

Ph.D. Qualifying Exam – Spring 2001

First problem

Let X be a metric space.

1. Suppose X is separable. Show that every open cover of X has a countable subcover.
2. Conversely, suppose that every open cover of X has a countable subcover. Show that X is separable.

Solution

[1] Suppose that X is separable and let $\{x_n \mid n \in \mathbb{N}\}$ be a countable dense subset of X . Let $(O_\alpha)_{\alpha \in A}$ be a collection of open sets, such that

$$X = \bigcup_{\alpha \in A} O_\alpha$$

For every integer n , there exists α_n such that $x_n \in O_{\alpha_n}$.

[2] Conversely, suppose that every open cover of X has a countable subcover. Obviously,

$$X = \bigcup_{\substack{x \in X \\ n \in \mathbb{N}^*}} \mathcal{B}\left(x, \frac{1}{n}\right)$$

This cover thus has a countable subcover; let $(x_j)_{j \in \mathbb{N}}$ be the centers of those balls. Then

$$X = \bigcup_{\substack{j \in \mathbb{N} \\ n \in \mathbb{N}^*}} \mathcal{B}\left(x_j, \frac{1}{n}\right)$$

If x is any point in X and ϵ is a positive real number, let n be big enough so that $\frac{1}{n} < \epsilon$. Then there exists a j such that $x \in \mathcal{B}(x_j, \frac{1}{n})$ which implies that $d(x, x_j) < \epsilon$. Thus $(x_j)_{j \in \mathbb{N}}$ is dense.

X is separable.

Second problem

Let T be the set of real number x with the following property:

$$\forall k \in \mathbb{N} \quad \exists N > k \quad \exists a \in \mathbb{N} \quad \left| x - \frac{a}{10^N} \right| \leq \frac{1}{20^N}$$

1. Prove that T is uncountable.
2. What is the Lebesgue measure of T ?

Solution

1 Let

$$T_1 = \bigcap_{k \in \mathbb{N}} A_k$$

where

$$\forall k \in \mathbb{N} \quad A_k = \bigcup_{N > k} \bigcup_{a \in \mathbb{Z}} \left(\frac{a}{10^N} - \frac{1}{20^N}, \frac{a}{10^N} + \frac{1}{20^N} \right)$$

so that

$$T_1 \subset T$$

We show that T_1 is uncountable, which in turn implies that T is uncountable.

First, let's talk about the $(A_k)_{k \in \mathbb{N}}$. Let k be a fixed integer. If x is any rational number with a finite decimal expansion, we can assume that this expansion has at least $k + 1$ decimals, by adding enough zeroes to the right of the expansion if needed. Which means that x can be written as $\frac{a}{10^N}$ for $N \geq k + 1$ and $a \in \mathbb{Z}$.

Thus every A_k contains rational numbers with a finite decimal expansion. And those are dense in \mathbb{R} since every real number has a (finite or infinite) decimal expansion. So A_k is dense in \mathbb{R} for every k .

Now, assume that T_1 is countable, say $T_1 = \{x_m \mid m \in \mathbb{N}\}$. Then

$$\emptyset = T_1 \cap T_1^c = \bigcap_{m \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} (A_k \cap \{x_m\}^c)$$

But removing a point off a dense set leaves it dense. Thus \emptyset is a countable intersection of dense sets. That's very unlikely, since \mathbb{R} is complete and we have Baire's lemma. T_1 is uncountable and as a consequence

T is uncountable.

2 We have

$$T = \bigcap_{k \in \mathbb{N}} \bigcup_{N > k} \bigcup_{a \in \mathbb{Z}} \left[\frac{a}{10^N} - \frac{1}{20^N}, \frac{a}{10^N} + \frac{1}{20^N} \right]$$

by definition. For every positive integer n , let

$$T_n = T \cap [-n, n] = \bigcap_{k \in \mathbb{N}} \bigcup_{N > k} \bigcup_{a \in \mathbb{Z}} \left(\left[\frac{a}{10^N} - \frac{1}{20^N}, \frac{a}{10^N} + \frac{1}{20^N} \right] \cap [-n, n] \right)$$

The intervals that appear on the righthandside above will have empty intersection as soon as

$$n < \frac{a}{10^N} - \frac{1}{20^N} \quad \text{or} \quad -n > \frac{a}{10^N} + \frac{1}{20^N}$$

that is
$$a > 10^N n + \frac{1}{2^N} \quad \text{or} \quad a < -10^N n - \frac{1}{2^N}$$

A sufficient condition for that is for example that

$$a > 10^N n + 1 \quad \text{or} \quad a < -10^N n - 1$$

Anyhow, it follows that

$$T_n \subset \bigcap_{k \in \mathbb{N}} \bigcup_{N > k} \bigcup_{\substack{a \in \mathbb{Z} \\ -10^N n - 1 \leq a \leq 10^N n + 1}} \left[\frac{a}{10^N} - \frac{1}{20^N}, \frac{a}{10^N} + \frac{1}{20^N} \right]$$

Thus
$$\forall k \in \mathbb{N} \quad T_n \subset \bigcup_{N > k} \bigcup_{\substack{a \in \mathbb{Z} \\ -10^N n - 1 \leq a \leq 10^N n + 1}} \left[\frac{a}{10^N} - \frac{1}{20^N}, \frac{a}{10^N} + \frac{1}{20^N} \right]$$

so
$$\forall k \in \mathbb{N} \quad \lambda(T_n) \leq \sum_{N > k} \sum_{a = -10^N n - 1}^{10^N n + 1} \frac{2}{20^N} = \sum_{N > k} \frac{2(10^N n + 3)}{20^N}$$

Since the righthandside tends to 0 as k tends to ∞ , T_n has Lebesgue measure 0. And T is the union of all the $(T_n)_{n \in \mathbb{N}}$'s, so

T has Lebesgue measure 0.

Third problem

Let X be a compact metric space and $\mathcal{C}(X)$ be the space of continuous real valued functions on X . Let $F : \mathcal{C}(X) \rightarrow \mathbb{R}$ be a continuous map such that $F(1) = 1$ and

$$\forall u, v \in \mathcal{C}(X) \quad \begin{cases} F(u + v) = F(u) + F(v) \\ F(uv) = F(u)F(v) \end{cases}$$

Show that there exists $x_0 \in X$ such that

$$\forall u \in \mathcal{C}(X) \quad F(u) = u(x_0)$$

Solution

This problem is easy once one know the structure of ideals of $\mathcal{C}(X)$. It is one of the first examples studied in any course on Banach algebras.

We first show that if I is a proper ideal of $\mathcal{C}(X)$, there exists $x_0 \in X$ such that every function in I vanishes at x_0 . Suppose it is not the case. Then for every $x \in X$, there exists $f_x \in I$ such that $f_x(x) \neq 0$. Up to replacing f by $-f$ (I is an ideal), we can assume that $f_x(x) > 0$. This allows us to form an open cover of X :

$$X = \bigcup_{x \in X} f_x^{-1} \left(\left(\frac{f_x(x)}{2}, +\infty \right) \right)$$

from which we extract a finite subcover. After renaming everything more conveniently,

$$X = \bigcup_{i=1}^n f_i^{-1} \left(\left(\frac{f_i(x_i)}{2}, +\infty \right) \right) \quad \text{with} \quad f_i(x_i) > 0 \quad (1)$$

Define then $\forall x \in X \quad g(x) = f_1(x)^2 + \dots + f_n(x)^2$

Since I is an ideal, it contains g . Now, if x is in X , there exists $i \in \{1, \dots, n\}$ (by (1)) such that

$$0 < \frac{f_i(x_i)}{2} < f_i(x)$$

Therefore $g(x) \geq f_i(x)^2 > 0$

which establishes that g is (strictly) positive. Then the function $\frac{1}{g}$ is in $\mathcal{C}(X)$. Because I is an ideal, $g \times \frac{1}{g} = 1$ is in I , so I is the whole ring $\mathcal{C}(X)$. By contrapositive, we showed what was announced:

Theorem *If I is a proper ideal in $\mathcal{C}(X)$, there exists $x_0 \in X$ such that*

$$\forall f \in I \quad f(x_0) = 0$$

Now, back to our problem. We first show that F is \mathbb{R} -linear. Let u be any continuous function on X . We have

$$\forall n \in \mathbb{N}^* \quad F(nu) = F(\underbrace{u + \cdots + u}_{n \text{ times}}) = \underbrace{F(u) + \cdots + F(u)}_{n \text{ times}} = nF(u)$$

Thus
$$\forall n \in \mathbb{N}^* \quad F(u) = F\left(\frac{n}{n}u\right) = nF\left(\frac{u}{n}\right)$$

so that
$$\forall n \in \mathbb{N}^* \quad F\left(\frac{u}{n}\right) = \frac{F(u)}{n}$$

and it follows that
$$\forall r = \frac{p}{q} \in \mathbb{Q}_+^* \quad F(ru) = F\left(p \frac{u}{q}\right) = pF\left(\frac{u}{q}\right) = \frac{p}{q}F(u) \quad (2)$$

Next,
$$F(0) = F(2 \times 0) = 2F(0)$$

so
$$F(0) = 0$$

As a consequence,
$$F(-u) + F(u) = F(0) = 0 \quad \text{and} \quad F(-u) = -F(u) \quad (3)$$

From (2) and (3),

$$\forall r \in \mathbb{Q} \quad F(ru) = rF(u)$$

Since F is continuous, this relation extends to real numbers and we obtain that F is \mathbb{R} -linear.

Because $F(1) = 1$, we get that $F(\lambda) = \lambda$ for all real number λ and therefore $\text{Im } F = \mathbb{R}$. Since F is a ring homomorphism, $\mathcal{C}(X)/\text{Ker } F \simeq \mathbb{R}$ which is a field, and $\text{Ker } F$ is a maximal ideal in $\mathcal{C}(X)$.

From the theorem established on the previous page, there exists $x_0 \in X$ such that

$$\forall u \in \text{Ker } F \quad u(x_0) = 0$$

Thus $\text{Ker } F$ is included in the ideal

$$I_0 = \{u \in \mathcal{C}(X) \mid u(x_0) = 0\}$$

And I_0 is proper since it does not contain the constant function equal to 1. Because $\text{Ker } F$ is maximal, it is equal to I_0 :

$$\text{Ker } F = \{u \in \mathcal{C}(X) \mid u(x_0) = 0\}$$

We now have all the ingredients to finish the proof. Let u be any element of $\mathcal{C}(X)$. The function $x \mapsto u(x) - u(x_0)$ takes the value 0 at x_0 , thus is in $\text{Ker } F$:

$$F(u - u(x_0) \times 1) = 0$$

Since F is \mathbb{R} -linear and $F(1) = 1$, it follows that $F(u) = u(x_0)$:

$$\boxed{\exists x_0 \in X \quad \forall u \in X \quad F(u) = u(x_0)}$$

Fourth problem

Prove:

1. The continuous image of a connected set is connected.
2. If X is compact, Y is Hausdorff and $f : X \rightarrow Y$ is injective and continuous, then f^{-1} is continuous.
3. The product of two compact sets is compact.

Solution

1 Let X and Y be two topological spaces, with X connected. Let $f : X \rightarrow Y$ be continuous. Let U and V be open subsets of $f(X)$, such that

$$U \cap V = \emptyset \quad \text{and} \quad U \cup V = f(X)$$

By definition of the subset topology, there are open sets U_1 and V_1 in Y , such that

$$U = U_1 \cap f(X) \quad \text{and} \quad V = V_1 \cap f(X)$$

We have

$$\begin{aligned} X &= f^{-1}(f(X)) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V) \\ &= \left(f^{-1}(U_1) \cap f^{-1}(f(X)) \right) \cup \left(f^{-1}(V_1) \cap f^{-1}(f(X)) \right) \\ &= f^{-1}(U_1) \cap f^{-1}(V_1) \end{aligned}$$

Similarly,

$$\emptyset = f^{-1}(U \cap V) = f^{-1}(U_1) \cap f^{-1}(V_1)$$

Since f is continuous, $f^{-1}(U_1)$ and $f^{-1}(V_1)$ are open in X . And because X is connected, one of those sets is X and the other one is empty. Let's say that

$$f^{-1}(U_1) = X \quad \text{and} \quad f^{-1}(V_1) = \emptyset$$

Then

$$f(f^{-1}(U_1)) = f(X) \quad \text{and} \quad f(f^{-1}(V_1)) = \emptyset$$

But we also have

$$f(f^{-1}(U_1)) = U_1 \cap f(X) = U \quad \text{and} \quad f(f^{-1}(V_1)) = V_1 \cap f(X) = V$$

Hence $f(X)$ is connected.

This problem is so simple that I believe all the details above (especially going through the definition of the subset topology of $f(X)$) are required.

2 Suppose X is compact, Y is Hausdorff and $f : X \rightarrow Y$ is a continuous injective map. Let C be any closed set in X . Then C is compact and therefore $f(C)$ is compact. But in a Hausdorff space, compact sets are closed so $f(C)$ is closed. Thus

$$\forall C \text{ closed in } X \quad f(C) \text{ is closed in } Y$$

$$\boxed{f^{-1} \text{ is continuous.}}$$

3 Let X and Y be two compact topological spaces. I'll start with a few reminders on the product topology. Remember that the product topology on $X \times Y$ is the weakest topology that makes both projections

$$p_X : (x, y) \mapsto x \quad \text{and} \quad p_Y : (x, y) \mapsto y$$

continuous. In other words, it is the weak topology on $X \times Y$ generated by p_X and p_Y . Notice that if O^X and O^Y are open sets in X and Y respectively, then

$$p_X^{-1}(O^X) = O^X \times Y \quad \text{and} \quad p_Y^{-1}(O^Y) = X \times O^Y$$

A set is open in $X \times Y$ if and only if it is a union of finite intersections of sets of the form $O^X \times Y$ and $X \times O^Y$ (see the first section of the handout on weak topologies if it is the first time you hear about this).

Handling intersections of cartesian products is not a problem. Indeed, if A^X, B^X are subsets of X , and A^Y, B^Y are subsets of Y , then

$$\begin{aligned} (A^X \times A^Y) \cap (B^X \times B^Y) &= \{(x, y) \in X \times Y \mid x \in A^X, y \in A^Y \text{ and } x \in B^X, y \in B^Y\} \\ &= (A^X \cap B^X) \times (A^Y \cap B^Y) \end{aligned}$$

So finite intersections of products are just products of the corresponding intersections, if you see what I mean. Therefore, a set is open in $X \times Y$ if and only if it is a union of sets of the form $O^X \times O^Y$, with O^X and O^Y open in X and Y respectively.

Finally, given any $x \in X$, we can define the bijection

$$\begin{aligned} i_x : Y &\longrightarrow X \times Y \\ y &\longmapsto (x, y) \end{aligned}$$

As one checks easily,

$$\forall y \in Y \quad p_X \circ i_x(y) = x \quad \text{and} \quad p_Y \circ i_x(y) = y$$

so the functions $p_X \circ i_x$ and $p_Y \circ i_x$ are continuous on Y : this implies (again, see the handout on weak topologies) that i_x is continuous on Y . This is not really important for our purpose, and is just displayed here to show how weak topologies are used.

Now, towards solving the problem. Consider an open cover of $X \times Y$; it looks like

$$X \times Y = \bigcup_{\alpha \in A} O_\alpha^X \times O_\alpha^Y$$

where the $(O_\alpha^X)_{\alpha \in A}$ and $(O_\alpha^Y)_{\alpha \in A}$ are open sets in X and Y respectively. Given $x \in X$, let

$$A(x) = \{\alpha \in A \mid x \in O_\alpha^X\}$$

so that

$$Y = i_x^{-1}(X \times Y) = \bigcup_{\alpha \in A} i_x^{-1}(O_\alpha^X \times O_\alpha^Y) = \bigcup_{\alpha \in A(x)} O_\alpha^Y$$

This is an open cover of Y . So we can extract a finite subcover: there exists $A_f(x) \subset A(x)$, finite, such that

$$Y = \bigcup_{\alpha \in A_f(x)} O_\alpha^Y \tag{1}$$

Next, for every x , $\bigcap_{\alpha \in A_f(x)} O_\alpha^X$ is open (finite intersection of open sets) and contains x .

Therefore,

$$X = \bigcup_{x \in X} \bigcap_{\alpha \in A_f(x)} O_\alpha^X$$

and we can extract a finite subcover: there exists $X_f \subset X$, finite, such that

$$X = \bigcup_{x \in X_f} \bigcap_{\alpha \in A_f(x)} O_\alpha^X \tag{2}$$

Now we claim that

$$X \times Y = \bigcup_{x \in X_f} \bigcup_{\alpha \in A_f(x)} O_\alpha^X \times O_\alpha^Y$$

Indeed, let $(x_0, y_0) \in X \times Y$. By (2), there exists $x \in X_f$ such that

$$\forall \alpha \in A_f(x) \quad x_0 \in O_\alpha^X$$

Now, by (1), there exists $\alpha \in A_f(x)$ such that $y_0 \in O_\alpha^Y$. Therefore

$$(x_0, y_0) \in O_\alpha^X \times O_\alpha^Y$$

$X \times Y$ is compact.

Fifth problem

Let S be a subset of $\mathcal{C}([0, 1])$, that is closed in $L^2(0, 1)$. Prove:

1. S is a closed subset of $\mathcal{C}([0, 1])$.
2. There exists $M > 0$ such that

$$\forall f \in S \quad \|f\|_2 \leq \|f\|_\infty \leq M\|f\|_2$$

3. For every $y \in [0, 1]$, there exists $K_y \in L^2(0, 1)$ such that

$$\forall f \in S \quad f(y) = \int_{[0,1]} K_y f$$

Solution

1 Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in S that converges (uniformly) to a (continuous) function f . Remember that

$$\forall g \in L^2(0, 1) \quad \|g\|_2^2 = \int_{[0,1]} |g|^2 \leq \|g\|_\infty^2 \tag{1}$$

Therefore $\forall n \in \mathbb{N} \quad \|f_n - f\|_2 \leq \|f_n - f\|_\infty$

Thus $(f_n)_{n \in \mathbb{N}}$ converges to f in $L^2(0, 1)$. Since S is closed in $L^2(0, 1)$, we deduce that $f \in S$.

S is closed in $\mathcal{C}([0, 1])$.

2 Since S is closed in $\mathcal{C}([0, 1])$ and $L^2(0, 1)$ it is a Banach space with either $\|\cdot\|_\infty$ and $\|\cdot\|_2$. And by **(1)**, the identity map

$$(S, \|\cdot\|_\infty) \longrightarrow (S, \|\cdot\|_2)$$

is continuous bijective. Therefore it has a continuous inverse, which means that

$$\exists M > 0 \quad \forall f \in S \quad \|f\|_\infty \leq M\|f\|_2$$

Conclusion:

$$\forall f \in S \quad \|f\|_2 \leq \|f\|_\infty \leq M\|f\|_2$$

3 Define $\forall y \in [0, 1] \quad \forall f \in S \quad J_y(f) = f(y)$

Of course, this map is perfectly well defined since functions in S are continuous. Furthermore, J_y is linear and

$$\forall f \in S \quad |J_y(f)| = |f(y)| \leq \|f\|_\infty \leq M\|f\|_2$$

So J_y is a bounded linear function on the Hilbert space $(S, \| \cdot \|_2)$. By the Riesz representation theorem,

$$\boxed{\exists K_y \in S \quad \forall f \in S \quad J_y(f) = \int_{[0,1]} K_y f}$$

Sixth problem

Suppose $(f_n)_{n \in \mathbb{N}}$ is a sequence of non-decreasing functions on $[0, 1]$ that converges pointwise to a continuous function f . Show that the convergence is uniform.

Solution

First, notice that the limit f is non-decreasing. Indeed, if $x \leq y$ are in $[0, 1]$,

$$\forall n \in \mathbb{N} \quad f_n(x) \leq f_n(y)$$

so

$$f(x) \leq f(y)$$

by taking limits in the inequality.

Let ϵ be a positive real number. Since f is continuous on the compact $[0, 1]$, it is uniformly continuous: there exists a positive η such that

$$\forall x, y \in [0, 1] \quad |x - y| \leq \eta \implies |f(x) - f(y)| \leq \epsilon \quad (1)$$

Let $0 = x_0 < x_1 < \dots < x_p = 1$ be a subdivision of $[0, 1]$, such that $x_{i+1} - x_i \leq \eta$ for every i . Since $(f_n)_{n \in \mathbb{N}}$ converges pointwise to f , we can find $N \in \mathbb{N}$ such that

$$\forall n \geq N \quad \forall i \in \{0, \dots, p\} \quad |f_n(x_i) - f(x_i)| \leq \epsilon \quad (2)$$

Now let x be any real number in $[0, 1]$ and squeeze it between x_i and x_{i+1} for some i . Then, since the functions involved are all non-decreasing,

$$\forall n \in \mathbb{N} \quad f_n(x_i) \leq f_n(x) \leq f_n(x_{i+1}) \quad (3)$$

and

$$f(x_i) \leq f(x) \leq f(x_{i+1}) \quad (4)$$

Now let $n \geq N$ and subtract (3) from (4):

$$f_n(x_i) - f(x_i) \leq f_n(x) - f(x) \leq f_n(x_{i+1}) - f(x_{i+1})$$

We have all that's needed to evaluate the terms on the left and on the right. For example, for the term on the right:

$$|f_n(x_{i+1}) - f(x_i)| \leq \underbrace{|f_n(x_{i+1}) - f(x_{i+1})|}_{\leq \epsilon \text{ by (2)}} + \underbrace{|f(x_{i+1}) - f(x_i)|}_{\leq \epsilon \text{ by (1)}}$$

Similarly

$$|f_n(x_i) - f(x_{i+1})| \leq 2\epsilon$$

so that

$$\forall n \geq N \quad |f_n(x) - f(x)| \leq 2\epsilon$$

$(f_n)_{n \in \mathbb{N}}$ converges uniformly to f .

Seventh problem

Let A and B be two closed subspaces of a Hilbert space H such that

$$\inf \{ \|x - y\| \mid (x, y) \in A \times B \quad \|x\| = \|y\| = 1 \} > 0$$

Prove that $A + B$ is complete.

Solution

Let δ be the Inf above. Then if x and y are in A and B respectively, with norm 1, we have

$$\delta \leq \|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x | y \rangle = 2 - 2\langle x | y \rangle$$

so that

$$\langle x | y \rangle \leq 1 - \frac{\delta}{2}$$

Now if x and y are non-zero in A and B respectively, we get by scaling

$$\langle x | y \rangle \leq \left(1 - \frac{\delta}{2}\right) \|x\| \|y\|$$

And of course, changing x into $-x$ gives an upper bound on $\langle x | y \rangle$:

$$-\left(1 - \frac{\delta}{2}\right) \|x\| \|y\| \leq \langle x | y \rangle \leq \left(1 - \frac{\delta}{2}\right) \|x\| \|y\| \tag{1}$$

Notice that $A \cap B = \{0\}$, otherwise there would exist $x \in A \cap B$ with norm 1, which would contradict the fact that $\delta > 0$. Therefore, A and B are in direct sum: any $z \in A + B$ can be written uniquely as $a + b$ with $a \in A$ and $b \in B$. We let p_A be the projection from $A + B$ onto A parallel to B , that is

$$p_A(z) = a$$

Similarly, the projection p_B from $A + B$ onto B parallel to A is defined by

$$p_B(z) = b = (I - p_A)z$$

We show that those are continuous by expanding the norm $\|a + b\|$, using (1) and completing a square:

$$\begin{aligned} \|z\|^2 &= \|a + b\|^2 = \|a\|^2 + \|b\|^2 + 2\langle a | b \rangle \\ &\geq \|a\|^2 + \|b\|^2 - 2\left(1 - \frac{\delta}{2}\right) \|a\| \|b\| \\ &\geq \left(\|a\| - \left(1 - \frac{\delta}{2}\right) \|b\|\right)^2 + \left(1 - \left(1 - \frac{\delta}{2}\right)^2\right) \|b\|^2 \end{aligned}$$

In particular, $\|b\|^2 = \|p_B z\|^2 \leq \frac{\|z\|^2}{1 - (1 - \delta/2)^2} \quad \text{if } \delta < 4$

But it turns out that δ is actually always less than 2 since

$$\forall(x, y) \in A \times B \quad \|x\| = \|y\| = 1 \quad \|x - y\| \leq \|x\| + \|y\| = 2$$

Anyhow, we see that p_B is bounded. The same holds for p_A since $p_A = I - p_B$. So if $(z_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $A + B$, then $(p_A z_n)_{n \in \mathbb{N}}$ and $(p_B z_n)_{n \in \mathbb{N}}$ are Cauchy as well, in A and B respectively. Since those are complete (closed in a Hilbert space), we have

$$\lim_{n \rightarrow \infty} p_A z_n = a \in A \quad \text{and} \quad \lim_{n \rightarrow \infty} p_B z_n = b \in B$$

Therefore

$$\lim_{n \rightarrow \infty} z_n = a + b \in A + B$$

Conclusion:

A + B is complete.

Eighth problem

Let \mathcal{A} be the space of Fourier transforms of $L^1(\mathbb{R})$ functions. Let $\mathcal{C}_0(\mathbb{R})$ be the space of continuous functions that tend to 0 at ∞ . Prove

1. $\mathcal{A} \subsetneq \mathcal{C}_0(\mathbb{R})$;
2. \mathcal{A} is dense in $\mathcal{C}_0(\mathbb{R})$.

Solution

1 There are various ways of solving this problem. You can either prove, as I did in lecture, that

$$\forall f \in L^1(\mathbb{R}) \quad \lim_{a \rightarrow +\infty} \int_1^a \frac{\widehat{f}(\xi) - \widehat{f}(-\xi)}{\xi} d\xi$$

exists. And deduce that any continuous function on \mathbb{R} that is odd and equal to $\frac{1}{\ln \xi}$ on $[2, +\infty)$ cannot be a Fourier transform, although it tends to 0 at ∞ .

Or you can look at the solution to **Problem 5** in the Spring 2005 exam. There, we take $\psi \in \mathcal{C}^\infty(\mathbb{R})$ with compact support in $[-1, 1]$ and define

$$\forall \lambda > 0 \quad \varphi_\lambda : x \mapsto e^{i\lambda x^2} \psi(x)$$

and we show that $\exists K > 0 \quad \forall \lambda \in \mathbb{R} \quad \|\widehat{\varphi_\lambda}\|_\infty \leq \frac{K}{\sqrt{\lambda}}$

If the Fourier transform is onto $\mathcal{C}_0(\mathbb{R})$, it has a continuous inverse since $\mathcal{C}_0(\mathbb{R})$ and $L^1(\mathbb{R})$ are Banach spaces. Thus there exists $M > 0$ such that

$$\forall f \in L^1(\mathbb{R}) \quad \|f\|_1 \leq M \|\widehat{f}\|_\infty$$

In particular $\forall \lambda > 0 \quad \|\psi\|_1 = \|\varphi_\lambda\| \leq \frac{MK}{\sqrt{\lambda}}$

which is impossible.

$\mathcal{A} \subsetneq \mathcal{C}_0(\mathbb{R})$

2 We know that the Fourier transform is a bijection from the Schwartz space onto itself. So $\mathcal{S}(\mathbb{R}) \subset \mathcal{A}(\mathbb{R})$. And of course, we know that $\mathcal{S}(\mathbb{R})$ is dense in $\mathcal{C}_0(\mathbb{R})$. Therefore,

\mathcal{A} is dense in $\mathcal{C}_0(\mathbb{R})$.

Ninth problem

Same as **Problem 6** in the Spring 2004 exam.

Tenth problem

Let f and g be continuous 2π -periodic functions. Prove that

$$\lim_{n \rightarrow \infty} \int_0^1 f(x)g(nx) dx = \int_{[0,1]} f \int_{[0,1]} g$$

Solution

Let g be a function in $L^\infty(0, 1)$ and define

$$\begin{aligned} \Phi: L^1(0, 1) &\longrightarrow \mathbb{C} \\ f &\longmapsto \widehat{f}(0)\widehat{g}(0) = \int_{(0,1)} f \int_{(0,1)} g \end{aligned}$$

Then Φ is clearly a linear functional, bounded since :

$$\forall f \in L^1(0, 1) \quad |(\Phi, f)| = |\widehat{g}(0)| \left| \int_{(0,1)} f \right| \leq |\widehat{g}(0)| \|f\|_1$$

We are going to show that

$$\forall f \in L^1(0, 1) \quad (\Phi, f) = \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int_{\mathbb{T}} f(t)g(nt) dt$$

by showing it first for trigonometric polynomials and then for $L^1(\mathbb{T})$ by density.

We define e_j to be the function $x \longmapsto e^{2\pi i j x}$ for every j in \mathbb{Z} .

• **If $j = 0$** : then we have :

$$\forall n \in \mathbb{N} \quad \int_{(0,1)} e_j(t)g(nt) dt = \int_0^1 g(nt) dt = \frac{1}{n} \int_0^n g(u) du$$

The last equality comes from the change of variable $u = nt$. Next, since g is 1-periodic, we have simply :

$$\frac{1}{n} \int_0^n g(u) du = \frac{1}{n} \times n \int_0^1 g(t) dt = \widehat{g}(0) = (\Phi, e_0)$$

Hence

$$(\Phi, e_0) = \lim_{n \rightarrow \infty} \int_{(0,1)} e_0(t)g(nt) dt$$

- For $j \neq 0$: take any integer n . Then we can write :

$$\begin{aligned} \int_{(0,1)} e_j(t)g(nt) dt &= \int_0^1 e^{2\pi ijt} g(nt) dt \stackrel{u=nt}{=} \frac{1}{n} \int_0^n e^{2\pi iju/n} g(u) du \\ &= \frac{1}{n} \sum_{k=0}^{n-1} \int_k^{k+1} e^{2\pi iju/n} g(u) du \end{aligned}$$

In each term of the sum, we make the change of variable $t = u - k$ and we use the 1-periodicity of g to get :

$$\begin{aligned} \int_{(0,1)} e_j(t)g(nt) dt &= \frac{1}{n} \sum_{k=0}^{n-1} \int_0^1 e^{2\pi ij(t+k)/n} g(t) dt \\ &= \frac{1}{n} \int_0^1 e^{2\pi iju/n} g(u) du \times \sum_{k=0}^{n-1} (e^{2\pi ij\pi/n})^k \end{aligned}$$

Now, if we take $n > |j|$, we have :

$$\sum_{k=0}^{n-1} (e^{2\pi ij\pi/n})^k = \frac{1 - e^{2\pi ij\pi}}{1 - e^{2\pi ij\pi/n}} = 0$$

so that $\forall n > |j| \quad \int_{(0,1)} e_j(t)g(nt) dt = 0 = \widehat{e}_j(0)\widehat{g}(0) = (\Phi, e_j)$

and $\lim_{n \rightarrow \infty} \int_{(0,1)} e_j(t)g(nt) dt = (\Phi, e_j)$

Since every trigonometric polynomial is a finite linear combination of the $(e_j)_{j \in \mathbb{Z}}$'s, we get :

$$\forall f \in \mathcal{P} \quad \lim_{n \rightarrow \infty} \int_{(0,1)} f(t)g(nt) dt = (\Phi, f)$$

Finally, let f be any function in $L^1(0, 1)$. If ϵ is any positive real number, we can find some P in \mathcal{P} such that $\|f - P\|_1 \leq \epsilon$. And from what precedes, there exists an integer N such that

$$\forall n \geq N \quad \left| (\Phi, P) - \int_{(0,1)} P(t)g(nt) dt \right| \leq \epsilon$$

Hence,

$$\begin{aligned} \forall n \geq N \quad \left| (\Phi, f) - \int_{(0,1)} f(t)g(nt) dt \right| &\leq |(\Phi, f) - (\Phi, P)| + \left| (\Phi, P) - \int_0^1 P(t)g(nt) dt \right| \\ &\quad + \left| \int_{(0,1)} (f - P)(t)g(nt) dt \right| \\ &\leq \|\Phi\| \|f - P\|_1 + \epsilon + \|g\|_\infty \|f - P\|_1 \end{aligned}$$

$$\forall n \geq N \quad \left| (\Phi, f) - \int_{(0,1)} f(t)g(nt) dt \right| \leq \epsilon (1 + \|\Phi\| + \|g\|_\infty)$$

which shows finally the result we want :

For every f in $L^1(0, 1)$, $\lim_{n \rightarrow \infty} \int_{(0,1)} f(t)g(nt) dt$ exists and equals (Φ, f) .