

Hidden Functional Equations for Rankin-Selberg transforms: New Results

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 - The Hilbert Modular Eisenstein Series
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 - Wrap It Up, David
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The $SL_2(\mathbb{Z})$ Eisenstein Series

- $E^*(z, s) = \frac{1}{2} \pi^{-s} \Gamma(s) \sum_{\substack{(m,n) \in \mathbb{Z}^2 \\ (m,n) \neq 0}} \frac{y^s}{|mz + n|^{2s}} \quad z \in \mathcal{H} \quad \text{Re } s > 1$
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Fourier Expansion

$$E^*(z, s) = y^s \zeta^*(2s) + y^{1-s} \zeta^*(2-2s) + 2\sqrt{y} \sum_{n \neq 0} \tau_{s-\frac{1}{2}}(|n|) K_{s-\frac{1}{2}}(2\pi|n|y) e^{2\pi i n x}$$

where

- $\tau_\omega(n) = n^{-\omega} \sum_{\substack{d|n \\ d>0}} d^{2\omega}$ is a divisor sum;
- $K_\omega(y) = \frac{1}{2} \int_0^{+\infty} e^{-\frac{y}{2}(t+\frac{1}{t})} t^\omega \frac{dt}{t}$ is a Bessel function.

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What You Need to Know About τ_ω and K_ω

- $\tau_\omega(n) = \tau_{-\omega}(n)$ and $K_\omega(y) = K_{-\omega}(y)$;
- K_ω has rapid decay:

$$\forall y > 2 \quad |K_\omega(y)| \leq M e^{-\frac{y}{2}}$$

As a consequence, $E^*(z, s)$ is the uniform limit of its Fourier series, can be analytically continued to $\mathbb{C} \setminus \{0, 1\}$ and satisfies the functional equation

$$\forall z \in \mathcal{H} \quad \forall s \neq 0, 1 \quad E^*(z, s) = E^*(z, 1 - s)$$

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The Setting

- $\mathbb{Q} \hookrightarrow K$ is a totally real field extension with ring of integers \mathfrak{o} and discriminant D .
- There are exactly N distinct embeddings $K \hookrightarrow \mathbb{C}$, denoted as $\sigma_1, \dots, \sigma_N$.
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where $\tau_\omega(\mathfrak{c}) = \mathbb{N}(\mathfrak{c})^{-\omega} \sum_{\substack{\mathfrak{b} \text{ ideal in } A \\ \mathfrak{b} | \mathfrak{c}}} \mathbb{N}(\mathfrak{b})^{2\omega}$ is a generalized divisor sum.

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Zagier's Theorem

Theorem

Let F be an $SL_2(\mathbb{Z})$ -automorphic function on \mathcal{H} . Suppose that there exists a function φ of the form

$$\varphi(y) = \sum_{i=1}^{\ell} c_i y^{\alpha_i} \ln^{n_i} y \quad \alpha_i \in \mathbb{C} \quad n_i \in \mathbb{N}$$

such that $F(z) - \varphi(y) = o(y^{-n})$ for every positive integer n . Then

$$R(s) = RN \int_{SL_2(\mathbb{Z}) \backslash \mathcal{H}} E^*(z, s) F(z) \frac{dz}{y^2} \stackrel{\text{def}}{=} \int_0^{+\infty} (a_0(y) - \varphi(y)) y^{s-1} \frac{dy}{y}$$

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Zagier's Theorem: Ze Continuation

Theorem

where $a_0(y) = \int_0^1 F(x + iy) dx$, is well defined as an absolutely convergent integral for $\text{Re } s$ big enough. It has analytic continuation to \mathbb{C} , except for poles at 0, 1, the α_j 's and the $(1 - \alpha_j)$'s. Furthermore, $R(s) = R(1 - s)$.

Previous Results

Were already studied by Professor BUMP:

- $RN \int_{SL_2(\mathbb{Z}) \backslash \mathcal{H}} E^*(z, s_0) E_{K,A}^*(z, s_1) \frac{dz}{y^2}$ when K is a totally real cubic field and A is an ideal class;
- $RN \int_{SL_2(\mathbb{Z}) \backslash \mathcal{H}} E^*(z, s_0) E^*(z, s_1) E^*(z, s_2) E^*(z, s_3) \frac{dz}{y^2}$.

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Our Goal

We suspect that a similar phenomenon occurs and want to identify the full group of functional equations for the renormalized Rankin-Selberg transform

$$R_A(s_0, s_1, s_2) = \int_{SL_2(\mathbb{Z}) \backslash \mathcal{H}} E^*(z, s_0) E^*(z, s_1) E_{K,A}^*(z, s_2) \frac{dz}{y^2}$$

when K is a real quadratic field and A is an ideal class. Zagier's theorem, together with our knowledge of the objects involved, predict 16 functional equations. In fact,

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Main Theorem

Theorem

Let K be a real quadratic field and A an ideal class. The function $R_A(s_0, s_1, s_2)$ has a group of functional equations of order 48. It is generated by the transformations

$$s_0 \mapsto 1 - s_0 \quad s_1 \mapsto 1 - s_1 \quad s_2 \mapsto 1 - s_2 \quad s_0 \longleftrightarrow s_1$$

and

$$\begin{bmatrix} s_0 \\ s_1 \\ s_2 \end{bmatrix} \xrightarrow{w} \begin{bmatrix} -\frac{s_0}{2} + \frac{s_1}{2} + s_2 \\ 1 - \frac{s_0}{2} + \frac{s_1}{2} - s_2 \\ 1 - \frac{s_0}{2} - \frac{s_1}{2} \end{bmatrix}.$$

Strategy

The polar divisor Π of R_A is the subset of \mathbb{C}^3 at which R_A is undefined. Zagier's theorem provides us with a complete description of Π . All we have to do is identify the part φ of F that is not of rapid decay and hope it looks like

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Reminder

Remember that

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Cartesian Equation of Π

By Zagier's Theorem, the polar divisor of R_A is the union of the 14 hyperplanes

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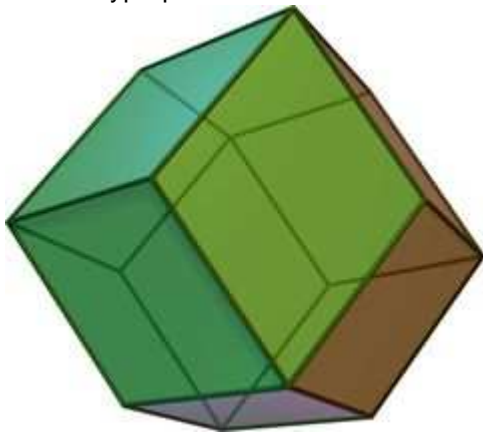
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So what does Π look like?

The 14 hyperplanes in \mathbb{C}^3 cut out a rhombic dodecahedron:



The Group of Π

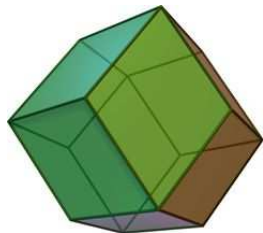
The group S_Π of transformations leaving Π invariant is computed through standard counting arguments.

For example, S_Π acts transitively on the set of vertices that are connected to 4 other vertices.

There are 6 such vertices.

Each of them is fixed by 8 transformations:
4 reflexions and 4 rotations.

So $|S_\Pi| = 6 \times 8 = 48$.



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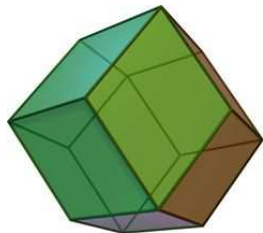
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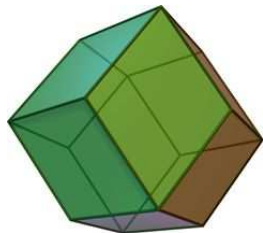
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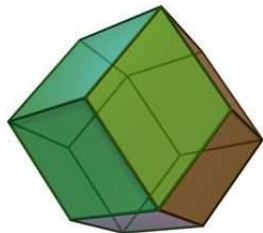
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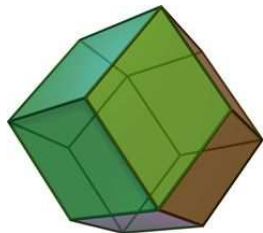
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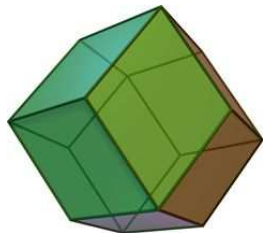
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Every functional equation for R_A should leave Π invariant.

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The $SL_3(\mathbb{Z})$ Eisenstein series I

- ν_1 and ν_2 are complex numbers with real part bigger than $\frac{2}{3}$.
- Ω is the set of quadruples $(\alpha, \beta, \xi, \eta) \in \mathfrak{a} \times (\mathfrak{a}\mathfrak{D})^{-1} \times \mathbb{Z} \times \mathbb{Z}$ with
 - α and ξ not simultaneously 0;
 - β and η not simultaneously 0;
 - $\text{Tr } \alpha\beta + \xi\eta = 0$.
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$$\begin{aligned} \text{Let } G_{\nu_1, \nu_2}(t_1, t_2) &= \frac{1}{4} \pi^{-\frac{3\nu_1}{2}} \Gamma\left(\frac{3\nu_1}{2}\right) \pi^{-\frac{3\nu_2}{2}} \Gamma\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1 + 3\nu_2 + 1) \\ &\times \sum_{(\alpha, \beta, \xi, \eta) \in \Omega} \left[(ct_1 \alpha^{(1)})^2 + (ct_2 \alpha^{(2)})^2 + \left(\frac{\xi}{t_1 t_2}\right) \right]^{-\frac{3\nu_1}{2}} \\ &\quad \times \left[\left(\frac{\beta^{(1)}}{ct_1}\right)^2 + \left(\frac{\beta^{(2)}}{ct_2}\right)^2 + (\eta t_1 t_2)^2 \right]^{-\frac{3\nu_2}{2}} \end{aligned}$$

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True facts about G_{ν_1, ν_2}

- $G_{\nu_1, \nu_2}(t_1, t_2)$ is well defined as a converging series when $\operatorname{Re} \nu_1$ and $\operatorname{Re} \nu_2$ are bigger than $\frac{2}{3}$;
- it has meromorphic continuation to \mathbb{C}^2 ;
- $G_{1-\nu_1-\nu_2, \nu_1}(t_1, t_2) = G_{\nu_1, \nu_2}(t_1, t_2)$.

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Action of \mathfrak{o}^\times on $(\mathbb{R}_+^*)^2$

Definition

If ϵ is a unit in \mathfrak{o} and t_1, t_2 are positive real numbers, we define

$$\epsilon(t_1, t_2) = (|\epsilon^{(1)}|t_1, |\epsilon^{(2)}|t_2)$$

Remarks

- $|\epsilon^{(1)}\epsilon^{(2)}| = |\mathbb{N}(\epsilon)|$ is a unit in \mathbb{Z} , so $|\epsilon^{(2)}| = |\epsilon|^{-1}$.
- As a consequence, $\epsilon(t_1, t_2) = (|\epsilon|t_1, |\epsilon|^{-1}t_2)$.
- By the Unit Theorem, there exists $\varepsilon \in \mathfrak{o}^\times$, such that

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Identifying $(\mathbb{R}_+^*)^2 / \mathfrak{o}^\times$

Theorem

The action of \mathfrak{o}^\times on $(\mathbb{R}_+^)^2$ has kernel $\{\pm 1\}$. A fundamental domain is given by*

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Action of \mathfrak{o}^\times on Ω

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$$\Omega = \left\{ (\alpha, \beta, \xi, \eta) \in \mathfrak{a} \times (\mathfrak{a}\mathfrak{D})^{-1} \times \mathbb{Z}^2 \mid \begin{cases} (\alpha, \xi) \neq 0 \\ (\beta, \eta) \neq 0 \\ \text{Tr } \alpha\beta + \xi\eta = 0 \end{cases} \right\}$$

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Identifying $\Omega/\mathfrak{o}^\times$

Let \mathcal{A} be a set of representatives of the principal ideals in \mathfrak{a} . In other words, $\mathcal{A} = \mathfrak{a}/\mathfrak{o}^\times$.

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The action of \mathfrak{o}^\times on Ω is faithful. A complete set of representatives for the orbits is

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The action of \mathfrak{o}^\times on Ω is faithful. A complete set of representatives for the orbits is

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Define

- $\Lambda = \frac{1}{4} \pi^{-\frac{3\nu_1}{2}} \Gamma\left(\frac{3\nu_1}{2}\right) \pi^{-\frac{3\nu_2}{2}} \Gamma\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1 + 3\nu_2 + 1)$ and
- $H(\alpha, \beta, \xi, \eta, t_1, t_2) = (t_1 t_2)^s \left[(ct_1 \alpha^{(1)})^2 + (ct_2 \alpha^{(2)})^2 + \left(\frac{\xi}{t_1 t_2}\right) \right]^{-\frac{3\nu_1}{2}} \times \left[\left(\frac{\beta^{(1)}}{ct_1}\right)^2 + \left(\frac{\beta^{(2)}}{ct_2}\right)^2 + (\eta t_1 t_2)^2 \right]^{-\frac{3\nu_2}{2}}$

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There are issues with this computation, though:

- We were not careful about justifying interverting integrals and sums.
- In fact, it is not valid at all: some terms in the sum are not even integrable on $(\mathbb{R}_+^*)^2 / o^\times$. For example,

$$H(0, 0, 1, 1, t_1, t_2) = (t_1 t_2)^{s+3\nu_1-3\nu_2}$$

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Troop Reorganization

We find it necessary to partition Ω into smaller subsets and sort out who is integrable and who is not.

The right way to do this is by conditioning the parameters α, β, ξ, η on whether they are 0 or not.

	$\alpha = 0 \quad \xi \neq 0$	$\alpha \neq 0 \quad \xi = 0$	$\alpha \neq 0 \quad \xi \neq 0$
$\beta = 0 \quad \eta \neq 0$	Excluded	Ω_6	Excluded
$\beta \neq 0 \quad \eta = 0$	Ω_5	Ω_4	Ω_2
$\beta \neq 0 \quad \eta \neq 0$	Excluded	Ω_3	Ω_1

Notice that each of these sets is stable under the action of σ^\times .

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Finally, Some Progress

Proposition

Let $s_0, s_1, s_2, \nu_1, \nu_2$ and s be complex numbers related by

$$\begin{cases} 3\nu_1 = s_0 + s_1 + 2s_2 - 1 \\ 3\nu_2 = s_0 - s_1 - 2s_2 + 2 \\ s = -2s_1 + 2s_2 \end{cases} \iff \begin{cases} \frac{3\nu_1 + 3\nu_2}{2} = s_0 + \frac{1}{2} \\ \frac{3\nu_1 - 3\nu_2 - 2s}{6} = s_1 - \frac{1}{2} \\ \frac{3\nu_1 - 2\nu_2 + s}{6} = s_2 - \frac{1}{2} \end{cases}$$

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Finally Some Progress II

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 &\times \sum_{\substack{\xi \in \mathfrak{D}^{-1}, n \in \mathbb{Z} \\ \xi, n \neq 0 \\ \text{Tr } \xi + n = 0}} \tau_{s_1 - \frac{1}{2}}(|n|) \tau_{s_2 - \frac{1}{2}}^{K, A}(\xi \mathfrak{D}) \\
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More Progress

Proposition

Same hypotheses as before. We have

$$\iint_{(\mathbb{R}_+^*)^2 / \mathfrak{o}^\times} G_{\nu_1, \nu_2}^{\Omega_2}(t_1, t_2) (t_1 t_2)^s \frac{dt_1 dt_2}{t_1 t_2} = \frac{2}{3} \zeta^*(2s_0) \zeta^*(2 - 2s_1)$$

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The Case of Ω_3

- Unfortunately, we run into trouble when trying to do a similar computation with $G_{\nu_1, \nu_2}^{\Omega_3}$.
- More precisely, along the way, we encounter

$$\int_0^{+\infty} e^{-x} x^{-\frac{3\nu_1 - 3\nu_2 - 2s}{6}} \frac{dx}{x}$$

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The Case of Ω_3

- Unfortunately, we run into trouble when trying to do a similar computation with $G_{\nu_1, \nu_2}^{\Omega_3}$.
- More precisely, along the way, we encounter

$$\int_0^{+\infty} e^{-x} x^{-\frac{3\nu_1 - 3\nu_2 - 2s}{6}} \frac{dx}{x}$$

- This requires $\operatorname{Re} \frac{3\nu_1 - 3\nu_2 - 2s}{6}$ to be negative, which contradicts the hypotheses made so far.

A Workaround: Poisson's Summation Formula

Poisson's Summation Formula

$$\forall t > 0 \quad \sum_{\eta \in \mathbb{Z}} e^{-\pi \eta^2 t} = \frac{1}{\sqrt{t}} \sum_{\eta \in \mathbb{Z}} e^{-\frac{\pi \eta^2}{t}}$$

This allows us to transform $G_{\nu_1, \nu_2}^{\Omega_3 \cup \Omega_4}$ and obtain a new expression on which the former strategy works.

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This allows us to transform $G_{\nu_1, \nu_2}^{\Omega_3 \cup \Omega_4}$ and obtain a new expression on which the former strategy works.

More Precisely...

Proposition

Same hypotheses as before. We have

$$\iint_{(\mathbb{R}_+^*)^2 / \mathfrak{o}^\times} (G_{\nu_1, \nu_2}^{\Omega_3 \cup \Omega_4}(t_1, t_2) - V_{\nu_1, \nu_2}(t_1, t_2))(t_1 t_2)^s \frac{dt_1 dt_2}{t_1 t_2} = \frac{2}{3} \zeta^*(2s_0) \zeta^*(2s_1)$$

$$\times \sum_{\substack{\xi \in \mathfrak{D}^{-1} \\ \xi \neq 0}} \tau_{s_2 - \frac{1}{2}}^{K, A}(\xi \mathfrak{D}) \int_0^{+\infty} y^{s_1 + 1} K_{s_2 - \frac{1}{2}}(2\pi y |\xi^{(1)}|) K_{s_2 - \frac{1}{2}}(2\pi y |\xi^{(2)}|) y^{s_0 - 1} \frac{dy}{y}$$

More Precisely II

Proposition

where

$$V_{\nu_1, \nu_2}(t_1, t_2) = \zeta^*(3\nu_1)\zeta^*(3\nu_2 - 1)(t_1 t_2)^{-\frac{3\nu_1 - 3\nu_2 + 3}{2}}$$

$$E^*\left(\frac{t_1\alpha_1^{(1)} - it_2\alpha_1^{(2)}}{t_1\alpha_2^{(1)} - it_2\alpha_2^{(2)}}, \frac{3\nu_1 + 3\nu_2 - 1}{2}\right)$$

and (α_1, α_2) is a \mathbb{Z} -basis of \mathfrak{a} .

What about Ω_5 and Ω_6 ?

It can be shown that

$$G_{\nu_1, \nu_2}^{\Omega_5}(t_1, t_2) = (t_1 t_2)^{3\nu_1 + \frac{3\nu_2}{2}} \zeta^*(3\nu_1) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\ \times E^*\left(\frac{t_1 \alpha_1^{(1)} + i t_2 \alpha_1^{(2)}}{t_1 \alpha_2^{(1)} + i t_2 \alpha_2^{(2)}}, \frac{3\nu_2}{2}\right)$$

$$G_{\nu_1, \nu_2}^{\Omega_6}(t_1, t_2) = (t_1 t_2)^{-3\nu_2 - \frac{3\nu_1}{2}} \zeta^*(3\nu_2) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\ \times E^*\left(\frac{t_1 \alpha_1^{(1)} + i t_2 \alpha_1^{(2)}}{t_1 \alpha_2^{(1)} + i t_2 \alpha_2^{(2)}}, \frac{3\nu_1}{2}\right)$$

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In Summary

Letting

$$\begin{aligned} f_{\nu_1, \nu_2}(t_1, t_2) = & \zeta^*(3\nu_1)\zeta^*(3\nu_2 - 1)(t_1 t_2)^{-\frac{3\nu_1 - 3\nu_2 + 3}{2}} E^* \left(\frac{t_1 \alpha_1^{(1)} - i t_2 \alpha_1^{(2)}}{t_1 \alpha_2^{(1)} - i t_2 \alpha_2^{(2)}}, \frac{3\nu_1 + 3\nu_2 - 1}{2} \right) \\ & + (t_1 t_2)^{3\nu_1 + \frac{3\nu_2}{2}} \zeta^*(3\nu_1)\zeta^*(3\nu_1 + 3\nu_1 - 1) E^* \left(\frac{t_1 \alpha_1^{(1)} + i t_2 \alpha_1^{(2)}}{t_1 \alpha_2^{(1)} + i t_2 \alpha_2^{(2)}}, \frac{3\nu_2}{2} \right) \\ & + (t_1 t_2)^{-3\nu_2 - \frac{3\nu_1}{2}} \zeta^*(3\nu_2)\zeta^*(3\nu_1 + 3\nu_2 - 1) E^* \left(\frac{t_1 \alpha_1^{(1)} + i t_2 \alpha_1^{(2)}}{t_1 \alpha_2^{(1)} + i t_2 \alpha_2^{(2)}}, \frac{3\nu_1}{2} \right) \end{aligned}$$

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In Summary II

Theorem

Same hypotheses as before. We have

$$R_A(s_0, s_1, s_2) = 6 \iint_{(\mathbb{R}_+^*)^2 / \mathfrak{o}^\times} (G_{\nu_1, \nu_2}(t_1, t_2) - f_{\nu_1, \nu_2}(t_1, t_2)) (t_1 t_2)^s \frac{dt_1 dt_2}{t_1 t_2}$$

What have we got so far?

- So, we have an identity linking our renormalized integral R_A , G_{ν_1, ν_2} and f_{ν_1, ν_2} .
- Although w is a functional equation for G_{ν_1, ν_2} , it does not leave f_{ν_1, ν_2} invariant.
- But, as it turns out, the integration over $(\mathbb{R}_+^*)^2 / \sigma^\times$ has the effect of creating this extra functional equation.
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Hmm... Kinda nasty...

$$\begin{aligned} \frac{R_A(s_0, s_1, s_2)}{3} = & \int_0^{+\infty} \left(\mathcal{G}_{\nu_1, \nu_2}(v) \right. \\ & - v^{-\frac{3\nu_1 - 3\nu_2 + 3}{2}} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_2 - 1) \\ & - v^{3\nu_1 + \frac{3\nu_2}{2}} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\ & \left. - v^{-3\nu_2 - \frac{3\nu_1}{2}} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2) \zeta^*(3\nu_1 + 3\nu_2 - 1) \right) v^s \frac{dv}{v} \end{aligned}$$

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Final Modifications

- Split this integral as $\int_0^1 + \int_1^{+\infty}$.
- Replace v by $\frac{1}{v}$ in the first one.
- Regroup into one integral from 1 to $+\infty$.
- Let $x = \sqrt{v}$, in order to get rid of those fractional powers.
- Shake.

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Nastier

$$\begin{aligned} \frac{R_A(s_0, s_1, s_2)}{3} = & \int_1^{+\infty} \left(\mathcal{G}_{\nu_1, \nu_2}(x^2) x^{2s} + \mathcal{G}_{\nu_1, \nu_2}(x^{-2}) x^{-2s} \right. \\ & - x^{-(3\nu_1 + 3\nu_2 + 3 - 2s)} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_2 - 1) \\ & - x^{6\nu_1 + 3\nu_2 + 2s} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\ & - x^{-(3\nu_1 + 6\nu_2 - 2s)} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\ & - x^{3\nu_1 - 3\nu_2 + 3 - 2s} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 - 2) \\ & - x^{-(6\nu_1 + 3\nu_2 + 2s)} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\ & \left. - x^{3\nu_1 + 6\nu_2 - 2s} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2) \zeta^*(3\nu_1 + 3\nu_2 - 1) \right) \frac{dx}{x} \end{aligned}$$

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$$\begin{aligned}
 \frac{R_A(s_0, s_1, s_2)}{3} = & \int_{-1}^{+\infty} \left(\mathcal{G}_{\nu_1, \nu_2}(x^2) x^{2s} + \mathcal{G}_{\nu_1, \nu_2}(x^{-2}) x^{-2s} \right. \\
 & - x^{6\nu_1 + 3\nu_2 + 2s} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\
 & - x^{3\nu_1 - 3\nu_2 + 3 - 2s} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 - 2) \\
 & \left. - x^{3\nu_1 + 6\nu_2 - 2s} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2) \zeta^*(3\nu_1 + 3\nu_2 - 1) \right) \frac{dx}{x} \\
 & - \frac{1}{3\nu_1 - 3\nu_2 + 3 - 2s} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_2 - 1) \\
 & - \frac{1}{3\nu_1 + 6\nu_2 - 2s} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\
 & - \frac{1}{6\nu_1 + 3\nu_2 + 2s} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 + 3\nu_2 - 1)
 \end{aligned}$$

Even More Nastier

$$\begin{aligned}
 \frac{R_A(s_0, s_1, s_2)}{3} = & \int_1^{+\infty} \left(\mathcal{G}_{\nu_1, \nu_2}(x^2) x^{2s} + \mathcal{G}_{\nu_1, \nu_2}(x^{-2}) x^{-2s} \right. \\
 & - x^{6\nu_1 + 3\nu_2 + 2s} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\
 & - x^{3\nu_1 - 3\nu_2 + 3 - 2s} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 - 2) \\
 & - x^{3\nu_1 + 6\nu_2 - 2s} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\
 & - x^{-(3\nu_1 + 6\nu_2 - 6 - 2s)} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2 - 1) \zeta^*(3\nu_1 + 3\nu_2 - 2) \\
 & - x^{-(3\nu_1 - 3\nu_2 - 3 - 2s)} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1 - 1) \zeta^*(3\nu_2) \\
 & \left. - x^{-(6\nu_1 + 3\nu_2 - 6 + 2s)} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1 + 3\nu_2 - 2) \zeta^*(3\nu_1 - 1) \right) \frac{dx}{x}
 \end{aligned}$$

Even More Nastier II

$$\begin{aligned}
 & -\frac{1}{3\nu_1-3\nu_2+3-2s}L\left(\frac{3\nu_1+3\nu_2-1}{2}\right)\zeta^*(3\nu_1)\zeta^*(3\nu_2-1) \\
 & -\frac{1}{3\nu_1+6\nu_2-2s}L\left(\frac{3\nu_1}{2}\right)\zeta^*(3\nu_2)\zeta^*(3\nu_1+3\nu_2-1) \\
 & -\frac{1}{6\nu_1+3\nu_2+2s}L\left(\frac{3\nu_2}{2}\right)\zeta^*(3\nu_1)\zeta^*(3\nu_1+3\nu_2-1) \\
 & +\frac{1}{3\nu_1+6\nu_2-6-2s}L\left(\frac{3\nu_1}{2}\right)\zeta^*(3\nu_2-1)\zeta^*(3\nu_1+3\nu_2-2) \\
 & +\frac{1}{3\nu_1-3\nu_2-3-2s}L\left(\frac{3\nu_1+3\nu_2-1}{2}\right)\zeta^*(3\nu_1-1)\zeta^*(3\nu_2) \\
 & +\frac{1}{6\nu_1+3\nu_2+2s-6}L\left(\frac{3\nu_2}{2}\right)\zeta^*(3\nu_1+3\nu_2-2)\zeta^*(3\nu_1-1).
 \end{aligned}$$

Even More Nastier

$$\frac{R_A(s_0, s_1, s_2)}{3} = \int_1^{+\infty} \left(\mathcal{G}_{\nu_1, \nu_2}(x^2) x^{2s} + \mathcal{G}_{\nu_1, \nu_2}(x^{-2}) x^{-2s} \right.$$

$$\begin{aligned} & - x^{6\nu_1 + 3\nu_2 + 2s} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\ & - x^{3\nu_1 - 3\nu_2 + 3 - 2s} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 - 2) \\ & - x^{3\nu_1 + 6\nu_2 - 2s} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\ & - x^{-(3\nu_1 + 6\nu_2 - 6 - 2s)} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2 - 1) \zeta^*(3\nu_1 + 3\nu_2 - 2) \\ & - x^{-(3\nu_1 - 3\nu_2 - 3 - 2s)} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1 - 1) \zeta^*(3\nu_2) \\ & \left. - x^{-(6\nu_1 + 3\nu_2 - 6 + 2s)} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1 + 3\nu_2 - 2) \zeta^*(3\nu_1 - 1) \right) \frac{dx}{x} \end{aligned}$$

Even More Nastier

$$\frac{R_A(s_0, s_1, s_2)}{3} = \int_1^{+\infty} \left(\mathcal{G}_{\nu_1, \nu_2}(x^2) x^{2s} + \mathcal{G}_{\nu_1, \nu_2}(x^{-2}) x^{-2s} \right.$$

$$\begin{aligned} & - x^{6\nu_1 + 3\nu_2 + 2s} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\ & - x^{3\nu_1 - 3\nu_2 + 3 - 2s} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 - 2) \\ & - x^{3\nu_1 + 6\nu_2 - 2s} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\ & - x^{-(3\nu_1 + 6\nu_2 - 6 - 2s)} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2 - 1) \zeta^*(3\nu_1 + 3\nu_2 - 2) \\ & - x^{-(3\nu_1 - 3\nu_2 - 3 - 2s)} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1 - 1) \zeta^*(3\nu_2) \\ & \left. - x^{-(6\nu_1 + 3\nu_2 - 6 + 2s)} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1 + 3\nu_2 - 2) \zeta^*(3\nu_1 - 1) \right) \frac{dx}{x} \end{aligned}$$

Even More Nastier

$$\frac{R_A(s_0, s_1, s_2)}{3} = \int_1^{+\infty} \left(\mathcal{G}_{\nu_1, \nu_2}(x^2) x^{2s} + \mathcal{G}_{\nu_1, \nu_2}(x^{-2}) x^{-2s} \right.$$

$$\begin{aligned} & - x^{6\nu_1 + 3\nu_2 + 2s} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\ & - x^{3\nu_1 - 3\nu_2 + 3 - 2s} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 - 2) \\ & - x^{3\nu_1 + 6\nu_2 - 2s} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\ & - x^{-(3\nu_1 + 6\nu_2 - 6 - 2s)} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2 - 1) \zeta^*(3\nu_1 + 3\nu_2 - 2) \\ & - x^{-(3\nu_1 - 3\nu_2 - 3 - 2s)} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1 - 1) \zeta^*(3\nu_2) \\ & \left. - x^{-(6\nu_1 + 3\nu_2 - 6 + 2s)} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1 + 3\nu_2 - 2) \zeta^*(3\nu_1 - 1) \right) \frac{dx}{x} \end{aligned}$$

Even More Nastier

$$\frac{R_A(s_0, s_1, s_2)}{3} = \int_1^{+\infty} \left(\mathcal{G}_{\nu_1, \nu_2}(x^2) x^{2s} + \mathcal{G}_{\nu_1, \nu_2}(x^{-2}) x^{-2s} \right.$$

$$\begin{aligned} & - x^{6\nu_1 + 3\nu_2 + 2s} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\ & - x^{3\nu_1 - 3\nu_2 + 3 - 2s} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 - 2) \\ & - x^{3\nu_1 + 6\nu_2 - 2s} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\ & - x^{-(3\nu_1 + 6\nu_2 - 6 - 2s)} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2 - 1) \zeta^*(3\nu_1 + 3\nu_2 - 2) \\ & - x^{-(3\nu_1 - 3\nu_2 - 3 - 2s)} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1 - 1) \zeta^*(3\nu_2) \\ & \left. - x^{-(6\nu_1 + 3\nu_2 - 6 + 2s)} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1 + 3\nu_2 - 2) \zeta^*(3\nu_1 - 1) \right) \frac{dx}{x} \end{aligned}$$

Even More Nastier

$$\frac{R_A(s_0, s_1, s_2)}{3} = \int_1^{+\infty} \left(\mathcal{G}_{\nu_1, \nu_2}(x^2) x^{2s} + \mathcal{G}_{\nu_1, \nu_2}(x^{-2}) x^{-2s} \right.$$

$$\begin{aligned} & -x^{6\nu_1+3\nu_2+2s} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1+3\nu_2-1) \\ & -x^{3\nu_1-3\nu_2+3-2s} L\left(\frac{3\nu_1+3\nu_2-1}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1-2) \\ & -x^{3\nu_1+6\nu_2-2s} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2) \zeta^*(3\nu_1+3\nu_2-1) \\ & -x^{-(3\nu_1+6\nu_2-6-2s)} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2-1) \zeta^*(3\nu_1+3\nu_2-2) \\ & -x^{-(3\nu_1-3\nu_2-3-2s)} L\left(\frac{3\nu_1+3\nu_2-1}{2}\right) \zeta^*(3\nu_1-1) \zeta^*(3\nu_2) \\ & \left. -x^{-(6\nu_1+3\nu_2-6+2s)} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1+3\nu_2-2) \zeta^*(3\nu_1-1) \right) \frac{dx}{x} \end{aligned}$$

Even More Nastier

$$\frac{R_A(s_0, s_1, s_2)}{3} = \int_1^{+\infty} \left(\mathcal{G}_{\nu_1, \nu_2}(x^2) x^{2s} + \mathcal{G}_{\nu_1, \nu_2}(x^{-2}) x^{-2s} \right.$$

$$\begin{aligned} & - x^{6\nu_1 + 3\nu_2 + 2s} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\ & - x^{3\nu_1 - 3\nu_2 + 3 - 2s} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 - 2) \\ & - x^{3\nu_1 + 6\nu_2 - 2s} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\ & - x^{-(3\nu_1 + 6\nu_2 - 6 - 2s)} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2 - 1) \zeta^*(3\nu_1 + 3\nu_2 - 2) \\ & - x^{-(3\nu_1 - 3\nu_2 - 3 - 2s)} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1 - 1) \zeta^*(3\nu_2) \\ & \left. - x^{-(6\nu_1 + 3\nu_2 - 6 + 2s)} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1 + 3\nu_2 - 2) \zeta^*(3\nu_1 - 1) \right) \frac{dx}{x} \end{aligned}$$

Even More Nastier

$$\frac{R_A(s_0, s_1, s_2)}{3} = \int_1^{+\infty} \left(\mathcal{G}_{\nu_1, \nu_2}(x^2) x^{2s} + \mathcal{G}_{\nu_1, \nu_2}(x^{-2}) x^{-2s} \right.$$

$$\begin{aligned} & - x^{6\nu_1 + 3\nu_2 + 2s} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\ & - x^{3\nu_1 - 3\nu_2 + 3 - 2s} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1 - 2) \\ & - x^{3\nu_1 + 6\nu_2 - 2s} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2) \zeta^*(3\nu_1 + 3\nu_2 - 1) \\ & - x^{-(3\nu_1 + 6\nu_2 - 6 - 2s)} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2 - 1) \zeta^*(3\nu_1 + 3\nu_2 - 2) \\ & - x^{-(3\nu_1 - 3\nu_2 - 3 - 2s)} L\left(\frac{3\nu_1 + 3\nu_2 - 1}{2}\right) \zeta^*(3\nu_1 - 1) \zeta^*(3\nu_2) \\ & \left. - x^{-(6\nu_1 + 3\nu_2 - 6 + 2s)} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1 + 3\nu_2 - 2) \zeta^*(3\nu_1 - 1) \right) \frac{dx}{x} \end{aligned}$$

Even More Nastier II

$$\begin{aligned}
 & -\frac{1}{3\nu_1-3\nu_2+3-2s} L\left(\frac{3\nu_1+3\nu_2-1}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_2-1) \\
 & -\frac{1}{3\nu_1+6\nu_2-2s} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2) \zeta^*(3\nu_1+3\nu_2-1) \\
 & -\frac{1}{6\nu_1+3\nu_2+2s} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1) \zeta^*(3\nu_1+3\nu_2-1) \\
 & +\frac{1}{3\nu_1+6\nu_2-6-2s} L\left(\frac{3\nu_1}{2}\right) \zeta^*(3\nu_2-1) \zeta^*(3\nu_1+3\nu_2-2) \\
 & +\frac{1}{3\nu_1-3\nu_2-3-2s} L\left(\frac{3\nu_1+3\nu_2-1}{2}\right) \zeta^*(3\nu_1-1) \zeta^*(3\nu_2) \\
 & +\frac{1}{6\nu_1+3\nu_2+2s-6} L\left(\frac{3\nu_2}{2}\right) \zeta^*(3\nu_1+3\nu_2-2) \zeta^*(3\nu_1-1)
 \end{aligned}$$

END

Acknowledgements

Thanks, yall!

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