } j = 1, \dots, m-1)$$

- For $m \leq n$, A_m and g_m are such that

$$\mu(A_m) \leq \delta_m \quad \text{and} \quad \int_{A_m} |g_m| d\mu > \epsilon$$

Using **Theorem 1** and (\star) , it is clear that we can construct δ_{n+1} , A_{n+1} and g_{n+1} . So the two properties listed above are actually satisfied for every n . Hope this is clear enough...

For every positive integer n , let

$$B_n = A_n \setminus \left(\bigcup_{j=1}^{n-1} A_j \right)$$

The sets $(B_n)_{n \in \mathbb{N}}$ are pairwise disjoint

Third problem

Let $f : (a, b) \rightarrow \mathbb{R}$ be an arbitrary function. Let

$$S = \{x \in (a, b) \mid f'(x) \text{ exists and equals } 0\}$$

Prove that the image $f(S)$ of S under f has measure 0.

Solution

Let ϵ be a positive real number. By definition, if x is in S , f is differentiable at x and $f'(x) = 0$. Thus there exists $h > 0$ such that

$$\forall y \in [x - h, x + h] \quad |f(y) - f(x)| \leq \epsilon|x - y| \leq 2\epsilon h$$

which means that $f([x - h, x + h]) \subset [f(x) - 2\epsilon h, f(x) + 2\epsilon h]$.

Now, let \mathcal{C} be the collection of all intervals of the form $I = [x - h, x + h] \subset (a, b)$, such that $x \in S$ and $f(I)$ is contained in an interval of length at most $2\epsilon h$. What we just explained shows that \mathcal{C} covers S . Thus, by the 5-times covering lemma, there exists a pairwise disjoint collection $(I_n)_{n \in \mathbb{N}} = ([x_n - h_n, x_n + h_n])_{n \in \mathbb{N}} \subset \mathcal{C}$ such that $(5I_n)_{n \in \mathbb{N}} \subset \mathcal{C}$ as well, and $S \subset \bigcup_{n \in \mathbb{N}} 5I_n$. Thus $f(S) \subset \bigcup_{n \in \mathbb{N}} f(5I_n)$ and since the Lebesgue measure is increasing and subadditive:

$$\mu(f(S)) \leq \sum_{n \in \mathbb{N}} \mu(f(5I_n))$$

For every integer n , because $5I_n \in \mathcal{C}$, $f(5I_n)$ is included in an interval of length $2\epsilon \times 5h_n$:

$$\mu(f(S)) \leq 5\epsilon \sum_{n \in \mathbb{N}} 2h_n = 5\epsilon \sum_{n \in \mathbb{N}} \mu(I_n) = 5\epsilon \mu\left(\bigcup_{n \in \mathbb{N}} I_n\right) \leq 5\epsilon(b - a)$$

This holds for every positive ϵ . Therefore,

$$\boxed{\mu(f(S)) = 0}$$

Fourth problem

Let X be a metric space.

1. Prove that if X is countably compact, then X is compact.
2. Prove that if every continuous function on X is bounded, then X is compact.

Solution

[1] Suppose that X is countably compact, which means that every countable open cover of X has a finite subcover. Then we can derive a property analogous to the finite intersection property: if $(F_n)_{n \in \mathbb{N}}$ is a (countable) collection of closed sets such that $\bigcap_{n \in \mathbb{N}} F_n = \emptyset$, there exists a finite subcollection F_{n_1}, \dots, F_{n_k} such that $F_{n_1} \cap \dots \cap F_{n_k} = \emptyset$.

This is proven exactly as the finite intersection property is proven: the $(F_n^c)_{n \in \mathbb{N}}$ form a countable open cover of X . Thus there is a finite subcover $F_{n_1}^c \cup \dots \cup F_{n_k}^c$ of X , and we have $F_{n_1} \cap \dots \cap F_{n_k} = \emptyset$.

Because X is a metric space, sequential compactness is equivalent to compactness. So let $(x_n)_{n \in \mathbb{N}}$ be a sequence of elements of X . The set of limit points of $(x_n)_{n \in \mathbb{N}}$ is

$$\mathcal{L} = \bigcap_{n \in \mathbb{N}} \overline{\{x_k \mid k \geq n\}}$$

Suppose that \mathcal{L} is empty. Then there exist integers $n_1 \leq \dots \leq n_j$ such that

$$\emptyset = \bigcap_{k=1}^j \overline{\{x_k \mid k \geq n_j\}}$$

But these closed sets form a decreasing sequence, therefore

$$\emptyset = \overline{\{x_k \mid k \geq n_j\}}$$

which is, of course, impossible. Hence \mathcal{L} is not empty: $(x_n)_{n \in \mathbb{N}}$ has convergent subsequences.

[2] Suppose that X is not compact. There exists a sequence $(x_n)_{n \in \mathbb{N}}$ of elements of X that has no convergent subsequence. Denote by \mathcal{X} the collection $(x_n)_{n \in \mathbb{N}}$. Then \mathcal{X} is closed: indeed, a convergent sequence of distinct points of \mathcal{X} would allow us to construct a convergent subsequence of $(x_n)_{n \in \mathbb{N}}$.

The same conclusion holds for $\mathcal{X}_n = \mathcal{X} \setminus \{x_n\}$, for every integer n : this set is closed. Since it does not contain x_n , the distance $\text{dist}(x_n, \mathcal{X}_n)$ is a positive real number r_n .

Define a function on \mathcal{X} as follows:

$$\forall n \in \mathbb{N} \quad f(x_n) = n$$

Then f is continuous on the closed set \mathcal{X} and by Tietze's extension theorem, it can be extended to a continuous function (still denoted f) on X . So there exists a continuous, unbounded, function on X .

If every continuous function on X is bounded, X is compact.

Fifth problem

Prove that there exists an orthonormal basis \mathcal{B} of $L^2([0, 1])$ such that

$$\forall f \in \mathcal{B} \quad \int_0^1 |f(x)| \frac{dx}{x} < \infty \quad \text{and} \quad \int_0^1 f(x) \frac{dx}{x} = 0$$

Solution

Let
$$\mathcal{S} = \left\{ f \in L^2(0, 1) \mid \int_0^1 |f(x)| \frac{dx}{x} < \infty \right\}$$

$$\forall f \in \mathcal{S} \quad \mathbb{T}f = \int_0^1 f(x) \frac{dx}{x}$$

and
$$\forall n \in \mathbb{N} \quad g_n = \mathbb{1}_{\left[\frac{1}{n}, 1\right]}$$

Then \mathcal{S} is dense in $L^2(0, 1)$. Indeed, let f be an L^2 function. For every positive integer n , the function $f_n = fg_n$ is in \mathcal{S} , and

$$\|f - f_n\|_2^2 = \int_0^{\frac{1}{n}} f(x) dx \xrightarrow[n \rightarrow \infty]{} 0$$

by dominated convergence.

Define now

$$\mathcal{Z} = \text{Ker } \mathbb{T}$$

and let's show that \mathcal{Z} is dense in \mathcal{S} . For this, we will need the value of $\mathbb{T}g_n$, which is not a problem:

$$\forall n \in \mathbb{N} \quad \mathbb{T}g_n = \int_{\frac{1}{n}}^1 \frac{dx}{x} = \ln n \xrightarrow[n \rightarrow \infty]{} +\infty$$

Let f be any function in \mathcal{S} . Then

$$\mathbb{T}\left(f - \frac{\mathbb{T}f}{\mathbb{T}g_n} g_n\right) = 0$$

which gives

$$f - \frac{\mathbb{T}f}{\mathbb{T}g_n} g_n \in \mathcal{Z}$$

But

$$\frac{|\mathbb{T}f|}{|\mathbb{T}g_n|} \|g_n\|_2 = \frac{\mathbb{T}f}{\ln n} \left(1 - \frac{1}{n}\right) \xrightarrow[n \rightarrow \infty]{} 0$$

and therefore

$$\lim_{n \rightarrow \infty} \left(f - \frac{\mathbb{T}f}{\mathbb{T}g_n} g_n\right) = f$$

f is indeed the limit of a sequence in \mathcal{Z} : the latter is dense in \mathcal{S} .

Since $L^2(0,1)$ is separable, \mathcal{Z} has a countable dense subset \mathcal{X} . Using the Gram-Schmidt process, we can extract an orthonormal sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{X}$, that spans the same subspace as \mathcal{X} . Thus

$$\mathcal{X} \subset \text{Span}(f_n)_{n \in \mathbb{N}}$$

and

$$L^2(0,1) = \overline{\text{Span}(f_n)_{n \in \mathbb{N}}}$$

Sixth problem

Let (X, \mathcal{B}, μ) be a finite measure space. Suppose that $(f_n)_{n \in \mathbb{N}}$ is a sequence of functions in $L^1(\mu)$, converging almost everywhere to an $L^1(\mu)$ function f . Suppose also that

$$\|f_n\|_1 \xrightarrow{n \rightarrow \infty} \|f\|_1$$

1. Prove that for every measurable set A , $\int_A |f_n| \, d\mu \xrightarrow{n \rightarrow \infty} \int_A |f|$.
2. Prove that $\|f_n - f\|_1 \xrightarrow{n \rightarrow \infty} 0$.

Solution

1 For every integer n , define

$$g_n = \text{Min}(|f_n|, |f|) = \frac{|f_n| + |f| - \left| |f_n| - |f| \right|}{2}$$

Since $(f_n)_{n \in \mathbb{N}}$ converges almost everywhere to f , it follows that $(g_n)_{n \in \mathbb{N}}$ converges almost everywhere to $|f|$ and is dominated by $|f|$.

Let A be any measurable set. Then $(g_n \mathbb{1}_A)_{n \in \mathbb{N}}$ converges almost everywhere to $|f| \mathbb{1}_A$ and is dominated by $|f| \mathbb{1}_A$. The dominated convergence theorem implies that

$$\lim_{n \rightarrow \infty} \int_A g_n \, d\mu = \int_A |f| \, d\mu$$

The same argument shows that

$$\lim_{n \rightarrow \infty} \int_X g_n \, d\mu = \int_X |f| \, d\mu$$

Now, by definition of g_n , the function $|f_n| - g_n$ is nonnegative for every integer n and thus we have

$$0 \leq (|f_n| - g_n) \mathbb{1}_A \leq |f_n| - g_n$$

so that

$$0 \leq \int_A (|f_n| - g_n) \, d\mu \leq \int_X (|f_n| - g_n) \, d\mu$$

Both terms on the righthandside tend to $\int_X |f| \, d\mu$, thus

$$\lim_{n \rightarrow \infty} \int_A (|f_n| - g_n) \, d\mu = 0$$

So

$$\forall n \in \mathbb{N} \quad \int_A |f_n| \, d\mu = \int_A (|f_n| - g_n) \, d\mu + \int_A g_n \, d\mu \xrightarrow{n \rightarrow \infty} \int_A |f| \, d\mu$$

2 For every integer n , the function $|f_n| + |f| - |f_n - f|$ is nonnegative, by the triangular inequality. Thus Fatou's lemma applies:

$$\int_X \liminf_{n \rightarrow \infty} (|f_n| + |f| - |f_n - f|) d\mu \leq \liminf_{n \rightarrow \infty} \int_X (|f_n| + |f| - |f_n - f|) d\mu$$

Since $(f_n)_{n \in \mathbb{N}}$ converges almost everywhere to f , we can actually simplify the lefthandside. And since $\int_X (|f_n| + |f|) d\mu$ does converge to $2 \int_X |f|$, we can pull it out of the liminf on the righthandside, after taking limits:

$$2 \int_X |f| \leq 2 \int_X |f| + \liminf_{n \rightarrow \infty} \left(- \int_X |f_n - f| d\mu \right)$$

Hence
$$\limsup_{n \rightarrow \infty} \int_X |f_n - f| d\mu = - \liminf_{n \rightarrow \infty} \left(- \int_X |f_n - f| d\mu \right) \leq 0$$

and

$$\boxed{\lim_{n \rightarrow \infty} \|f_n - f\|_1 = 0}$$

We used here the following manipulation involving liminfs:

Lemma: If $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ are two sequences of real numbers such that the first one converges to some $u \in \mathbb{R}$, we have

$$\liminf_{n \rightarrow \infty} (u_n + v_n) = u + \liminf_{n \rightarrow \infty} v_n$$

Note that in general, if we don't assume that $(u_n)_{n \in \mathbb{N}}$ converges, then all we can say is that

$$\liminf_{n \rightarrow \infty} (u_n + v_n) \geq \liminf_{n \rightarrow \infty} u_n + \liminf_{n \rightarrow \infty} v_n$$

which is not sufficient for us to solve the problem.

Proof: Define

$$\forall n \in \mathbb{N} \quad U_n = \inf_{k \geq n} u_k \quad V_n = \inf_{k \geq n} v_k \quad \text{and} \quad w_n = \inf_{k \geq n} (u_k + v_k)$$

so that
$$\liminf_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} U_n = u \quad \liminf_{n \rightarrow \infty} v_n = \lim_{n \rightarrow \infty} V_n$$

and
$$\liminf_{n \rightarrow \infty} (u_n + v_n) = \lim_{n \rightarrow \infty} w_n$$

Let ϵ be a positive real number. Then there exists an integer N such that

$$\forall n \geq N \quad u - \epsilon < u_n < u + \epsilon$$

Let n be an integer bigger than N , let k be an integer bigger than n . Then we have in particular

$$u_k + v_k < u + \epsilon + v_k$$

But

$$u_k + v_k \geq \inf_{p \geq n} (u_p + v_p) = w_n$$

Therefore

$$w_n < u + \epsilon + v_k$$

This holds for every $k \geq n$, so taking the infimum over k yields

$$\forall n \geq N \quad w_n < u + \epsilon + V_n$$

Let n tend to ∞ in order to obtain

$$\liminf_{n \rightarrow \infty} (u_n + v_n) \leq u + \epsilon + \liminf_{n \rightarrow \infty} v_n$$

Since this holds for every positive ϵ , we get

$$\liminf_{n \rightarrow \infty} (u_n + v_n) \leq u + \liminf_{n \rightarrow \infty} v_n$$

But the converse inequality holds in general, as stated earlier. Hence

$$\liminf_{n \rightarrow \infty} (u_n + v_n) = u + \liminf_{n \rightarrow \infty} v_n$$

□

Seventh problem

Suppose that $(f_n)_{n \in \mathbb{N}}$ is a decreasing sequence of continuous functions on $[0, 1]$, converging pointwise to a continuous function f . Prove that the convergence is uniform.

Solution

Let ϵ be a positive real number. Define

$$\forall n \in \mathbb{N} \quad O_n = \{x \in [0, 1] \mid f_n(x) - f(x) < \epsilon\}$$

The sequence $(f_n)_{n \in \mathbb{N}}$ is decreasing. Thus, if $p < n$ are integers, and x is in O_p , we have

$$f_n(x) - f(x) < f_p(x) - f(x) < \epsilon$$

which proves that $x \in O_n$. So the sequence $(O_n)_{n \in \mathbb{N}}$ is increasing.

Since the $(f_n)_{n \in \mathbb{N}}$'s and f are continuous, all the $(O_n)_{n \in \mathbb{N}}$'s are open. Because $(f_n)_{n \in \mathbb{N}}$ converges pointwise to f , every $x \in [0, 1]$ belongs to at least one of these sets. So $(O_n)_{n \in \mathbb{N}}$ is an open cover of the compact set $[0, 1]$. Thus there is a finite subcover: there exist integers $n_1 \leq \dots \leq n_k$ such that

$$[0, 1] = (O_{n_1} \cup \dots \cup O_{n_k}) \subset (O_{n_k} \cup \dots \cup O_{n_k}) = O_{n_k}$$

But using again the fact that the $(O_n)_{n \in \mathbb{N}}$ are increasing, we get

$$\forall n \geq n_k \quad [0, 1] = O_n$$

In other words $\forall n \geq n_k \quad \forall x \in [0, 1] \quad 0 \leq f_n(x) - f(x) < \epsilon$

Conclusion

$(f_n)_{n \in \mathbb{N}}$ converges uniformly to f .

Eighth problem

Let $(c_n)_{n \in \mathbb{N}}$ be a sequence of complex numbers. Assume there are positive integers $N_k \xrightarrow[k \rightarrow \infty]{} \infty$ such that the trigonometric polynomials

$$P_k(t) = \sum_{j=-N_k}^{N_k} \left(1 - \frac{|j|}{N_k + 1}\right) c_j e^{ijt}$$

are nonnegative for all t . Prove that there exists a positive measure μ on \mathbb{T} such that

$$\forall n \in \mathbb{Z} \quad c_n = \int_{\mathbb{T}} e^{-int} d\mu(t)$$

Solution

For every integer k , let μ_k be the measure with density P_k relatively to the Lebesgue measure. Then μ_k is a positive measure and

$$\|\mu_k\| = \mu_k(\mathbb{T}) = \int_{\mathbb{T}} P_k(t) dt = 2\pi c_0$$

So $(\mu_k)_{k \in \mathbb{N}}$ is a bounded sequence in $\mathcal{M}(\mathbb{T})$ (the Banach space of signed measures on \mathbb{T}). Up to multiplying the $(\mu_k)_{k \in \mathbb{N}}$ by the constant $\frac{1}{2\pi c_0}$, we can assume that $\|\mu_k\| = 1$.

Since $\mathcal{M}(\mathbb{T})$ is the dual of the separable Banach space $\mathcal{C}(\mathbb{T})$ (this is Riesz' representation theorem), the unit ball of $\mathcal{M}(\mathbb{T})$, together with the topology $\star - \sigma(\mathcal{C}(\mathbb{T}), \mathcal{M}(\mathbb{T}))$, is metrizable. And by Banach-Alaoglu-Bourbaki, it is also compact. Thus any bounded sequence there, such as $(\mu_k)_{k \in \mathbb{N}}$, has a weak- \star convergent subsequence. So we can assume also that $(\mu_k)_{k \in \mathbb{N}}$ converges weakly to a measure μ . In particular, we have

$$\forall n \in \mathbb{N} \quad \int_{\mathbb{T}} e^{-int} d\mu(t) = \lim_{k \rightarrow \infty} \int_{\mathbb{T}} P_k(t) e^{-int} d\mu_k(t)$$

But $\forall k \in \mathbb{N} \quad \int_{\mathbb{T}} P_k(t) e^{-int} dt = \begin{cases} 0 & \text{if } N_k < n \\ \left(1 - \frac{|n|}{N_k + 1}\right) c_n & \text{if } N_k \geq n \end{cases}$

Hence

$$\boxed{\forall n \in \mathbb{Z} \quad \int_{\mathbb{T}} e^{-int} d\mu(t) = c_n}$$

Ninth problem

Let X be a vector space.

1. Prove that X has a Hamel basis (or algebraic basis).
2. If X is an infinite dimensional Banach space, prove that any Hamel basis must be uncountable.

Solution

1 Remember that a finite collection (x_1, \dots, x_n) of vectors in X is called linearly independent if and only if

$$\forall (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n \quad \left(\sum_{k=1}^n \lambda_k x_k = 0 \implies \lambda_1 = \dots = \lambda_n = 0 \right)$$

A set $L \subset X$ is called linearly independent if and only if every finite subset of L is linearly independent. A Hamel basis for X is a linearly independent subset of X that spans X .

Define $\mathcal{L} = \{L \subset X \mid L \text{ linearly independent}\}$

\mathcal{L} is partially ordered by inclusion. Let $\mathcal{L}' \subset \mathcal{L}$ be totally ordered, that is: every two elements of \mathcal{L}' can be compared and consider

$$\mathcal{L} = \bigcup_{L \in \mathcal{L}'} L$$

Let's show that $\mathcal{L} \in \mathcal{L}$, or in other words, that \mathcal{L} is linearly independent. Let $\{x_1, \dots, x_n\}$ be any finite subset of \mathcal{L} . Then

$$\forall k \in \{1, \dots, n\} \quad \exists L_k \in \mathcal{L}' \quad x_k \in L_k$$

Since \mathcal{L}' is totally ordered, we can suppose, up to renaming the x 's, that $L_1 \subset L_2 \subset \dots \subset L_n$. Therefore, $\{x_1, \dots, x_n\} \subset L_n$ is linearly independent. Which proves that \mathcal{L} is linearly independent. Thus we showed that \mathcal{L} is inductive: every totally ordered subset \mathcal{L}' of \mathcal{L} has an upper bound.

By Zorn's lemma, \mathcal{L} has a maximal element: there exists a maximal linearly independent family \mathcal{B} in X . Let's check that \mathcal{B} is a basis. Of course, by definition, \mathcal{B} is linearly independent. If $x \in X$ is not in $\text{Span } \mathcal{B}$, then $\mathcal{B} \subsetneq (\mathcal{B} \cup \{x\}) \in \mathcal{L}$, which contradicts the maximality of \mathcal{B} .

Span $\mathcal{B} = X$: X has a Hamel basis.

2 Let X be an infinite dimensional Banach space and suppose it has a countable Hamel basis $(x_n)_{n \in \mathbb{N}^*}$. Let

$$\forall n \in \mathbb{N}^* \quad X_n = \text{Span}(x_1, \dots, x_n)$$

so that

$$X = \bigcup_{n \in \mathbb{N}^*} X_n$$

Each X_n is closed. Indeed, let $(u_p)_{p \in \mathbb{N}}$ be a sequence in X_n , converging to some $u \in X$. Then $(u_p)_{p \in \mathbb{N}}$ is Cauchy; but we know that finite dimensional vector spaces are complete, thus $(u_p)_{p \in \mathbb{N}}$ has a limit $v \in X_n$. By unicity of the limit, $u = v$ and therefore $u \in X_n$.

By Baire's lemma, there has to be a positive integer N such that X_N has nonempty interior: there exists $x \in X_N$ and $\epsilon > 0$, such that $\mathcal{B}(x, 2\epsilon) \subset X_N$. We should deduce from this that $X_N = X$. Let $y \in X$ be nonzero. Then $x + \frac{\epsilon}{\|y\|}y$ is in the ball $\mathcal{B}(x, 2\epsilon) \subset X_N$. Therefore, since X_N is a vector space,

$$y = \left[\frac{\|y\|}{\epsilon} \left(x + \frac{\epsilon}{\|y\|}y \right) - \frac{\|y\|}{\epsilon}x \right] \in X_N$$

which proves

$$X \subset X_N$$

So X is finite dimensional and we have a contradiction.

A Hamel basis for an infinite dimensional Banach space is uncountable.

Tenth problem

Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded function such that

$$\exists K > 0 \quad \forall x, h \in \mathbb{R} \quad |f(x+h) + f(x-h) - 2f(x)| \leq K|h|$$

Prove that

$$\forall x, y \in \mathbb{R} \quad |f(x) - f(y)| \leq c \left(1 + \ln^+ \frac{1}{|x-y|} \right) |x-y|$$

where c is a constant depending only on K and $\|f\|_\infty$.

Solution

Fix an $x \in \mathbb{R}$ and define

$$\forall z \in \mathbb{R} \quad g(z) = f(z+x) - f(x)$$

Then

$$g(0) = 0 \quad \|g\|_\infty \leq 2\|f\|_\infty$$

$$\forall h \in \mathbb{R} \quad |g(z+h) + g(z-h) - 2g(z)| = |f(z+x+h) + f(z+x-h) - 2f(z+x)| \leq K|h|$$

Fix a $z \in \mathbb{R}$, let n be any integer; by (1):

$$|g(2^n z + 2^n z) + g(2^n z - 2^n z) - 2g(2^n z)| \leq 2^n K |z|$$

or

$$\left| \frac{g(2^{n+1}z)}{2^{n+1}} - \frac{g(2^n z)}{2^n} \right| \leq \frac{K|z|}{2}$$

after dividing both sides by 2^{n+1} . This is true for every integer n , therefore we can sum all these inequalities between $n = 0$ and $n = N - 1$ for some positive integer N :

$$\left| \sum_{n=0}^{N-1} \left(\frac{g(2^{n+1}z)}{2^{n+1}} - \frac{g(2^n z)}{2^n} \right) \right| \leq \sum_{n=0}^{N-1} \left| \frac{g(2^{n+1}z)}{2^{n+1}} - \frac{g(2^n z)}{2^n} \right| \leq \frac{KN|z|}{2}$$

The lefthandside is a telescoping series and we are left with

$$\left| \frac{g(2^N z)}{2^N} - g(z) \right| \leq \frac{KN|z|}{2}$$

Now use the triangle inequality $|a - b| \geq |a| - |b|$:

$$|g(z)| - \frac{|g(2^N z)|}{2^N} \leq \frac{KN|z|}{2}$$

and finally

$$|g(z)| \leq \frac{\|g\|_\infty}{2^N} + \frac{KN|z|}{2} \leq \frac{2\|f\|_\infty}{2^N} + \frac{KN|z|}{2} \tag{2}$$

We consider now two cases, depending on $|z|$:

- **If $|z| \geq 1$:** Just take $N = 1$ in **(2)** in order to get

$$|g(z)| \leq \|f\|_\infty + \frac{K|z|}{2} \leq \left(\|f\|_\infty + \frac{K}{2}\right)|z|$$

- **If $|z| < 1$:** Choose N to be the only positive integer such that

$$\frac{1}{2^{N+1}} < |z| \leq \frac{1}{2^N}$$

that is

$$N \leq \frac{\ln 1/|z|}{\ln 2} < N + 1$$

and see what happens in **(2)**:

$$|g(z)| \leq 4\|f\|_\infty|z| + \frac{K|z|}{2\ln 2} \ln \frac{1}{|z|}$$

Letting c be some bigass constant, for example

$$c = \text{Max}\left(4\|f\|_\infty, \frac{K}{2\ln 2}, \|f\|_\infty + \frac{K}{2}\right)$$

we get an inequality that combines both cases:

$\forall x, z \in \mathbb{R} \quad g(z) = f(x+z) - f(x) \leq c\left(1 + \ln^+ \frac{1}{ z }\right) z $
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