

Ph.D. Qualifying Exam – Spring 2002

First problem

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be any function. Let E be the set of points $x \in \mathbb{R}$ such that

$$f'_+(x) = \lim_{\substack{h \rightarrow 0 \\ h > 0}} \frac{f(x+h) - f(x)}{h} \quad \text{and} \quad \lim_{\substack{h \rightarrow 0 \\ h < 0}} \frac{f(x+h) - f(x)}{h}$$

both exist and are different. Prove that E is countable.

Solution

Let $F = \{x \in \mathbb{R} \mid f'_+(x), f'_-(x) \text{ exist and } f'_+(x) > f'_-(x)\}$

and $G = \{x \in \mathbb{R} \mid f'_+(x), f'_-(x) \text{ exist and } f'_+(x) < f'_-(x)\}$

so that $E = F \cup G$

We will show first that F is countable. If $x \in F$, since the inequality $f'_+(x) > f'_-(x)$ is strict, the interval $(f'_-(x), f'_+(x))$ is nonempty and thus contains a rational q . Because it is open, there is a positive ω such that $(q - \omega, q + \omega) \subset (f'_-(x), f'_+(x))$. Thus

$$F \subset \bigcup_{\substack{q \in \mathbb{Q} \\ \omega > 0}} F_{q,\omega}$$

where $F_{q,\omega} = \{x \in \mathbb{R} \mid f'_+(x), f'_-(x) \text{ exist and } f'_+(x) > q - \omega > q + \omega > f'_-(x)\}$

Fix a rational number q and let x_0 be in $F_{q,\omega}$ for some positive ω . That is, $x \in \bigcup_{\omega > 0} F_{q,\omega}$.

Consider the auxiliary function

$$\forall x \in \mathbb{R} \quad g(x) = f(x) - qx$$

Then $g'_+(x_0)$ and $g'_-(x_0)$ both exist since $x \mapsto qx$ is differentiable at x_0 , and we have

$$g'_+(x_0) = f'_+(x_0) - q \quad g'_-(x_0) = f'_-(x_0) - q$$

so that $g'_+(x_0) > \omega > -\omega > g'_-(x_0)$ (1)

This implies that g has a strict local minimum at x_0 . Indeed, the existence of $g'_+(x_0)$ and $g'_-(x_0)$ tell us that there exists a positive η such that

$$\forall h \in (0, \eta) \quad \frac{g(x_0 + h) - g(x_0)}{h} - g'_+(x_0) > -\frac{\omega}{2} \quad (2)$$

and $\forall h \in (-\eta, 0) \quad \frac{g(x_0 - h) - g(x_0)}{h} - g'_-(x_0) < \frac{\omega}{2}$ (3)

From (1) and (2) $\forall h \in (0, \eta) \quad g(x_0 + h) - g(x_0) > h\left(g'_+(x_0) - \frac{\omega}{2}\right) > \frac{h\omega}{2} > 0$

while from (1) and (3) $\forall h \in (-\eta, 0) \quad g(x_0 + h) - g(x_0) > h\left(g'_-(x_0) + \frac{\omega}{2}\right)$
 $> -\frac{h\omega}{2} > 0$

Notice how there was a change in the direction of inequality (3) when multiplying both sides by h , due to the fact that $h < 0$. Anyhow, we obtained

$$\forall h \in (-\eta, \eta) \quad h \neq 0 \quad g(x_0 + h) - g(x_0) > 0$$

which confirms that x_0 is a strict minimum for g in $(x_0 - \eta, x_0 + \eta)$. Of course, there exists an integer n big enough so that $(x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \subset (-\eta, \eta)$. Therefore, we showed that

$$\left(\bigcup_{\omega > 0} F_{g, \omega}\right) \subset \bigcup_{n \in \mathbb{N}^*} M_n$$
 (4)

where $M_n = \left\{x \in \mathbb{R} \mid g \text{ has a strict minimum at } x \text{ in } \left(x - \frac{1}{n}, x + \frac{1}{n}\right)\right\}$

Finally, we show that each M_n is countable. If x and y are in M_n , such that $|x - y| < \frac{1}{n}$, we know that, unless $x = y$, we have

$$g(x) < g(y) \quad \text{and} \quad g(y) < g(x)$$

which is impossible. Thus if x and y are distinct in M_n , they are at least at distance $\frac{1}{n}$ apart. Which proves that M_n is countable. By (4), $\bigcup_{\omega > 0} F_{g, \omega}$ is countable, and therefore F is countable. Applying this result to $-f$, we get that G is countable as well. Therefore,

E is countable.

Second problem

Consider functions $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ in $L^2(0,1)$ such that $f_n \xrightarrow[n \rightarrow \infty]{} f$ and $g_n \xrightarrow[n \rightarrow \infty]{} g$ weakly in $L^2(0,1)$.

1. Show that the L^2 norms of the $(f_n)_{n \in \mathbb{N}}$'s are uniformly bounded.
2. Show by example that $(f_n g_n)_{n \in \mathbb{N}}$ need not converge to fg in the weak star topology of $L^1(0,1)$.
3. Suppose $(h_n)_{n \in \mathbb{N}}$ and h are in $L^1(0,1)$ and that the L^1 norms of the $(h_n)_{n \in \mathbb{N}}$ are uniformly bounded. Show that $h_n \xrightarrow[n \rightarrow \infty]{} h$ in the weak star topology of L^1 if and only if each Fourier coefficient of h_n converges to the corresponding Fourier coefficient of h .
4. Suppose that the Fourier coefficients of the $(f_n)_{n \in \mathbb{N}}$'s and $(g_n)_{n \in \mathbb{N}}$'s are 0 at negative integers. Show that $f_n g_n \xrightarrow[n \rightarrow \infty]{} fg$ in the weak star topology of $L^1(0,1)$.

Solution

[1] This is a consequence of the Banach-Steinhaus theorem, as well as the fact that $L^2(0,1)$ is its own dual, through Riesz's representation theorem for Hilbert spaces.

Indeed, the $(f_n)_{n \in \mathbb{N}}$ define a collection of bounded linear functionals on $L^2(0,1)$ as follows:

$$\varphi_n : g \longmapsto \langle f_n | g \rangle$$

and

$$\|\varphi_n\| = \sup_{\substack{g \in L^2 \\ \|g\|=1}} (\varphi_n, g) = \sup_{\substack{g \in L^2 \\ \|g\|=1}} \langle f_n | g \rangle = \|f_n\|$$

Furthermore, the $(f_n)_{n \in \mathbb{N}}$ converge weakly, which means in particular that the sequence $(\langle f_n | g \rangle)_{n \in \mathbb{N}}$ is bounded for every g . Since $L^2(0,1)$ is complete, the principle of uniform boundedness tells us that the sequence $(\varphi_n)_{n \in \mathbb{N}}$ is bounded. It follows that $(f_n)_{n \in \mathbb{N}}$ is bounded.

[2] Let $\forall n \in \mathbb{N} \quad f_n : x \longmapsto e^{-2\pi i n x} \quad \text{and} \quad g_n : x \longmapsto e^{2\pi i n x}$

The Riemann-Lebesgue lemma tells us that

$$\forall h \in L^1(0,1) \quad \lim_{n \rightarrow \infty} \int_{[0,1]} f_n h = \lim_{n \rightarrow \infty} \int_{[0,1]} g_n h = 0$$

In particular $\forall h \in \mathcal{C}([0,1]) \quad \lim_{n \rightarrow \infty} \int_{[0,1]} f_n h = \lim_{n \rightarrow \infty} \int_{[0,1]} g_n h = 0$

so $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ both converge to 0 in the weak- \star topology of $L^1(0, 1)$. But

$$\forall n \in \mathbb{N} \quad f_n g_n = 1$$

so $(f_n g_n)_{n \in \mathbb{N}}$ converges to 1 in the weak- \star topology of $L^1(0, 1)$.

3 Suppose $(h_n)_{n \in \mathbb{N}}$ is a bounded sequence in $L^1(0, 1)$, let h be another L^1 function.

First, if $(h_n)_{n \in \mathbb{N}}$ converges to h in the weak- \star topology of $L^1(0, 1)$, then in particular,

$$\forall p \in \mathbb{Z} \quad \widehat{h_n}(p) = \int_0^1 h_n(t) e^{-2\pi i p t} dt \xrightarrow{n \rightarrow \infty} \int_0^1 h(t) e^{-2\pi i p t} dt = \widehat{h}(p) \quad (1)$$

since the functions $t \mapsto e^{-2\pi i p t}$ are continuous on $[0, 1]$.

Conversely, suppose that (1) holds. Then because integrating and taking limits are linear operations, we get

$$\forall g \in \mathcal{P} \quad \int_{[0,1]} h_n g \xrightarrow{n \rightarrow \infty} \int_{[0,1]} h g$$

Here, \mathcal{P} is the set of 1-periodic trigonometric polynomials. Let f be any continuous function such that $f(0) = f(1)$. We know that \mathcal{P} is dense in the set of such functions (the convolutions $K_n \star f$ converge to f uniformly, where K_n is the n -th Fejér kernel), so if ϵ is a positive number, there exists a trigonometric polynomial g such that $\|f - g\|_\infty < \epsilon$. Now, we have for every integer n :

$$\begin{aligned} \left| \int_{[0,1]} (h_n - h) f \right| &\leq \left| \int_{[0,1]} (h_n - h)(f - g) \right| + \left| \int_{[0,1]} (h_n - h) g \right| \\ &\leq \|h_n - h\|_1 \|f - g\|_\infty + \left| \int_{[0,1]} (h_n - h) g \right| \end{aligned} \quad (2)$$

Because the sequence $(h_n)_{n \in \mathbb{N}}$ is bounded, we can find a positive real number M such that

$$\|h\|_1 \leq M \quad \text{and} \quad \forall n \in \mathbb{N} \quad \|h_n\|_1 \leq M$$

And because g is a trigonometric polynomial,

$$\lim_{n \rightarrow \infty} \int_{[0,1]} (h_n - h) g = 0$$

Therefore, taking the limsup in (2) yields

$$\limsup_{n \rightarrow \infty} \left| \int_{[0,1]} (h_n - h) g \right| \leq 2M\epsilon$$

This holds for every positive ϵ . Thus

$$\lim_{n \rightarrow \infty} \int_{[0,1]} h_n f = \int_{[0,1]} h f$$

Finally, if we want to show something similar for functions f that are continuous, but don't satisfy the endpoint condition $f(0) = f(1)$, we can proceed as in the 2003 exam,

problem 5: first let f be the indicator function of an interval and approximate it by functions satisfying the endpoint condition. Then use the fact that linear combinations of indicators of intervals are dense in $\mathcal{C}([0, 1])$.

$$\boxed{(h_n)_{n \in \mathbb{N}} \text{ converges to } h \text{ in the weak-}\star \text{ topology of } L^1.}$$

[4] We go back to the initial situation: $(f_n)_{n \in \mathbb{N}}$ and $(g_n)_{n \in \mathbb{N}}$ are sequences in $L^2(0, 1)$, converging weakly respectively to f and g . We suppose that the $(f_n)_{n \in \mathbb{N}}$'s and $(g_n)_{n \in \mathbb{N}}$'s have their Fourier coefficients equal to 0 at negative integers. Then f (and g) have the same property since

$$\forall p \in \mathbb{Z} \quad \widehat{f}(p) = \int_0^1 f(t) e^{-2\pi i p t} dt = \lim_{n \rightarrow \infty} \int_0^1 f_n(t) e^{-2\pi i p t} dt = \lim_{n \rightarrow \infty} \widehat{f}_n(p) = 0 \quad (3)$$

Define $\forall n \in \mathbb{N} \quad h_n = f_n g_n \quad \text{and} \quad h = f g$

As we saw in the first question, there exists a positive number M such that

$$\forall n \in \mathbb{N} \quad \|f_n\|_2, \|g_n\|_2 \leq M$$

Therefore $\forall n \in \mathbb{N} \quad \|h_n\|_1 \leq \|f_n\|_2 \|g_n\|_2 \leq M^2$

Since $h_n = f_n g_n$, its Fourier coefficients are obtained as the convolution of the Fourier coefficients of f_n and g_n :

$$\forall p \in \mathbb{N} \quad \widehat{h}_n(p) = \sum_{k \in \mathbb{Z}} \widehat{f}_n(k) \widehat{g}_n(p - k) = \sum_{k=0}^p \widehat{f}_n(k) \widehat{g}_n(p - k) \quad (4)$$

and if $p < 0$, $\widehat{h}_n(p) = \sum_{k \in \mathbb{Z}} \widehat{f}_n(k) \widehat{g}_n(p - k) = \sum_{k \in \mathbb{N}} \widehat{f}_n(k) \underbrace{\widehat{g}_n(p - k)}_{=0} = 0$

Similarly $\forall p \in \mathbb{Z} \quad \widehat{h}(p) = \begin{cases} 0 & \text{if } p < 0 \\ \sum_{k=0}^p \widehat{f}(k) \widehat{g}(p - k) & \text{if } p \geq 0 \end{cases} \quad (5)$

Because of (3), and because (4) and (5) only involve finite sums, it follows that

$$\forall p \in \mathbb{Z} \quad \widehat{h}_n(p) \xrightarrow[n \rightarrow \infty]{} \widehat{h}(p)$$

Using the result from the third question,

$$\boxed{(h_n)_{n \in \mathbb{N}} \text{ converges to } h \text{ in the weak-}\star \text{ topology of } L^1(0, 1).}$$

Third problem

Consider a \mathcal{C}^∞ function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that for every $x \in \mathbb{R}$, there exists an integer k such that $f^{(k)}(x) = 0$. Let

$$U = \{x \in \mathbb{R} \mid f \text{ is a polynomial in a neighbourhood of } x\}$$

1. Prove that U is a dense open subset of \mathbb{R} .
2. Prove that the complement of U contains no isolated points.

Solution

1 We first show that U is open. Let $x_0 \in U$. Then, by definition of U , there exists a positive ϵ and a polynomial function P such that $f = P$ on $I = (x_0 - \epsilon, x_0 + \epsilon)$.

If x is any point in that interval, letting $r = |x - x_0|$, the interval $(x - (\epsilon - r), x + (\epsilon - r))$ is included in I , on which $f = P$. Thus f is a polynomial in a neighbourhood of x , which shows that x is in U as well, and therefore $I \subset U$:

U is open.

Next we show that U is dense in \mathbb{R} . Let O be a nonempty open set in \mathbb{R} . Because every point x in \mathbb{R} is such that $f^{(k)}(x) = 0$ for some k , we have

$$\mathbb{R} = \bigcup_{k \in \mathbb{N}} F_k$$

where $\forall k \in \mathbb{N} \quad F_k = \{x \in \mathbb{R} \mid f^{(k)}(x) = 0\}$

Therefore $\overline{O} = \bigcup_{k \in \mathbb{N}} (\overline{O} \cap F_k)$

By Baire's lemma, there exists a k such that $\overline{O} \cap F_k$ has nonempty interior. This interior is actually $O \cap \overset{\circ}{F}_k$.

Remember indeed that $\text{Int}(A \cap B) = \overset{\circ}{A} \cap \overset{\circ}{B}$. The similar property with unions, however, is false. Take for example

$$A = \mathbb{Q} \quad B = \mathbb{R} \setminus \mathbb{Q} \quad \overset{\circ}{A} = \emptyset \quad \overset{\circ}{B} = \emptyset$$

Then $\overset{\circ}{A} \cup \overset{\circ}{B} = \emptyset$ while $\text{Int}(A \cup B) = \overset{\circ}{\mathbb{R}} = \mathbb{R}$

Similarly, for all sets A and B , we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$, but the property with the intersections is false in general.

So there is an open ball $\mathcal{B}(x_0, \epsilon)$ in $O \cap \overset{\circ}{F}_k$. Then in particular

$$\forall x \in (x_0 - \epsilon, x_0 + \epsilon) \quad f^{(k)}(x) = 0$$

Since f is obtained by multiple integrations of $f^{(k)}$, it follows that f is a polynomial of degree at most $k - 1$ in $(x_0 - \epsilon, x_0 + \epsilon)$. Which proves that $x_0 \in U$, so that $O \cap U$ is not empty. Every open subset of \mathbb{R} intersects U nontrivially, or in other words,

U is dense in \mathbb{R} .

2 We first show that if (a, b) is an open interval contained in U , then f is a polynomial in (a, b) . For this, let $x_0 \in (a, b) \subset U$. Then we know that for some positive ϵ , there exists a polynomial function P such that $f = P$ on $(x_0 - \epsilon, x_0 + \epsilon)$. Let

$$M = \text{Sup} \{x \in (x_0, b) \mid f = P \text{ in } (x_0, x)\}$$

Then

$$\epsilon \leq M \leq b$$

and we will show that M is actually equal to b . Suppose it is not the case. Then we know that $f = P$ on (x_0, M) and $M \in (a, b) \subset U$. Therefore, there exists a positive η and a polynomial Q such that $f = Q$ on $(M - \eta, M + \eta)$. But then

$$\forall x \in (M - \eta, M) \quad P(x) - Q(x) = f(x) - f(x) = 0$$

$P - Q$ is a polynomial with infinitely many zeroes, so it has to be the 0 polynomial: $P = Q$. It follows that $f = P$ on $(x_0, M + \eta)$, which contradicts the definition of M . Thus $M = b$.

Similarly, we would show that

$$\text{Inf} \{x \in (a, x_0) \mid f = P \text{ in } (x, x_0)\} = a$$

It follows that $f = P$ on all of (a, b) .

Now, going back to the question asked. Suppose that $x \in \mathbb{R} \setminus U$ is isolated. Then there exists a positive ϵ such that $(x_0 - \epsilon, x_0)$ and $(x_0, x_0 + \epsilon)$ sit entirely inside U . From what we just showed, there are polynomials P and Q such that

$$\forall x \in (x_0 - \epsilon, x_0) \quad f(x) = P(x)$$

and

$$\forall x \in (x_0, x_0 + \epsilon) \quad f(x) = Q(x)$$

Let k be the biggest number of $\deg(P)$ and $\deg(Q)$. Then $P^{(k+1)} = 0$ and $Q^{(k+1)} = 0$. As a consequence,

$$\forall x \in (x_0 - \epsilon, x_0 + \epsilon) \setminus \{x_0\} \quad f^{(k+1)}(x) = P^{(k+1)}(x) \text{ or } Q^{(k+1)}(x) = 0$$

Because $f^{(k+1)}$ is continuous at x_0 , it follows that $f^{(k+1)}(x_0) = 0$ as well. So $f^{(k+1)} = 0$ on the whole interval $(x_0 - \epsilon, x_0 + \epsilon)$. By integrating $k + 1$ times, it follows that f is a polynomial function in that interval. Thus x_0 is in U , which gives us a contradiction.

The complement of U has no isolated points.

Fourth problem

Prove the inequality

$$\forall a \in \ell^2(\mathbb{Z}) \quad \sum_{m \in \mathbb{Z}} \left| \sum_{\substack{n \in \mathbb{Z} \\ n \neq m}} \frac{a_n}{n - m} \right|^2 \leq \pi^2 \|a\|^2$$

Show also that π^2 cannot be replaced by a smaller constant.

Solution

Let f be the 1-periodic function coinciding with $x \mapsto x - \frac{1}{2}$ on $(0, 1)$ and let's compute its Fourier coefficients. Obviously, since f is symmetric around the point $\frac{1}{2}$, we'll have $\widehat{f}(0) = 0$. So let $n \in \mathbb{Z}$ nonzero and integrate by parts:

$$\begin{aligned} \widehat{f}(n) &= \int_0^1 \left(x - \frac{1}{2}\right) e^{-2\pi i n x} dx = \int_0^1 x e^{-2\pi i n x} dx \\ &= \frac{1}{-2\pi i n} [x e^{-2\pi i n x}]_0^1 + \frac{1}{2\pi i n} \int_0^1 e^{-2\pi i n x} dx \\ \widehat{f}(n) &= -\frac{1}{2\pi i n} \end{aligned}$$

Let a be any sequence in $\ell^2(\mathbb{Z})$ and let g be the $L^2(0, 1)$ function whose Fourier coefficients are the terms of the sequence a :

$$g = \sum_{n \in \mathbb{Z}} a_n e_n \quad \text{where} \quad e_n : x \mapsto e^{2\pi i n x}$$

We know that fg is $L^2(0, 1)$ (since f is bounded), whose Fourier coefficients are obtained by convolution of f and g 's coefficients:

$$\forall m \in \mathbb{N} \quad \widehat{fg}(m) = \sum_{n \in \mathbb{Z}} a_n \widehat{f}(m - n) = -\frac{1}{2\pi i} \sum_{\substack{n \in \mathbb{Z} \\ n \neq m}} \frac{a_n}{m - n}$$

And Parseval's formula tells us that

$$\|fg\|_2^2 = \sum_{m \in \mathbb{Z}} |\widehat{fg}(m)|^2 = \frac{1}{4\pi^2} \sum_{m \in \mathbb{Z}} \left| \sum_{\substack{n \in \mathbb{Z} \\ m \neq n}} \frac{a_n}{m - n} \right|^2$$

Thus

$$\sum_{m \in \mathbb{Z}} \left| \sum_{\substack{n \in \mathbb{Z} \\ m \neq n}} \frac{a_n}{m - n} \right|^2 = 4\pi^2 \|fg\|_2^2 \leq 4\pi^2 \|f\|_\infty^2 \|g\|_2^2 = \pi^2 \sum_{m \in \mathbb{Z}} |a_m|^2$$

To show that the inequality is sharp, it is enough to show that

$$\forall g \in L^2 \quad \|fg\|_2^2 \leq \frac{\|g\|_2^2}{4}$$

is optimal. All we have to do is find a sequence of $L^2(0,1)$ functions $(g_n)_{n \in \mathbb{N}}$ that will pick up the value $\frac{1}{4}$, which is the supremum of f^2 on $(0,1)$. So let

$$g_n = \sqrt{\frac{n}{2}} (\mathbb{1}_{(0, \frac{1}{n})} + \mathbb{1}_{(1-\frac{1}{n}, 1)})$$

Then

$$|g_n|^2 = \frac{n}{2} (\mathbb{1}_{(0, \frac{1}{n})} + \mathbb{1}_{(1-\frac{1}{n}, 1)})$$

so that

$$\|g_n\|_2^2 = 1$$

Now, we also have, using the fact that f^2 and g_n^2 are symmetric around the line $x = \frac{1}{2}$:

$$\|fg_n\|_2^2 = 2 \int_{[0, \frac{1}{2}]} f^2 g_n^2 = n \int_0^{\frac{1}{n}} \left(x - \frac{1}{2}\right)^2 dx = \frac{n}{3} \left(\left(\frac{1}{n} - \frac{1}{2}\right) - \frac{1}{8} \right) = \frac{1}{3n^2} - \frac{1}{2n} + \frac{1}{4}$$

This tends to $\frac{1}{4}$ as n goes to ∞ .

Fifth problem

Let K be an L^1 function on the unit square $[0, 1]^2$. Suppose that for every continuous function f on $[0, 1]$, we have

$$\int_{[0,1]} K(x, y)f(y) dy \quad \text{for almost every } x$$

Prove that $K = 0$ almost everywhere.

Solution

Denote the unit square $[0, 1]^2$ by Q and consider the subspace \mathcal{F} of $\mathcal{C}(Q)$ spanned by functions of the type

$$(x, y) \mapsto g(x)f(y) \quad f, g \in \mathcal{C}([0, 1])$$

It is easy to check that a product of finite linear combinations of such functions is still a finite linear combination of such functions. Thus \mathcal{F} is a subalgebra of $\mathcal{C}([0, 1])$. It separates points, since if (x_1, y_1) and (x_2, y_2) are distinct points in Q , they are separated by the function $(x, y) \mapsto (x - x_1)^2 + (y - y_1)^2$. Also, \mathcal{F} contains the constant function 1. Therefore, \mathcal{F} is dense in $\mathcal{C}(Q)$.

Now, let f and g be any two continuous functions on $[0, 1]$. Since K is $L^1(Q)$,

$$\int_0^1 \int_0^1 |K(x, y)g(x)f(y)| dy dx \leq \|f\|_\infty \|g\|_\infty \|K\|_1 < \infty$$

and therefore
$$\int_Q K(x, y)g(x)f(y) d\lambda(x, y) = \int_0^1 g(x) \left(\int_0^1 K(x, y)f(y) dy \right) dx = 0$$

by Fubini and by hypothesis on K . By linearity,

$$\forall f \in \mathcal{F} \quad \int_Q Kf = 0$$

But
$$\|K\|_1 = \sup_{\substack{f \in \mathcal{C}(Q) \\ \|f\|_\infty=1}} \int_Q Kf = \sup_{\substack{f \in \mathcal{F} \\ \|f\|_\infty=1}} \int_Q Kf$$

since \mathcal{F} is dense in $\mathcal{C}(Q)$. Thus $\|K\|_1 = 0$, and

K is 0 almost everywhere.

Sixth problem

Show that there do not exist measurable sets A and B with positive measure such that $A \cap (B - r) = \emptyset$ for every rational number r .

Solution

Suppose that such sets exist. We might as well assume that A and B have positive *finite* measure, by intersecting them with a big enough compact set. Then $\mathbb{1}_A$ and $\mathbb{1}_B$ are both L^2 functions, and $\mathbb{1}_A \star \mathbb{1}_B$ is continuous.

This is a remarkable property of the convolution of L^2 functions. Indeed, if f and g are L^2 , then

$$\begin{aligned} |f \star g(x+h) - f \star g(x)| &= \left| \int_{\mathbb{R}} f(y)(g(x+h-y) - g(x-y)) dy \right| \\ &\leq \|f\|_2 \left(\int_{\mathbb{R}} |g(y-h) - g(y)|^2 dy \right)^{\frac{1}{2}} \end{aligned}$$

and we know that translations are continuous on $L^2(\mathbb{R})$ so the righthandside tends to 0.

Another way of seeing it is using the Fourier transform:

$$\widehat{f \star g} = \widehat{f} \widehat{g}$$

But the Fourier transform is an isometry of L^2 onto itself, therefore $\widehat{f \star g}$ is L^1 . So $f \star g$, as the inverse Fourier transform of an L^1 function, is continuous.

In fact, more generally, the convolution of an L^p and an $L^{p'}$ function yields a continuous function.

We have $\forall x \in \mathbb{R} \quad \mathbb{1}_A \star \mathbb{1}_B(x) = \int_{\mathbb{R}} \mathbb{1}_A(y) \mathbb{1}_B(x-y) dy$

But $\mathbb{1}_A(y) \mathbb{1}_B(x-y)$ is 1 if and only if

$$\begin{cases} y \in A \\ x-y \in B \end{cases} \quad \text{that is} \quad y \in A \cap (x+B)$$

and is 0 otherwise. Therefore, $\mathbb{1}_A \star \mathbb{1}_B$ is 0 at every rational number. But this function is continuous, and rationals are dense in \mathbb{R} , therefore

$$\forall x \in \mathbb{R} \quad \mathbb{1}_A \star \mathbb{1}_B(x) = 0$$

However $\|\mathbb{1}_A \star \mathbb{1}_B\|_1 = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_A(y) \mathbb{1}_B(x-y) dy dx = \mu(A)\mu(B) \neq 0$

We have our contradiction.

Seventh problem

Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of Lebesgue measurable functions on $[0, 1]$. Let

$$E = \left\{ x \in [0, 1] \mid \sum_{n \in \mathbb{N}} f_n(x) \text{ converges} \right\}$$

Show that for every positive ϵ , there is a set F and an integer k such that

- F is in the ring of sets generated by sets of the form $f_i^{-1}(A)$ for $i \leq k$ and A Borel;
- $m(E \Delta F) < \epsilon$.

Solution

Using the fact that a sequence of real or complex numbers converges if and only if it is Cauchy, we see that

$$E = \bigcap_{p \in \mathbb{N}^*} \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} \bigcap_{m \in \mathbb{N}} \left\{ \left| \sum_{k=n}^{n+m} f_k \right| < \frac{1}{p} \right\}$$

For every integers N , n and m , we let

$$A_{p,n,m} = \left\{ \left| \sum_{k=n}^{n+m} f_k \right| < \frac{1}{p} \right\}$$

Notice that $A_{p,n,m}$ is in the σ -algebra generated by sets of the form $f_i^{-1}(B)$ where $i \leq n+m$ and B Borel.

In case you are not sure anymore why this is true, here is a way to do it. Let f and g be two measurable functions, let a be any real number. Then one checks easily that

$$\{f + g < a\} = \{f < a - g\} = \bigcup_{r \in \mathbb{Q}} (\{f < r\} \cap \{g < a - r\})$$

which shows that $\{f + g < a\}$ is in the σ -algebra generated by sets of the form $f^{-1}(B), g^{-1}(B)$ with B Borel. Inductively, this generalizes to sums of an arbitrary number of functions.

For every pair of integers K and M , we let

$$E_{K,M} = \bigcap_{p \in \mathbb{N}^*} \bigcup_{N \in \mathbb{N}} \bigcap_{n=N}^K \bigcap_{m=0}^M \left\{ \left| \sum_{k=n}^{n+m} f_k \right| < \frac{1}{p} \right\}$$

This set is in the σ -algebra generated by the $f_i^{-1}(B)$ for B Borel and $i \leq K + M$.

It can probably be checked that $\mathbb{1}_{E_{K,M}}$ converges pointwise to $\mathbb{1}_E$ as K and M tend to ∞ . Moreover, all the $E_{K,M}$'s are dominated by the constant function equal to 1, which is integrable on $[0, 1]$. By the dominated convergence theorem,

$$\lim_{K,M \rightarrow \infty} \mathbb{1}_{E_{K,M}} = \mathbb{1}_E \quad \text{in } L^1(0, 1)$$

or in other words, $0 = \lim_{K,M \rightarrow \infty} \|\mathbb{1}_{E_{K,M}} - \mathbb{1}_E\|_1 = \lim_{K,M \rightarrow \infty} \mu(E \Delta E_{K,M})$

So if ϵ is a positive real number, just take K and M big enough, let $F = E_{K,M}$, so that

$$\boxed{\mu(E \Delta F) \leq \epsilon}$$

Eighth problem

Let $C \in (0, 1)$. Show that there are numbers δ_N , depending on C , with the following properties:

- If A_1, \dots, A_N are measurable sets in $[0, 1]$ each with measure C , then

$$m(A_i \cap A_j) \geq (1 - \delta_N)C^2$$

for some $i \neq j$;

- $\delta_N \xrightarrow{N \rightarrow \infty} 0$.

Solution

Let N be a positive integer and A_1, \dots, A_N be measurable sets in $[0, 1]$, each with measure C . Consider the function $F = \sum_{n=1}^N \mathbb{1}_{A_n}$. Then

$$F^2 = \sum_{m,n=1}^N \mathbb{1}_{A_n} \mathbb{1}_{A_m} = \sum_{n=1}^N \mathbb{1}_{A_n} + \sum_{n=1}^N \sum_{\substack{m=1 \\ m \neq n}}^N \mathbb{1}_{A_n \cap A_m}$$

and

$$\int_{[0,1]} F^2 = NC + \sum_{n=1}^N \sum_{\substack{m=1 \\ m \neq n}}^N m(A_n \cap A_m)$$

Now let i and j be distinct and such that

$$m(A_i \cap A_j) = \text{Max} \{m(A_n \cap A_m) \mid m \neq n\}$$

Then

$$NC + N(N-1)m(A_i \cap A_j) \geq \int_{[0,1]} F^2 \geq \left(\int_{[0,1]} F \right)^2 = N^2 C^2$$

This last step is just Hölder's inequality applied to $F = F \times 1$. Anyhow, it follows that

and

$$m(A_i \cap A_j) \geq \frac{N^2 C^2 - NC}{N(N-1)} = \frac{N - \frac{1}{C}}{N-1} C^2$$

so take

$$\delta_N = \frac{1 - C}{NC - C}$$

Ninth problem

Suppose the Banach space X is uniformly convex. That is, for every positive ϵ , there exists a positive δ such that

$$\forall x, y \in X \quad \left(\|x\| = \|y\| = 1 \quad \text{and} \quad \left\| \frac{x+y}{2} \right\| > 1 - \delta \right) \implies \|x - y\| \leq \epsilon$$

Let f be a bounded linear functional on X with norm 1. Show that there is a unique point $x \in X$ with $\|x\| = 1$ such that $f(x) = 1$.

Solution

Since f has norm 1, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ in the unit sphere of X such that

$$\lim_{n \rightarrow \infty} (f, x_n) = 1 \tag{1}$$

Let ϵ be a positive real number; uniform convexity of X provides us with a δ such that

$$\forall x, y \in X \quad \left(\|x\| = \|y\| = 1 \quad \text{and} \quad \left\| \frac{x+y}{2} \right\| > 1 - \delta \right) \implies \|x - y\| \leq \epsilon$$

And (1) provides us with a positive integer N such that

$$\forall n \geq N \quad 1 - \delta < (f, x_n)$$

Thus $\forall n \geq N \quad \forall p \in \mathbb{N} \quad 1 - \delta < \left(f, \frac{x_n + x_{n+p}}{2} \right)$

which implies in turn that $\left\| \frac{x_n + x_{n+p}}{2} \right\| \geq 1 - \delta$

Therefore $\forall n \geq N \quad \forall p \in \mathbb{N} \quad \|x_{n+p} - x_n\| \leq \epsilon$

The sequence $(x_n)_{n \in \mathbb{N}}$ is Cauchy, and therefore converges to some x . Of course,

$$\|x\| = 1 \quad \text{and} \quad (f, x) = \lim_{n \rightarrow \infty} (f, x_n) = 1$$

Now let $y \in X$ satisfy this same property:

$$\|y\| = 1 \quad \text{and} \quad (f, y) = 1$$

Then $\left\| \frac{x+y}{2} \right\| \geq \left(f, \frac{x+y}{2} \right) = 1$

which implies, by uniform convexity, that $\|x - y\| \leq \epsilon$ for every positive ϵ .

There exists a unique $x \in X$ such that $(f, x) = 1$.

Tenth problem

1. Let μ be a finite measure on \mathbb{R} and let ν be the measure with density e^{-x^2} with respect to μ . Show that the Fourier transform of ν is the restriction to \mathbb{R} of an entire function F .
2. Express the n -th derivative $F^{(n)}(0)$ in terms of μ .
3. Show that the set

$$S = \{p(x)e^{-x^2} \mid p \text{ polynomial}\}$$

is dense in $\mathcal{C}_0(\mathbb{R})$, the space of continuous functions tending to 0 at $\pm\infty$.

Solution

1 For every complex number z , define

$$F(z) = \int_{\mathbb{R}} e^{-izx} d\nu(x) = \int_{\mathbb{R}} e^{-x^2-izx} d\mu(x)$$

Since the function $x \mapsto e^{-x^2+x\text{Im}z}$ is continuous and tends to 0 at $\pm\infty$, it is bounded by some $M > 0$ and we have

$$\int_{\mathbb{R}} |e^{-x^2-izx}| d\mu(x) = \int_{\mathbb{R}} e^{-x^2+x\text{Im}z} d\mu(x) \leq M\|\mu\| < \infty$$

Therefore, F is well defined at every $z \in \mathbb{C}$.

Now, let γ be any closed curve in \mathbb{C} . Since γ is compact, there exists a positive number C such that

$$\forall z \in \gamma \quad \text{Im}z \leq C$$

Therefore $\int_{\gamma} \int_{\mathbb{R}} |e^{-x^2-izx}| d\mu(x) d|z| \leq \int_{\gamma} \int_{\mathbb{R}} e^{-x^2+Cx} d\mu(x) d|z| \leq F(C)\text{length}(\gamma) < \infty$

which allows us to Fubini the double integral

$$\int_{\gamma} F(z) dz = \int_{\mathbb{R}} e^{-x^2} \int_{\gamma} e^{-izx} dz d\mu(x) = 0$$

since $z \mapsto e^{-izx}$ is holomorphic in a neighbourhood of γ . Since this holds for every closed curve γ , F is analytic in \mathbb{C} . And of course, the restriction to \mathbb{R} of F is the Fourier transform of ν .

$\widehat{\nu}$ is the restriction to \mathbb{R} of an entire function.

In the next question, we will see another strategy that could have been used to solve that question, and gives us at the same time the derivatives of F at 0. I simply wanted to present both methods because each of them is interesting and worth seeing at least once.

2 The function $x \mapsto e^{|zx|-x^2}$ is continuous and tends to 0 at $\pm\infty$, thus is bounded. We let $M > 0$ be such that

$$\forall x \in \mathbb{R} \quad e^{|zx|-x^2} \leq M$$

Thus
$$\int_{\mathbb{R}} \sum_{n \in \mathbb{N}} \frac{|zx|^n}{n!} e^{-x^2} d\mu(x) = \int_{\mathbb{R}} e^{|zx|-x^2} d\mu(x) \leq M \|\mu\| < \infty$$

which allows us, thanks to Fubini, to compute as we wish the double integral:

$$F(z) = \int_{\mathbb{R}} e^{-x^2-izx} d\mu(x) = \int_{\mathbb{R}} \sum_{n \in \mathbb{N}} \frac{(-izx)^n}{n!} e^{-x^2} d\mu(x) = \sum_{n \in \mathbb{N}} \frac{z^n}{n!} \int_{\mathbb{R}} (-ix)^n e^{-x^2} d\mu(x)$$

This gives us a power series expansion of F around zero. In particular, we get for free the derivatives of F at 0:

$$\forall n \in \mathbb{N} \quad F^{(n)}(0) = \int_{\mathbb{R}} (-ix)^n e^{-x^2} d\mu(x)$$

Of course, this is the method you would use during the exam, since it overkills questions 1 and 2.

3 Let
$$S = \{p(x)e^{-x^2} \mid p \text{ polynomial}\}$$

and let μ be a finite measure on \mathbb{R} that vanishes on S :

$$\forall p \in \mathbb{C}[X] \quad \int_{\mathbb{R}} p(x)e^{-x^2} d\mu(x) = 0$$

In particular
$$\forall n \in \mathbb{N} \quad \int_{\mathbb{R}} (-ix)^n e^{-x^2} d\mu = 0$$

Hence, by questions 1 and 2, the Fourier transform of $e^{-x^2} d\mu$ is 0. And, by injectivity of the Fourier transform, this measure itself is 0:

$$\forall f \in \mathcal{C}_c(\mathbb{R}) \quad \int_{\mathbb{R}} f(x) e^{-x^2} d\mu(x) = 0$$

Now, if f is continuous with compact support, $x \mapsto f(x)e^{x^2}$ is also continuous with compact support and as a consequence,

$$\int_{\mathbb{R}} f d\mu = \int_{\mathbb{R}} (f(x) e^{x^2}) e^{-x^2} d\mu = 0$$

So μ vanishes on $\mathcal{C}_c(\mathbb{R})$. This set is dense in $\mathcal{C}_0(\mathbb{R})$, so μ also vanishes there and it follows that $\mu = 0$. Therefore,

S is dense in $\mathcal{C}_0(\mathbb{R})$.