

Ph.D. Qualifying Exam – Spring 2003

First problem

Let f be a continuous function on the unit square $Q = [0, 1]^2$ and for $s \in [0, 1]$, let

$$g(s) = \text{Max} \{f(s, t) \mid t \in [0, 1]\}$$

1. Show that g is a continuous function on $[0, 1]$.
2. Prove that if $|f(x) - f(y)| \leq M|x - y|$ for x and y in Q , then $|g(s_1) - g(s_2)| \leq M|s_1 - s_2|$ for s_1 and s_2 in $[0, 1]$.
3. Give an example in which f is $\mathcal{C}^1(Q)$ but g is not $\mathcal{C}^1([0, 1])$.

Solution

[a] f is continuous on the compact Q , therefore it is uniformly continuous. So if ϵ is a given positive number, there exists $\eta > 0$ such that

$$\forall (s_1, t_1), (s_2, t_2) \in Q \quad (|s_1 - s_2| \leq \eta \quad \text{and} \quad |t_1 - t_2| \leq \eta) \implies |f(s_1, t_1) - f(s_2, t_2)| \leq \epsilon$$

Let s_1 and s_2 be any two elements of $[0, 1]$, such that $|s_1 - s_2| \leq \eta$. By definition of $g(s_1)$,

$$\forall t \in [0, 1] \quad f(s_1, t) \leq g(s_1)$$

By uniform continuity of f ,

$$\forall t \in [0, 1] \quad |f(s_1, t) - f(s_2, t)| \leq \epsilon$$

$$\text{In particular} \quad \forall t \in [0, 1] \quad f(s_2, t) \leq f(s_1, t) + \epsilon \leq g(s_1) + \epsilon$$

Taking the max over t in the lefthandside yields

$$g(s_2) \leq g(s_1) + \epsilon$$

Since s_1 and s_2 play symmetric roles, we obtain in the same way that

$$g(s_1) \leq g(s_2) + \epsilon$$

$$\text{Therefore} \quad |g(s_1) - g(s_2)| \leq \epsilon \quad \text{as soon as} \quad |s_1 - s_2| \leq \eta$$

Conclusion:

$$g \text{ is uniformly continuous on } [0, 1].$$

[b] Suppose now that f is M -Lipschitz on Q , which means that

$$\forall (s_1, t_1), (s_2, t_2) \in Q \quad |f(s_1, t_1) - f(s_2, t_2)| \leq M(|s_1 - s_2| + |t_1 - t_2|)$$

Let s_1 and s_2 be in $[0, 1]$. Since $t \mapsto f(s_1, t)$ is continuous on the compact set $[0, 1]$, the Max in the definition of g is attained (this is actually implicit in the definition) at some $t_1 \in [0, 1]$. Then

$$|g(s_1) - f(s_2, t_1)| = |f(s_1, t_1) - f(s_2, t_1)| \leq M|s_1 - s_2|$$

In particular $g(s_1) \leq M|s_1 - s_2| + f(s_2, t_1) \leq M|s_1 - s_2| + g(s_2)$

Since s_1 and s_2 play symmetric roles, we could show as well that

$$g(s_2) \leq M|s_1 - s_2| + g(s_1)$$

Therefore

$$\boxed{\forall s_1, s_2 \in [0, 1] \quad |g(s_1) - g(s_2)| \leq M|s_1 - s_2|}$$

[c] Take $\forall (x, y) \in [0, 1]^2 \quad f(x, y) = \left(\frac{1}{2} - x\right)\left(\frac{1}{2} - y\right)$

Then $\forall x \in [0, 1] \quad g(x) = \frac{1}{2} \left| \frac{1}{2} - x \right|$

which is not \mathcal{C}^1 , while f is.

Second problem

Suppose (X, d) is a metric space without isolated points, such that every continuous function $f : X \rightarrow [0, 1]$ is uniformly continuous. Show that X is compact.

Solution

Suppose that X is not compact. Then there is a sequence $(u_n)_{n \in \mathbb{N}} \subset X$ that has no convergent subsequence; up to removing multiple occurrences of terms, we can suppose that all terms of the sequence are pairwise distinct.

Denote by \mathcal{U} the collection $\{u_n \mid n \in \mathbb{N}\}$. This set is closed. Indeed, suppose that a sequence of elements of \mathcal{U} converges to some $x \in X$. If this sequence contains infinitely many elements of \mathcal{U} , then we can construct a convergent subsequence of $(u_n)_{n \in \mathbb{N}}$, which is impossible. Therefore our sequence only takes finitely many values; since it converges, it is eventually constant and therefore $x \in \mathcal{U}$.

Let p be an integer and define $\mathcal{U}_p = \mathcal{U} \setminus \{u_p\}$. Then \mathcal{U}_p is closed as well (the sequence $(u_n)_{n \neq p}$ has no convergent subsequence) and the distance of u_p to \mathcal{U}_p is thus positive. Let ϵ_p be a positive number such that

$$0 < \epsilon_p < \text{Min}\left(\frac{1}{p}, d(u_p, \mathcal{U}_p)\right)$$

Because u_p is not isolated, the ball $\mathcal{B}(u_p, \epsilon_p)$ contains some v_p other than u_p .

The sequence $(v_p)_{p \in \mathbb{N}}$ constructed has no convergent subsequence (if it had, since $d(v_p, u_p) < \frac{1}{p}$, then $(u_p)_{p \in \mathbb{N}}$ would have a convergent subsequence) and therefore the set $\mathcal{V} = \{v_p \mid p \in \mathbb{N}\}$ is closed. A union of closed sets is closed and therefore $\mathcal{F} = \mathcal{U} \cup \mathcal{V}$ is closed. Define a function f on \mathcal{F} as follows:

$$\forall n \in \mathbb{N} \quad f(u_p) = 0 \quad \text{and} \quad f(v_p) = 1$$

This function is continuous on \mathcal{F} since it only takes the values 0 and 1, and

$$f^{-1}(\{1\}) = \mathcal{V} \quad \text{and} \quad f^{-1}(\{0\}) = \mathcal{U}$$

which are closed sets. By Tietze's extension theorem there exists a continuous function extending f . For convenience, this new function is still denoted by f .

We finally remark that f cannot be uniformly continuous. Indeed, if η is a positive real number, there exists p such that $\epsilon_p < \eta$. Then

$$|u_p - v_p| < \epsilon_p < \eta \quad \text{and} \quad |f(u_p) - f(v_p)| = 1$$

This contradicts the hypothesis on X , that every continuous function is uniformly continuous. Hence

X is compact.

Third problem

Suppose X and Y are Banach spaces and $T : X \rightarrow Y$ is linear. Prove that T is bounded in each of the following cases:

1. If there is a family \mathcal{F} of real continuous linear functionals on Y such that $f \circ T$ is continuous for each $f \in \mathcal{F}$ and $\bigcap_{f \in \mathcal{F}} f^{-1}(0) = \{0\}$.
2. If there are closed sets $(A_n)_{n \in \mathbb{N}}$ with $\bigcup_{n \in \mathbb{N}} A_n = X$ and with $T(A_n)$ bounded subset of Y for every integer n .

Solution

[a] Suppose such a family \mathcal{F} exists. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in X that converges to some $x \in X$ and such that $(Tx_n)_{n \in \mathbb{N}}$ converges to some $y \in Y$. Because $f \circ T$ is continuous for every $f \in \mathcal{F}$, we have

$$\forall f \in \mathcal{F} \quad \lim_{n \rightarrow \infty} f \circ T(x_n) = f \circ T(x)$$

Also, because $\lim_{n \rightarrow \infty} Tx_n = y$ and all f 's in \mathcal{F} are continuous, we have

$$\forall f \in \mathcal{F} \quad \lim_{n \rightarrow \infty} f \circ T(x_n) = f(y)$$

Therefore $\forall f \in \mathcal{F} \quad f(Tx - y) = 0$

which means that $Tx - y \in \bigcap_{f \in \mathcal{F}} f^{-1}(0) = \{0\}$

Hence $Tx = y$

which shows that T has closed graph.

T is continuous.

[b] Suppose we are given those closed sets $(A_n)_{n \in \mathbb{N}}$. By Baire's theorem, since their union is equal to X , one of them, say A_N , has nonempty interior:

$$\exists x_0 \in A_N \quad \exists \epsilon > 0 \quad \mathcal{B}(x_0, \epsilon) \subset A_N$$

Because A_N is closed, the closed ball $\overline{\mathcal{B}(x_0, \epsilon)}$ is also included in A_N . We know as well that $T(A_N)$ is bounded:

$$\exists M > 0 \quad \forall x \in A_N \quad \|Tx\| \leq M$$

Let x be a nonzero element of X . Then $x_0 + \frac{\epsilon x}{\|x\|}$ is in the closed ball of center x and radius ϵ . Thus

$$\left\| \mathbb{T}x_0 + \frac{\epsilon}{\|x\|} \mathbb{T}x \right\| \leq M$$

By the triangle inequality $\|a\| - \|b\| \leq \|a - b\|$,

$$\frac{\epsilon}{\|x\|} \|\mathbb{T}x\| - \|\mathbb{T}x_0\| \leq M$$

and

$$\|\mathbb{T}x\| \leq \frac{M + \|\mathbb{T}x_0\|}{\epsilon} \|x\| \leq \frac{2M}{\epsilon} \|x\|$$

Conclusion:

T is bounded.

Fourth problem

Suppose $T : X \rightarrow Y$ is a compact operator between Banach spaces. Prove that the adjoint operator $T^* : Y^* \rightarrow X^*$ is also compact.

Solution

Since T is compact, the set $\mathcal{K} = \overline{T(\mathcal{B}_X)}$ is compact. The proof of the compactness of T^* relies on the following important observation. If f is a bounded linear functional on Y , then f is in particular continuous on the compact set \mathcal{K} . Thus it makes sense to consider $\|f\|_\infty = \sup_{y \in \mathcal{K}} |f(y)|$. But since \mathcal{K} is the closure of $T(\mathcal{B}_X)$, we also have

$$\|f\|_\infty = \sup_{y \in T(\mathcal{B}_X)} |f(y)| = \sup_{x \in \mathcal{B}_X} |f(Tx)| = \sup_{x \in \mathcal{B}_X} |T^*(f)x|$$

and we recognize here the norm of T^*f as an element of the dual of X . So our important observation is that

$$\|f\|_\infty = \|T^*f\|_{X^*}$$

Now, consider a sequence $(f_n)_{n \in \mathbb{N}}$ of elements of the unit ball of Y^* . First notice that

$$\forall n \in \mathbb{N} \quad \|f_n\|_\infty = \|T^*f_n\| \leq \|T^*\| \|f_n\| \leq \|T^*\|$$

so $(f_n)_{n \in \mathbb{N}}$ is bounded in $\mathcal{C}(\mathcal{K})$. Furthermore, this sequence is uniformly equicontinuous on \mathcal{K} since

$$\forall y_1, y_2 \in \mathcal{K} \quad |f_n(y_1) - f_n(y_2)| = |f_n(y_1 - y_2)| \leq \|f_n\| \|y_1 - y_2\| \leq \|y_1 - y_2\|$$

So by Arzela-Ascoli, there $(f_n)_{n \in \mathbb{N}}$ has a converging subsequence in $\mathcal{C}(\mathcal{K})$. So $(T^*f_n)_{n \in \mathbb{N}}$ has a subsequence which is Cauchy, and therefore convergent in X^* . This achieves showing that

T^* is compact.

Fifth problem

A sequence $(\xi_j)_{j \in \mathbb{N}} \subset [0, 1]$ is said to be uniformly distributed in $[0, 1]$ if

$$\forall f \in \mathcal{C}([0, 1]) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n f(\xi_j) = \int_{[0,1]} f$$

Prove that $(\xi_j)_{j \in \mathbb{N}}$ is uniformly distributed in $[0, 1]$ if

$$\forall m \in \mathbb{Z} \setminus \{0\} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n e^{2\pi i m \xi_j} = 0$$

Solution

First suppose that $(\xi_j)_{j \in \mathbb{N}}$ is uniformly distributed, let m be a nonzero integer. Then, since $x \mapsto e^{2\pi i m x}$ is continuous on $[0, 1]$, we know that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n e^{2\pi i m \xi_j} = \int_0^1 e^{2\pi i m x} dx = 0$$

Conversely, suppose that this property holds for every $m \neq 0$. When $m = 0$, we get:

$$\forall n \in \mathbb{N} \quad \frac{1}{n} \sum_{j=1}^n e^{2\pi i \times 0 \times \xi_j} = 1$$

so we can actually write in general that

$$\forall m \in \mathbb{Z} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n e^{2\pi i m \xi_j} = \int_0^1 e^{2\pi i m x} dx$$

or

$$\forall g \in \mathcal{P} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n g(\xi_j) = \int_{[0,1]} g$$

where \mathcal{P} is the set of trigonometric polynomials on $[0, 1]$ (finite linear combinations of functions of the type $x \mapsto e^{2\pi i m x}$).

Now, we know that \mathcal{P} is dense in the set of continuous functions on $[0, 1]$ that take the same value at 0 and 1. And

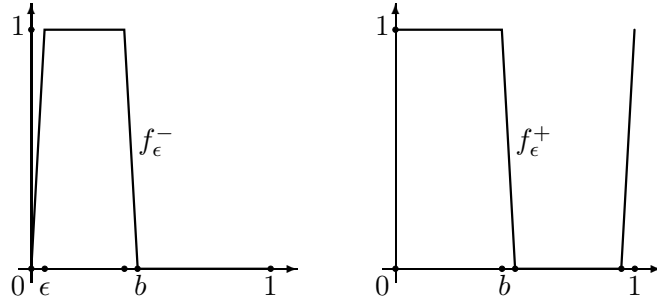
$$g \mapsto \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n g(\xi_j)$$

is a bounded linear functional on \mathcal{P} , coinciding with integration against Lebesgue measure. Therefore, it extends to $\overline{\mathcal{P}}$ and

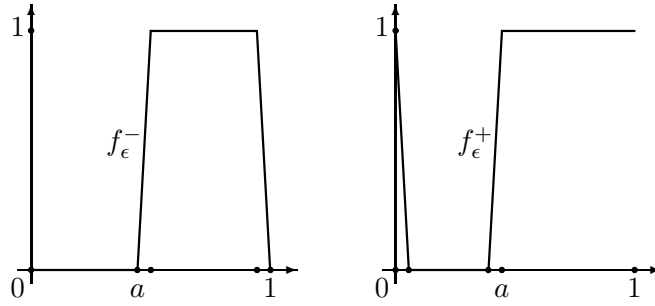
$$\forall f \in \mathcal{C}([0, 1]) \quad f(0) = f(1) \quad \implies \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n f(\xi_j) = \int_{[0,1]} f$$

Now we have to extend this somehow to all continuous functions on $[0, 1]$, regardless of the values at 0 and 1. Let $0 \leq a < b \leq 1$ and $f = \mathbb{1}_{[a,b]}$. If $a = 0$ and $b = 1$, we have the constant function equal to 1 and we know that everything works fine. So suppose we are not in this case. We define function f_ϵ^- and f_ϵ^+ as follows:

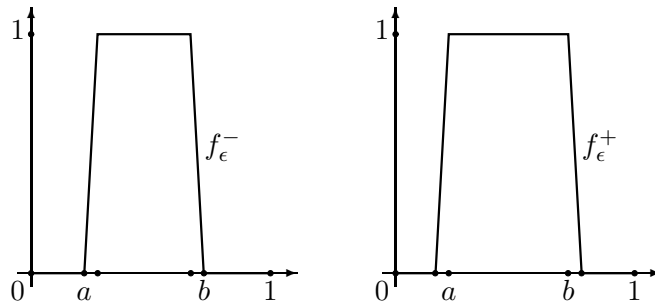
- **If $a = 0$:** Take ϵ small enough so that $0 < \epsilon < b - \epsilon < b + \epsilon < 1 - \epsilon$ and define



- **If $b = 1$:** Take ϵ small enough so that $0 < \epsilon < a - \epsilon < a + \epsilon < 1$ and define



- **If $0 < a < b < 1$:** Take ϵ small enough so that $0 < a - \epsilon < a + \epsilon < b - \epsilon < b + \epsilon < 1$ and define



In any case, the functions f_ϵ^- and f_ϵ^+ have been chosen so that

$$f_\epsilon^-(0) = f_\epsilon^-(1) \quad f_\epsilon^+(0) = f_\epsilon^+(1) \tag{1}$$

$$\forall x \in [0, 1] \quad f_\epsilon^-(x) < f(x) < f_\epsilon^+(x) \tag{2}$$

and

$$\int_{[0,1]} f_\epsilon^- = \int_{[0,1]} f - \epsilon \quad \int_{[0,1]} f_\epsilon^+ = \int_{[0,1]} f + \epsilon \tag{3}$$

From (2),

$$\forall n \in \mathbb{N}^* \quad \frac{1}{n} \sum_{j=1}^n f_\epsilon^-(\xi_j) < \frac{1}{n} \sum_{j=1}^n f(\xi_j) < \frac{1}{n} \sum_{j=1}^n f_\epsilon^+(\xi_j)$$

Because of (1), the left- and righthand side have limits as n tends to ∞ that are respectively $\int f_\epsilon^-$ and $\int f_\epsilon^+$. Thus

$$\int_{[0,1]} f_\epsilon^- \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n f(\xi_j) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n f(\xi_j) \leq \int_{[0,1]} f_\epsilon^+$$

And finally, using (3):

$$-\epsilon + \int_{[0,1]} f \leq \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n f(\xi_j) \leq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n f(\xi_j) \leq \epsilon + \int_{[0,1]} f$$

Since this is true for every (small enough) positive ϵ , we deduce that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n f(\xi_j) \text{ exists and } \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n f(\xi_j) = \int_{[0,1]} f$$

Let \mathcal{S} be the set of step functions on $[0, 1]$, that is linear combinations of indicator functions of intervals. Because integrating and taking limits are linear operations, and a, b were arbitrary, it follows that

$$\forall f \in \mathcal{S} \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n f(\xi_j) = \int_{[0,1]} f$$

Finally, \mathcal{S} is dense in $\mathcal{C}([0, 1])$. Therefore,

$$\forall f \in \mathcal{C}([0, 1]) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^n f(\xi_j) = \int_{[0,1]} f$$

In other words

The sequence $(\xi_j)_{j \in \mathbb{N}}$ is uniformly distributed.

Sixth problem

If X is a finite dimensional real vector space, prove that all norms on X are equivalent.

Solution

Call n the dimension of X and let (e_1, \dots, e_n) be a basis for X . Any vector $x \in X$ has a unique decomposition long this basis:

$$\exists! (x_1, \dots, x_n) \in \mathbb{R}^n \quad x = \sum_{i=1}^n x_i e_i$$

If we define $\|x\|_\infty = \text{Max} \{|x_1|, \dots, |x_n|\}$

we know that $\|\cdot\|_\infty$ is a norm on X , and that closed bounded sets are compact for this norm.

Let $\|\cdot\|$ be any other norm. Then by the triangle inequality

$$\|x\| \leq |x_1| \|e_1\| + \dots + |x_n| \|e_n\| \leq C \|x\|_\infty$$

with

$$C = \sum_{i=1}^n \|e_i\|$$

We deduce from this that

$$\forall x, y \in X \quad \left| \|x\| - \|y\| \right| \leq \|x - y\| \leq C \|x - y\|_\infty$$

which implies that $\|\cdot\|$ is continuous on the normed space $(X, \|\cdot\|_\infty)$. Since the unit sphere in X for $\|\cdot\|_\infty$ is closed and bounded, it is compact, and $\|\cdot\|$ has a minimum on this set:

$$\exists m \geq 0 \quad \forall x \in X \quad \|x\|_\infty = 1 \quad \implies \quad m \leq \|x\|$$

We know that m is attained, which excludes the possibility for it to be 0 (since $\|a\| = 0$ implies that $a = 0$); thus $m > 0$.

Now let x be any nonzero elements of X . Then $\frac{x}{\|x\|_\infty}$ has ∞ -norm equal to 1 and therefore

$$m \leq \left\| \frac{x}{\|x\|_\infty} \right\| = \frac{\|x\|}{\|x\|_\infty}$$

Conclusion: $\forall x \in X \quad m \|x\|_\infty \leq \|x\| \leq C \|x\|_\infty$

and the two norms $\|\cdot\|_\infty$ and $\|\cdot\|$ are equivalent. Since equivalence between norms is actually an equivalence relationship, we deduce that

All norms on X are equivalent.

Seventh problem

1. Prove that a weakly compact subset of a normed space is bounded.
2. In the Hilbert space $L^2(0, 1)$, give an example of a countable closed bounded subset that is not weakly closed. Justify your answer (no kidding?).

Solution

1 Remember that the weak topology on X is the weakest topology that makes all bounded linear functionals continuous. We also know from class that X injects isometrically in X^{**} by defining, for every $x \in X$:

$$\forall f \in X^* \quad Tx(f) = f(x)$$

We then have $\|x\|_X = \|Tx\|_{X^{**}}$

Let K be a weakly compact set. If $f \in X^*$, since it is (weakly) continuous on K (weakly) compact, it is bounded on this set:

$$\forall f \in X^* \quad \exists M_f > 0 \quad \forall x \in K \quad |f(x)| = |T(x)f| \leq M_f$$

Applying the Banach-Steinhaus theorem (principle of uniform boundedness), we get that the family of operators $(Tx)_{x \in K}$ is uniformly bounded:

$$\exists M > 0 \quad \forall x \in K \quad \|Tx\| \leq M$$

But since T is an isometry, as explained earlier, this means exactly that K is bounded.

A weakly compact set is bounded.

Let us recall the statement of the Banach-Steinhaus theorem:

Theorem: Let E be a normed space and B be a Banach space. Let $(T_a)_{a \in A}$ be any family of bounded linear maps from B to E , such that

$$\forall x \in B \quad \sup_{a \in A} \|T_a x\| < \infty$$

Then $(T_a)_{a \in A}$ is uniformly bounded:

$$\exists M > 0 \quad \forall a \in A \quad \|T_a\| \leq M$$

In this problem, we apply this theorem with $B = X^*$ (which is always complete, even if X is not), $E = \mathbb{R}$, and the family $(Tx)_{x \in K}$.

b For every integer n , let $e_n : x \mapsto e^{2\pi i n x}$ and $E = \{e_n \mid n \in \mathbb{N}\}$. E is closed (all points are isolated) and bounded since $\|e_n\| = 1$ for every n . But E is not weakly closed: indeed, by the L^2 version of Riemann-Lebesgue's lemma, $(e_n)_{n \in \mathbb{N}}$ converges weakly to 0 in $L^2(0, 1)$, but 0 is not in E .

Eighth problem

Let μ be a finite positive Borel measure on $[0, 1]$.

1. Prove that there is an increasing function α on $(0, 1)$ such that

$$\forall f \in \mathcal{C}_c^1(0, 1) \quad \int_{[0,1]} f \, d\mu = - \int_{[0,1]} f' \alpha$$

2. In case μ is non-atomic, prove that α is unique up to an additive constant, and is continuous.

Solution

1 Simply define

$$\forall t \in [0, 1] \quad \alpha(t) = \mu([0, t])$$

Since μ is positive, α is of course an increasing function.

If f is a \mathcal{C}^1 function on $(0, 1)$, with compact support, we have in particular $f(1) = 0$ and therefore

$$\forall x \in [0, 1] \quad f(x) = - \int_x^1 f'(t) \, dt = \int_{[0,1]} f' \mathbf{1}_{[x,1]}$$

Then
$$\int_{[0,1]} f \, d\mu = - \int_{[0,1]} \int_{[0,1]} f'(t) \mathbf{1}_{[x,1]}(t) \, dt \, d\mu(x) \tag{1}$$

Remark that
$$\forall x, t \in [0, 1] \quad \mathbf{1}_{[x,1]}(t) = \mathbf{1}_{[0,t]}(x)$$

All that is left is showing that we can Fubini the double integral in (1):

$$\int_{[0,1]} \int_{[0,1]} |f'(t) \mathbf{1}_{[x,1]}(t)| \, dt \, d\mu(x) \leq \|f'\|_\infty \mu([0, 1]) < \infty$$

Therefore
$$\int_{[0,1]} f \, d\mu = - \int_{[0,1]} \int_{[0,1]} f'(t) \mathbf{1}_{[0,t]}(x) \, d\mu(x) \, dt = - \int_0^1 f'(t) \alpha(t) \, dt$$

2 First we establish that α , defined in part 1, is continuous, under the assumption that μ does not charge points. Remember that

$$\forall t \in [0, 1] \quad \alpha(t) = \mu([0, t])$$

Let $t \in [0, 1]$ and define

$$\forall h \in [0, 1 - t] \quad \varphi(h) = \alpha(t + h) - \alpha(t) = \mu((t, t + h])$$

Then φ is increasing, because μ is a positive measure, bounded below by 0 at 0, and it thus has a non-negative limit ℓ at 0. Now that we know that this limit exists, we can write that

$$\ell = \lim_{n \rightarrow \infty} g\left(\frac{1}{n}\right) = \lim_{n \rightarrow \infty} \mu\left(\left(t, t + \frac{1}{n}\right]\right) = \mu\left(\bigcap_{n \in \mathbb{N}} \left(t, t + \frac{1}{n}\right]\right) = \mu(\emptyset) = 0$$

Hence
$$\lim_{h \rightarrow 0^+} (\alpha(t+h) - \alpha(t)) = 0$$

which shows that α is right-continuous at t . To prove left-continuity, we do the same job and find that

$$\lim_{h \rightarrow 0^-} (\alpha(t+h) - \alpha(t)) = \mu(\{t\}) = 0$$

α is continuous on $[0, 1]$

Next we show that two functions satisfying the same property as α differ by a constant. Let β be an increasing function on $[0, 1]$ such that

$$\forall f \in \mathcal{C}_c^1(0, 1) \quad \int_{[0,1]} f \, d\mu = - \int_{[0,1]} f' \beta$$

Then
$$\forall t \in [0, 1] \quad \beta(0) \leq \beta(t) \leq \beta(1)$$

so that
$$\forall t \in [0, 1] \quad |\beta(t)| \leq \text{Max}(|\beta(0)|, |\beta(1)|)$$

Therefore β is integrable on $[0, 1]$. Similarly, the α we found in part 1 is also integrable on $[0, 1]$. Therefore, $\gamma = \alpha - \beta \in L^1(0, 1)$. Since

$$\forall f \in \mathcal{C}_c^1(0, 1) \quad \int_{[0,1]} f \, d\mu = - \int_{[0,1]} f' \alpha = - \int_{[0,1]} f' \beta$$

we have
$$\forall f \in \mathcal{C}_c^1(0, 1) \quad \int_{[0,1]} f' \gamma = 0$$

Let f be any function in $\mathcal{C}_c(0, 1)$ and define

$$\forall t \in [0, 1] \quad g(t) = \int_0^t f(x) \, dx - t \int_0^1 f(x) \, dx$$

Then g is in $\mathcal{C}_c^1(0, 1)$ and

$$\forall t \in [0, 1] \quad g'(t) = f(t) - \int_0^1 f(x) \, dx$$

Then
$$0 = \int_{[0,1]} g' \gamma = \int_0^1 f(t) \gamma(t) \, dt - \int_0^1 \int_0^1 f(x) \, dx \gamma(t) \, dt$$

Intervert the order of integration and combine the integrals again:

$$0 = \int_0^1 f(t) \left(\gamma(t) - \int_0^1 \gamma(x) dx \right) dt$$

This holds for every continuous function with compact support in $(0, 1)$, therefore $\gamma - \int_0^1 \gamma(x) dx = 0$ almost everywhere. That is, γ is almost everywhere constant, or $\beta = \alpha + C$ almost everywhere.

Since $\alpha + C$ is continuous and β is increasing, it is clear that β has to be continuous as well (this can be checked properly with ϵ 's if you want, I won't do it here).

The α found in part 1 is continuous, and unique up to constants.

Ninth problem

Prove that the following integrals converge to 0 as n goes to ∞ :

1. $\int_0^n \frac{\cos nx \, dx}{\sqrt{x(1+n^2x^2)}}$;
2. $\int_0^1 \frac{n(1-x)^2}{(1+nx)\ln^2 x} \cos nx \, dx$.

Solution

[1] Let's first justify that the integral exists. Let n be a fixed positive integer. The function

$$f_n : x \mapsto \frac{\cos nx}{\sqrt{x(1+n^2x^2)}}$$

is continuous on $(0, n]$ and

$$|f_n(x)| \underset{x \rightarrow 0}{\sim} \frac{1}{\sqrt{x}}$$

which is integrable at 0. Therefore, f_n is integrable on $(0, n]$. Let $I_n = \int_0^n f_n(x) \, dx$ and make the substitution $u = nx$:

$$I_n = \int_0^{n^2} \frac{\cos u}{\sqrt{\frac{u}{n}(1+u^2)}} \frac{du}{n} = \frac{1}{\sqrt{n}} \int_0^{n^2} \frac{\cos u}{\sqrt{u(1+u^2)}} \, du$$

The function

$$f : u \mapsto \frac{\cos u}{\sqrt{u(1+u^2)}}$$

is continuous on $(0, +\infty)$, integrable because

$$|f(u)| \underset{u \rightarrow 0}{\sim} \frac{1}{\sqrt{u}} \quad \text{and} \quad |f(u)| \underset{u \rightarrow +\infty}{=} O\left(\frac{1}{u^{3/2}}\right)$$

Therefore

$$|I_n| \leq \frac{\|f\|_1}{\sqrt{n}}$$

and

$$\lim_{n \rightarrow \infty} \int_0^n \frac{\cos nx}{\sqrt{x(1+n^2x^2)}} \, dx = 0$$

[2] Again, we start by justifying the existence of the considered integral. If n is an integer, define

$$\forall x \in [0, 1] \quad f_n(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{n}{1+n} & \text{if } x = 1 \\ \frac{n(1-x)^2}{(1+nx)\ln^2 x} & \text{otherwise} \end{cases}$$

Then f_n is continuous on $(0, 1)$ by theorems about operations on continuous functions. It is also continuous at 0 since

$$f_n(x) \underset{x \rightarrow 0}{\sim} \frac{1}{\ln^2 x} \xrightarrow{x \rightarrow 0} 0$$

and at 1 because

$$f_n(x) \underset{x \rightarrow 1}{\sim} \frac{n}{1+n} \left(\frac{1-x}{\ln x} \right)^2 \xrightarrow{x \rightarrow 1} \frac{n}{1+n}$$

$$\left| \text{Remember that} \quad \ln x = \ln(1 + (x-1)) = x-1 + o(x-1) \right.$$

So f_n actually is continuous on $[0, 1]$ and therefore $L^1(0, 1)$. Let's see now what happens as n goes to ∞ :

$$\forall x \in (0, 1) \quad f_n(x) = \frac{n(1-x)^2}{(1+nx)\ln^2 x} \xrightarrow{n \rightarrow \infty} \frac{(1-x)^2}{x \ln^2 x}$$

Define the righthand side to be f and notice that f is in $L^1(0, 1)$.

$$\left| \text{An antiderivative of } \frac{1}{x \ln^2 x} \text{ is } -\frac{1}{\ln x}, \text{ which has the limit } 0 \text{ at } 0. \right.$$

We then showed that $(f_n)_{n \in \mathbb{N}}$ converges almost everywhere to f . We also have that, for fixed x , $(f_n(x))_{n \in \mathbb{N}}$ is increasing. This follows at once from the identity

$$\frac{n}{1+nx} = \frac{1}{x} - \frac{1}{x(1+nx)}$$

From the monotone convergence theorem $(f_n)_{n \in \mathbb{N}}$ actually converges in L^1 to f .

Now let's get back to our problem. The integrals we are looking at are the

$$I_n = \int_0^1 f_n(x) \cos nx \, dx = \int_0^{2\pi} f_n(x) \chi(x) \cos nx \, dx \quad \text{where} \quad \chi = \mathbf{1}_{[0,1]}$$

We recognize here the real part of $\widehat{f_n \chi}(n)$, the n -th Fourier coefficient of $f_n \chi$. We have then

$$|\widehat{f_n \mathbf{1}_{[0,1]}}(n)| \leq |(\widehat{f - f_n})\chi(n)| + |\widehat{f \chi}(n)| \leq \|f - f_n\|_1 + |\widehat{f \chi}(n)|$$

The first term tends to 0 since $f_n \chi \xrightarrow[n \rightarrow \infty]{L^1} f$; the second term tends to 0 by Riemann-Lebesgue. Hence

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{n(1-x)^2}{(1+nx) \ln^2 x} \cos nx \, dx = 0$$

Tenth problem

Prove that if $\alpha \in (0, 1)$ and if f is an $L^2(\mathbb{T})$ function such that $\sum_{|n| \geq N} |\widehat{f}(n)| \leq N^{-\alpha}$ for each $N \geq 1$, then the L^2 class of f has a Hölder continuous representative with exponent α .

Solution

We start by proving the hint, which was:

Proposition: *There exists a $C > 0$, depending only on α , such that*

$$\forall N \in \mathbb{N} \quad \sum_{|n| \leq N} |n\widehat{f}(n)| \leq CN^{1-\alpha}$$

Proof: For every positive integer n , define

$$S_n = \sum_{|p| \geq n} |\widehat{f}(p)|$$

so that we know that

$$S_n \leq \frac{1}{n^\alpha}$$

Then

$$S_{n+1} = S_n - |\widehat{f}(n)| - |\widehat{f}(-n)|$$

or

$$|\widehat{f}(n)| + |\widehat{f}(-n)| = S_n - S_{n+1}$$

Let N be a positive integer. We have

$$\begin{aligned} \sum_{|n| \leq N} |n\widehat{f}(n)| &= \sum_{n=1}^N n(|\widehat{f}(n)| + |\widehat{f}(-n)|) = \sum_{n=1}^N n(S_n - S_{n+1}) \\ &= \sum_{n=1}^N nS_n - \sum_{n=1}^N nS_{n+1} = \sum_{n=1}^N nS_n - \sum_{n=2}^{N+1} (n-1)S_n \\ &= \sum_{n=2}^N S_n + S_1 - S_{N+1} \leq \sum_{n=1}^N S_n \\ \sum_{|n| \leq N} |n\widehat{f}(n)| &\leq \sum_{n=1}^N \frac{1}{n^\alpha} \end{aligned}$$

This manipulation is called the Abel transform, and it can be regarded as a discrete integration by parts. It is a very powerful technique for estimating partial sums of, or studying, series.

The sum on the righthandside is estimated by comparison with an integral: on the interval $[n-1, n]$, the function $x \mapsto \frac{1}{x^\alpha}$ is decreasing, hence bigger than $\frac{1}{n^\alpha}$. Therefore

$$\sum_{n=1}^N \frac{1}{n^\alpha} \leq \sum_{n=1}^N \int_{n-1}^n \frac{dx}{x^\alpha} \leq \int_0^N \frac{dx}{x^\alpha} = \frac{N^{1-\alpha}}{1-\alpha}$$

This achieves the proof of the proposition; our C is $\frac{1}{1-\alpha}$. \square

Now let's get back to our problem. Since

$$\forall N \in \mathbb{N} \quad S_N \leq \frac{1}{N^\alpha}$$

we know that the series $\sum |\widehat{f}(n)|$ converges. This implies that $\sum \widehat{f}(n) e^{inx}$ converges absolutely and uniformly on \mathbb{T} and thus defines a continuous function f_0 . Since f_0 has the same Fourier coefficients as f , these functions are equal:

The equivalence class of f in $L^2(\mathbb{T})$ contains a continuous function f_0 .

We are going to show that this representative is Hölder continuous with exponent α . Let x and h be in $[0, 2\pi]$, let N be a positive integer. We split the sum defining $f_0(x+h) - f_0(x)$ at N and study each term separately:

$$|f_0(x+h) - f_0(x)| = \sum_{|n| \leq N} \widehat{f}(n) (e^{in(x+h)} - e^{inx}) + \sum_{|n| \geq N+1} \widehat{f}(n) (e^{in(x+h)} - e^{inx})$$

The second term is roughly estimated as follows:

$$\left| \sum_{|n| \geq N} \widehat{f}(n) (e^{in(x+h)} - e^{inx}) \right| \leq 2 \sum_{|n| \geq N+1} |\widehat{f}(n)| \leq \frac{2}{N^\alpha}$$

The first term will require a bit more work. First, we have

$$|e^{in(x+h)} - e^{inx}| = |e^{inh} - 1| = 2 \left| \sin \frac{nh}{2} \right| \leq |nh|$$

Thus
$$\left| \sum_{|n| \leq N} \widehat{f}(n) (e^{in(x+h)} - e^{inx}) \right| \leq |h| \sum_{|n| \leq N} |n \widehat{f}(n)| \leq C|h|N^{1-\alpha}$$

using the proposition. So

$$|f_0(x+h) - f_0(x)| \leq C|h|N^{1-\alpha} + \frac{2}{N^\alpha}$$

Now is time to choose N , which was undetermined until now. Just take

$$N = \left\lceil \frac{1}{|h|} \right\rceil \quad N \leq \frac{1}{h} < N+1 \quad \frac{1}{N} \leq \frac{2}{N+1} \leq 2|h|$$

and obtain

$$|f_0(x+h) - f_0(x)| \leq (C + 2^{\alpha+1})|h|^\alpha$$

f_0 is Hölder continuous, of exponent α .