

Matrix elements in J-Coupling scheme

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February 22, 2017

Matrix element of Ω^{40}

Starting from the fully antisymmetric matrix elements, while conserving only the six-terms of the one-body part of this operator.

$$\tilde{\Omega}_{k_1 k_2 k_3 k_4}^{40} = \langle 0 | \tilde{\Omega} | k_1 k_2 k_3 k_4 \rangle \quad (1)$$

$$\begin{aligned} \tilde{\Omega}_{k_1 k_2 k_3 k_4}^{40} = & \sum_{l_1 l_2 l_3 l_4} [\Theta_{l_1 l_2 l_3 l_4} (U_{l_1 k_1}^* U_{l_2 k_2} U_{l_3 k_3}^* V_{l_4 k_4}^* - U_{l_1 k_1} U_{l_2 k_2}^* U_{l_3 k_3} V_{l_4 k_4}^* \\ & - V_{l_4 k_1}^* U_{l_2 k_2}^* U_{l_1 k_3}^* V_{l_2 k_4}^* + U_{l_1 k_1}^* V_{l_4 k_2}^* V_{l_3 k_3}^* U_{l_2 k_4}^* \\ & + V_{l_4 k_1}^* U_{l_2 k_2}^* V_{l_3 k_3}^* U_{l_1 k_4}^* + V_{l_4 k_1}^* V_{l_3 k_2}^* U_{l_1 k_3}^* U_{l_2 k_4}^*)] \end{aligned} \quad (2)$$

Where $\{U, V\}$ stands for the usual Bogoliubov matrices. One may define a set of real reduced Bogoliubov matrices $\{U, V\}$, defined as:

$$U_{\mathbf{1}, \mathbf{2}} = \delta_{\pi_1}^{\pi_2} \delta_{j_1}^{j_2} \delta_{m_1}^{m_2} \delta_{t_1}^{t_2} \tilde{U}_{n_1, n_2}^{(ljt)_2} \quad (3)$$

$$V_{\mathbf{1}, \mathbf{2}} = (-1)^{(j_1 - m_1)} \delta_{\pi_1}^{\pi_2} \delta_{j_1}^{j_2} \delta_{m_1}^{m_2} \delta_{t_1}^{t_2} \tilde{V}_{n_1, n_2}^{(ljt)_2} \quad (4)$$

Where $\mathbf{1}$ is a shorthand to describe a quantum state labeled by $\{n_1, l_1, j_1, m_1, t_1, \pi_1\}$

Time-reversal

In order to deal directly with a matrix instead of a (4,0) tensor one may define Ω^{40} as:

$$\langle 0 | \tilde{\Omega} | k_1 k_2 \bar{k}_3 \bar{k}_4 \rangle = \langle \bar{k}_3 \bar{k}_4 | \Omega^{40} | k_1 k_2 \rangle \quad (5)$$

Where \bar{k} denote the time reversed k-state. Using the reduced set of Bogoliubov matrices, the time-reversal operation will turn $\{n_1, l_1, j_1, m_1, t_1, \pi_1\}$ to $\{n_1, l_1, j_1, -m_1, t_1, \pi_1\}$

$$\begin{aligned} \check{\Omega}_{k_1 k_2 \bar{k}_3 \bar{k}_4}^{40} = & \sum_{r_1 r_2 r_3 r_4} [\Theta_{r_1 r_2 r_3 r_4} (\delta_{r_1}^{\delta_{k_1}} \delta_{m_{r_1}}^{m_{k_1}} \delta_{r_2}^{\delta_{k_2}} \delta_{m_{r_2}}^{m_{k_2}} \delta_{r_3}^{\delta_{k_3}} \delta_{m_{r_3}}^{m_{k_3}} \delta_{r_4}^{\delta_{k_4}} \delta_{m_{r_4}}^{m_{k_4}} (-1)^{(j_{r_4} - m_{r_4} + j_{r_3} - m_{r_3})} \\ & \tilde{U}_{n_{r_1} n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2} n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_4} n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3} n_{k_4}}^{(ljt)_{k_4}} \\ & - \delta_{r_1}^{\delta_{k_1}} \delta_{m_{r_1}}^{m_{k_1}} \delta_{r_4}^{\delta_{k_2}} \delta_{m_{r_4}}^{m_{k_2}} \delta_{r_2}^{\delta_{k_3}} \delta_{m_{r_2}}^{m_{k_3}} \delta_{r_3}^{\delta_{k_4}} \delta_{m_{r_3}}^{m_{k_4}} (-1)^{(j_{r_4} - m_{r_4} + j_{r_3} - m_{r_3})} \\ & \tilde{U}_{n_{r_1} n_{k_1}}^{(ljt)_{k_1}} \tilde{V}_{n_{r_4} n_{k_2}}^{(ljt)_{k_2}} \tilde{U}_{n_{r_2} n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3} n_{k_4}}^{(ljt)_{k_4}} \\ & - \delta_{r_4}^{\delta_{k_1}} \delta_{m_{r_4}}^{m_{k_1}} \delta_{r_2}^{\delta_{k_2}} \delta_{m_{r_2}}^{m_{k_2}} \delta_{r_3}^{\delta_{k_3}} \delta_{m_{r_3}}^{m_{k_3}} \delta_{r_1}^{\delta_{k_4}} \delta_{m_{r_1}}^{m_{k_4}} (-1)^{(j_{r_4} - m_{r_4} + j_{r_3} - m_{r_3})} \\ & \tilde{V}_{n_{r_4} n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2} n_{k_2}}^{(ljt)_{k_2}} \tilde{U}_{n_{r_1} n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3} n_{k_4}}^{(ljt)_{k_4}} \\ & + \delta_{r_1}^{\delta_{k_1}} \delta_{m_{r_1}}^{m_{k_1}} \delta_{r_4}^{\delta_{k_2}} \delta_{m_{r_4}}^{m_{k_2}} \delta_{r_3}^{\delta_{k_3}} \delta_{m_{r_3}}^{m_{k_3}} \delta_{r_2}^{\delta_{k_4}} \delta_{m_{r_2}}^{m_{k_4}} (-1)^{(j_{r_4} - m_{r_4} + j_{r_3} - m_{r_3})} \\ & \tilde{U}_{n_{r_1} n_{k_1}}^{(ljt)_{k_1}} \tilde{V}_{n_{r_4} n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_3} n_{k_3}}^{(ljt)_{k_3}} \tilde{U}_{n_{r_2} n_{k_4}}^{(ljt)_{k_4}} \\ & + \delta_{r_4}^{\delta_{k_1}} \delta_{m_{r_4}}^{m_{k_1}} \delta_{r_2}^{\delta_{k_2}} \delta_{m_{r_2}}^{m_{k_2}} \delta_{r_3}^{\delta_{k_3}} \delta_{m_{r_3}}^{m_{k_3}} \delta_{r_1}^{\delta_{k_4}} \delta_{m_{r_1}}^{m_{k_4}} (-1)^{(j_{r_4} - m_{r_4} + j_{r_3} - m_{r_3})} \\ & \tilde{V}_{n_{r_4} n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2} n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_3} n_{k_3}}^{(ljt)_{k_3}} \tilde{U}_{n_{r_1} n_{k_4}}^{(ljt)_{k_4}} \\ & + \delta_{r_4}^{\delta_{k_1}} \delta_{m_{r_4}}^{m_{k_1}} \delta_{r_3}^{\delta_{k_2}} \delta_{m_{r_3}}^{m_{k_2}} \delta_{r_1}^{\delta_{k_3}} \delta_{m_{r_1}}^{m_{k_3}} \delta_{r_2}^{\delta_{k_4}} \delta_{m_{r_2}}^{m_{k_4}} (-1)^{(j_{r_4} - m_{r_4} + j_{r_3} - m_{r_3})} \\ & \tilde{V}_{n_{r_4} n_{k_1}}^{(ljt)_{k_1}} \tilde{V}_{n_{r_3} n_{k_2}}^{(ljt)_{k_2}} \tilde{U}_{n_{r_1} n_{k_3}}^{(ljt)_{k_3}} \tilde{U}_{n_{r_2} n_{k_4}}^{(ljt)_{k_4}} \end{aligned} \quad (6)$$

Where $\delta_r^k = \delta_{\pi_r} \delta_{j_r}^{j_k} \delta_{t_r}^{t_k} \delta_{l_r}^{l_k}$

As the interaction is independent under rotation one can directly sum over $\{l, j, m, t\}$, thus we can write the six terms.

$$\sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(mlj)_{k_1}, n_{r_2}(mlj)_{k_2}} (-1)^{(j_{k_3}+j_{k_4}-m_{k_3}-m_{k_4})} \tilde{U}_{n_{r_1}n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2}n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_4}n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3}n_{k_4}}^{(ljt)_{k_4}} \quad (7a)$$

$$- \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(mlj)_{k_1}, n_{r_2}(-mlj)_{k_3}} (-1)^{(j_{k_2}+j_{k_4}+m_{k_2}-m_{k_4})} \tilde{U}_{n_{r_1}n_{k_1}}^{(ljt)_{k_1}} \tilde{V}_{n_{r_4}n_{k_2}}^{(ljt)_{k_2}} \tilde{U}_{n_{r_2}n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3}n_{k_4}}^{(ljt)_{k_4}} \quad (7b)$$

$$- \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(-mlj)_{k_3}, n_{r_2}(mlj)_{k_2}} (-1)^{(j_{k_1}+j_{k_4}+m_{k_1}-m_{k_4})} \tilde{V}_{n_{r_4}n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2}n_{k_2}}^{(ljt)_{k_2}} \tilde{U}_{n_{r_1}n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3}n_{k_4}}^{(ljt)_{k_4}} \quad (7c)$$

$$\sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(mlj)_{k_1}, n_{r_2}(-mlj)_{k_4}} (-1)^{(j_{k_2}+j_{k_3}+m_{k_2}-m_{k_3})} \tilde{U}_{n_{r_1}n_{k_1}}^{(ljt)_{k_1}} \tilde{V}_{n_{r_4}n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_3}n_{k_3}}^{(ljt)_{k_3}} \tilde{U}_{n_{r_2}n_{k_4}}^{(ljt)_{k_4}} \quad (7d)$$

$$\sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(mlj)_{k_4}, n_{r_2}(mlj)_{k_2}} (-1)^{(j_{k_1}+j_{k_3}+m_{k_1}-m_{k_3})} \tilde{V}_{n_{r_4}n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2}n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_3}n_{k_3}}^{(ljt)_{k_3}} \tilde{U}_{n_{r_1}n_{k_4}}^{(ljt)_{k_4}} \quad (7e)$$

$$\sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(-mlj)_{k_3}, n_{r_2}(-mlj)_{k_4}} (-1)^{(j_{k_1}+j_{k_2}+m_{k_1}+m_{k_2})} \tilde{V}_{n_{r_4}n_{k_1}}^{(ljt)_{k_1}} \tilde{V}_{n_{r_3}n_{k_2}}^{(ljt)_{k_2}} \tilde{U}_{n_{r_1}n_{k_3}}^{(ljt)_{k_3}} \tilde{U}_{n_{r_2}n_{k_4}}^{(ljt)_{k_4}} \quad (7f)$$

J-Coupling

Each term shall now be expressed in J-coupled scheme:

Useful relations and definitions

For two Clebsch-Gordan coefficients this orthogonality relation holds

$$\sum_{m_1, m_2} C_{(jm)_1, (jm)_2}^{JM} C_{(jm)_1, (jm)_2}^{J'M'} = \delta_J^{J'} \delta_M^{M'} \quad (8)$$

Moreover the so-called mirror symmetry relations will be of a great use in the following

$$\overline{C_{(jm)_1, (jm)_2}^{JM}} = (-1)^{j_1+j_2-J} C_{(j_2m_2), (j_1m_1)}^{(J-M)} = (-1)^{j_1+j_2-J} C_{(j_1-m_1), (j_2-m_2)}^{(J-M)} \quad (9)$$

One shall also recall the relation with the 3jm symbols and the Clebsch-Gordan coefficients

$$\begin{aligned} C_{(jm)_1, (jm)_2}^{JM} &= (-1)^{(j_1-j_2+M)} \sqrt{2J+1} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} \\ &= (-1)^{(j_1-j_2+M)} \Pi_J \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} \end{aligned} \quad (10)$$

Where $\Pi_J = \sqrt{2J+1}$

One shall also note that under an odd number of permutation or under time-reversal a 3j symbol will transform as:

$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{pmatrix} = (-1)^{(j_1+j_2+J)} \begin{pmatrix} j_2 & j_1 & J \\ m_2 & m_1 & M \end{pmatrix} \quad (11)$$

$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{pmatrix} = (-1)^{(j_1+j_2+J)} \begin{pmatrix} j_1 & j_2 & J \\ -m_1 & -m_2 & -M \end{pmatrix} \quad (12)$$

One might also write a sum of four 3j-Wigner symbols in term of 6j-Wigner symbols using:

$$\begin{aligned} (-1)^{(J-M)} \left\{ \begin{matrix} j_1 & j_2 & J \\ j_4 & j_3 & J' \end{matrix} \right\} \delta_{J''}^J \delta_{M''}^M &= \Pi_J^2 \sum_{M'} \sum_{m_1, m_2, m_3, m_4} (-1)^{(j_1+j_2+j_3+j_4-m_1-m_2-m_3-m_4+J'-M')} \\ &\begin{pmatrix} j_2 & J & j_1 \\ -m_2 & -M & -m_1 \end{pmatrix} \begin{pmatrix} j_1 & j_3 & J' \\ m_1 & -m_3 & M' \end{pmatrix} \begin{pmatrix} j_3 & J'' & j_4 \\ m_3 & M'' & m_4 \end{pmatrix} \begin{pmatrix} j_4 & j_2 & J' \\ -m_4 & m_2 & -M' \end{pmatrix} \end{aligned} \quad (13)$$

Term One

For (7a)

$$\begin{aligned} &\sum_{J'M'} \sum_{\substack{m_{k_1}, m_{k_2} \\ m_{k_3}, m_{k_4}}} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(mlj)_{k_1}, n_{r_2}(mlj)_{k_2}, n_{r_4}(mlj)_{k_3}, n_{r_3}(mlj)_{k_4}} (-1)^{(j_{k_3}+j_{k_4}-m_{k_3}-m_{k_4})} \\ C_{(jm)_{k_1}, (jm)_{k_2}}^{JM} C_{(jm)_{k_3}, (jm)_{k_4}}^{J''M''} C_{(jm)_{k_1}, (jm)_{k_2}}^{J'M'} C_{(jm)_{k_4}, (jm)_{k_3}}^{J'M'} \tilde{U}_{n_{r_1}n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2}n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_4}n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3}n_{k_4}}^{(ljt)_{k_4}} \end{aligned} \quad (14)$$

Using the orthogonality of the Clebsch-Gordan Coefficients (8) and the mirror symmetry (9) Moreover one might and a mirror symmetry of the Clebsch-Gordan, thus (14) gives:

$$\begin{aligned} &\sum_{J'M'} \sum_{\substack{m_{k_3}, m_{k_4} \\ n_{r_3}, n_{r_4}}} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}(lj)_{k_4}, n_{r_4}(lj)_{k_3}}} \Theta_{n_{r_1}(lj)_{k_1}, n_{r_2}(lj)_{k_2}, n_{r_3}(lj)_{k_4}, n_{r_4}(lj)_{k_3}} (-1)^{(j_{k_3}+j_{k_4}-m_{k_3}-m_{k_4})} (-1)^{(j_{k_3}+j_{k_4}-J)} \\ &C_{(jm)_{k_3}, (jm)_{k_4}}^{JM} C_{(jm)_{k_3}, (jm)_{k_4}}^{J'M'} \tilde{U}_{n_{r_1}n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2}n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_4}n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3}n_{k_4}}^{(ljt)_{k_4}} \delta_J^{J''} \delta_M^{M''} \end{aligned} \quad (15)$$

Then using again (8),

$$\sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(lj)_{k_1}, n_{r_2}(lj)_{k_2}, n_{r_3}(lj)_{k_4}, n_{r_4}(lj)_{k_3}} (-1)^{(J-M)} \tilde{U}_{n_{r_1}n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2}n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_4}n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3}n_{k_4}}^{(ljt)_{k_4}} \delta_J^{J''} \delta_M^{M''} \quad (16)$$

Term Two

For (7b) with $M' = m_{k_4} - m_{k_2}$

$$\begin{aligned}
& - \sum_{J'M'} \sum_{\substack{m_{k_1}, m_{k_2} \\ m_{k_3}, m_{k_4}}} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(mlj)_{k_1}, n_{r_2}(-mlj)_{k_3}}^{J'} (-1)^{(j_{k_2} + j_{k_4} - M')} \\
& C_{(jm)_{k_1}, (jm)_{k_2}}^{JM} C_{(jm)_{k_3}, (jm)_{k_4}}^{J''M''} C_{(j-m)_{k_1}, (j-m)_{k_3}}^{J'M'} C_{(jm)_{k_4}, (j-m)_{k_2}}^{J'M'} \tilde{U}_{n_{r_1}n_{k_1}}^{(ljt)_{k_1}} \tilde{V}_{n_{r_4}n_{k_2}}^{(ljt)_{k_2}} \tilde{U}_{n_{r_2}n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3}n_{k_4}}^{(ljt)_{k_4}} \quad (17)
\end{aligned}$$

Utilizing (10)

$$\begin{aligned}
& - \sum_{J'M'} \sum_{\substack{m_{k_1}, m_{k_2} \\ m_{k_3}, m_{k_4}}} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(mlj)_{k_1}, n_{r_2}(-mlj)_{k_3}}^{J'} (-1)^{(2j_{k_1} - j_{k_2} + j_{k_4} + M + M' + M'')} \Pi_{J'}^2 \Pi_J \Pi_{J''} \\
& \begin{pmatrix} j_{k_2} & J & j_{k_1} \\ m_{k_2} & -M & m_{k_2} \end{pmatrix} \begin{pmatrix} j_{k_4} & J'' & j_{k_3} \\ m_{k_4} & -M'' & m_{k_3} \end{pmatrix} \begin{pmatrix} j_{k_1} & j_{k_3} & J' \\ m_{k_1} & -m_{k_3} & -M' \end{pmatrix} \begin{pmatrix} j_{k_4} & j_{k_2} & J' \\ m_{k_4} & -m_{k_2} & -M' \end{pmatrix} \\
& \tilde{U}_{n_{r_1}n_{k_1}}^{(ljt)_{k_1}} \tilde{V}_{n_{r_4}n_{k_2}}^{(ljt)_{k_2}} \tilde{U}_{n_{r_2}n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3}n_{k_4}}^{(ljt)_{k_4}} \quad (18)
\end{aligned}$$

Using twice (12) and once (11):

$$\begin{aligned}
& - \sum_{J'M'} \sum_{\substack{m_{k_1}, m_{k_2} \\ m_{k_3}, m_{k_4}}} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(mlj)_{k_1}, n_{r_2}(-mlj)_{k_3}}^{J'} (-1)^{(2j_{k_1} - j_{k_2} + j_{k_4} + M + M' + M'')} \Pi_{J'}^2 \Pi_J \Pi_{J''} \\
& \begin{pmatrix} j_{k_2} & J & j_{k_1} \\ m_{k_2} & -M & m_{k_2} \end{pmatrix} \begin{pmatrix} j_{k_3} & J'' & j_{k_4} \\ -m_{k_3} & M'' & -m_{k_4} \end{pmatrix} \begin{pmatrix} j_{k_1} & j_{k_3} & J' \\ -m_{k_1} & m_{k_3} & M' \end{pmatrix} \begin{pmatrix} j_{k_4} & j_{k_2} & J' \\ m_{k_4} & -m_{k_2} & -M' \end{pmatrix} \\
& (-1)^{(j_{k_1} + j_{k_3} + J')} \tilde{U}_{n_{r_1}n_{k_1}}^{(ljt)_{k_1}} \tilde{V}_{n_{r_4}n_{k_2}}^{(ljt)_{k_2}} \tilde{U}_{n_{r_2}n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3}n_{k_4}}^{(ljt)_{k_4}} \quad (19)
\end{aligned}$$

Thus with (13) taking the sum over m_{k_i} and M'

$$\begin{aligned}
& - \sum_{J'} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(lj)_{k_1}, n_{r_2}(lj)_{k_3}}^{J'} (-1)^{(2j_{k_1})} \Pi_J'^2 (-1)^{J-M} \begin{Bmatrix} j_{k_1} & j_{k_2} & J \\ j_{k_4} & j_{k_3} & J' \end{Bmatrix} \tilde{U}_{n_{r_1}n_{k_1}}^{(ljt)_{k_1}} \tilde{V}_{n_{r_4}n_{k_2}}^{(ljt)_{k_2}} \tilde{U}_{n_{r_2}n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3}n_{k_4}}^{(ljt)_{k_4}} \delta_J^{J''} \delta_M^{M''} \\
& \quad (20)
\end{aligned}$$

Term Three

For term (7c), with $M' = m_{k_4} - m_{k_1}$

$$\begin{aligned}
& - \sum_{J'M'} \sum_{\substack{m_{k_1}, m_{k_2} \\ m_{k_3}, m_{k_4}}} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(-mlj)_{k_3}, n_{r_2}(mlj)_{k_2}}^{J'} (-1)^{(j_{k_1} + j_{k_4} - M')} \\
& C_{(jm)_{k_1}, (jm)_{k_2}}^{JM} C_{(jm)_{k_3}, (jm)_{k_4}}^{J''M''} C_{(j-m)_{k_3}, (jm)_{k_2}}^{J'M'} C_{(jm)_{k_4}, (j-m)_{k_1}}^{J'M'} \tilde{V}_{n_{r_4}n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2}n_{k_2}}^{(ljt)_{k_2}} \tilde{U}_{n_{r_1}n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3}n_{k_4}}^{(ljt)_{k_4}} \quad (21)
\end{aligned}$$

Utilizing (10)

$$\begin{aligned}
& - \sum_{J'M'} \sum_{\substack{m_{k_1}, m_{k_2} \\ m_{k_3}, m_{k_4}}} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(-mlj)_{k_3}, n_{r_2}(mlj)_{k_2}}^{J'} (-1)^{(j_{k_1} + j_{k_4} - M')} \Pi_{J'}^2 \Pi_J \Pi_{J''} \\
& \begin{pmatrix} j_{k_1} & j_{k_2} & J \\ m_{k_1} & m_{k_2} & -M \end{pmatrix} \begin{pmatrix} j_{k_3} & j_{k_4} & J'' \\ m_{k_3} & m_{k_4} & -M'' \end{pmatrix} \begin{pmatrix} j_{k_3} & j_{k_2} & J' \\ -m_{k_3} & m_{k_2} & -M' \end{pmatrix} \begin{pmatrix} j_{k_4} & j_{k_1} & J' \\ m_{k_4} & -m_{k_1} & -M' \end{pmatrix} \\
& (-1)^{(-2j_{k_2} + 2j_{k_3} + M + 2M' - M'')} \tilde{V}_{n_{r_4}n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2}n_{k_2}}^{(ljt)_{k_2}} \tilde{U}_{n_{r_1}n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3}n_{k_4}}^{(ljt)_{k_4}} \quad (22)
\end{aligned}$$

Using the fact that a 3j symbol is invariant under an even number of permutations:

$$\begin{aligned}
& - \sum_{J'M'} \sum_{\substack{m_{k_1}, m_{k_2} \\ m_{k_3}, m_{k_4}}} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(-mlj)_{k_3}, n_{r_2}(mlj)_{k_2}}^{J'} (-1)^{(j_{k_1}+j_{k_4}-M')} \Pi_J^2 \Pi_J \Pi_{J''} \\
& \begin{pmatrix} j_{k_2} & J & j_{k_1} \\ m_{k_2} & -M & m_{k_1} \end{pmatrix} \begin{pmatrix} j_{k_1} & j_{k_4} & J' \\ m_{k_1} & -m_{k_4} & M' \end{pmatrix} \begin{pmatrix} j_{k_4} & J'' & j_{k_3} \\ m_{k_4} & -M'' & m_{k_3} \end{pmatrix} \begin{pmatrix} j_{k_3} & j_{k_2} & J' \\ -m_{k_3} & m_{k_2} & -M' \end{pmatrix} (-1)^{(j_{k_1}+j_{k_2}+J)} \\
& (-1)^{(-2j_{k_2}+2j_{k_3}+M+2M'-M'')} \tilde{V}_{n_{r_4}n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2}n_{k_2}}^{(ljt)_{k_2}} \tilde{U}_{n_{r_1}n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3}n_{k_4}}^{(ljt)_{k_4}} \quad (23)
\end{aligned}$$

Thus one can get

$$\begin{aligned}
& - \sum_{J'M'} \sum_{\substack{m_{k_1}, m_{k_2} \\ m_{k_3}, m_{k_4}}} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(-mlj)_{k_3}, n_{r_2}(mlj)_{k_2}}^{J'} (-1)^{(j_{k_1}+j_{k_4}-M')} \Pi_J^2 \Pi_J \Pi_{J''} \\
& \begin{pmatrix} j_{k_2} & J & j_{k_1} \\ m_{k_2} & -M & m_{k_1} \end{pmatrix} \begin{pmatrix} j_{k_1} & j_{k_4} & J' \\ m_{k_1} & -m_{k_4} & M' \end{pmatrix} \begin{pmatrix} j_{k_4} & J'' & j_{k_3} \\ m_{k_4} & -M'' & m_{k_3} \end{pmatrix} \begin{pmatrix} j_{k_3} & j_{k_2} & J' \\ -m_{k_3} & m_{k_2} & -M' \end{pmatrix} (-1)^{(j_{k_1}+j_{k_2}+J)} \\
& (-1)^{(-2j_{k_2}+2j_{k_3}+M+2M'-M'')} \tilde{V}_{n_{r_4}n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2}n_{k_2}}^{(ljt)_{k_2}} \tilde{U}_{n_{r_1}n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3}n_{k_4}}^{(ljt)_{k_4}} \quad (24)
\end{aligned}$$

Inserting

$$(-1)^{(2(J+j_{k_2}+j_{k_3}))} = 1 \quad (25)$$

Thus with (13) taking the sum over m_{k_i} and M'

$$\begin{aligned}
& - \sum_{J'} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(lj)_{k_3}, n_{r_2}(lj)_{k_2}}^{J'} (-1)^{(j_{k_1}+2j_{k_2}-j_{k_3}+J+J')} \Pi_J^2 (-1)^{J+M} \left\{ \begin{matrix} j_{k_1} & j_{k_2} & J \\ j_{k_3} & j_{k_4} & J' \end{matrix} \right\} \\
& \tilde{V}_{n_{r_4}n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2}n_{k_2}}^{(ljt)_{k_2}} \tilde{U}_{n_{r_1}n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3}n_{k_4}}^{(ljt)_{k_4}} \delta_J^{J''} \delta_M^{M''} \quad (26)
\end{aligned}$$

Term Four

For term (7d), with $M' = m_{k_3} - m_{k_2}$

$$\begin{aligned}
& - \sum_{J'M'} \sum_{\substack{m_{k_1}, m_{k_2} \\ m_{k_3}, m_{k_4}}} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(mlj)_{k_1}, n_{r_2}(-mlj)_{k_4}}^{J'} (-1)^{(j_{k_2}+j_{k_3}-M')} \\
& C_{(jm)_{k_1}, (jm)_{k_2}}^{JM} C_{(jm)_{k_3}, (jm)_{k_4}}^{J''M''} C_{(jm)_{k_1}, (j-m)_{k_4}}^{J'M'} C_{(jm)_{k_3}, (j-m)_{k_2}}^{J''M''} \tilde{U}_{n_{r_1}n_{k_1}}^{(ljt)_{k_1}} \tilde{V}_{n_{r_4}n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_3}n_{k_3}}^{(ljt)_{k_3}} \tilde{U}_{n_{r_2}n_{k_4}}^{(ljt)_{k_4}} \quad (27)
\end{aligned}$$

Utilizing (10)

$$\begin{aligned}
& \sum_{J'M'} \sum_{\substack{m_{k_1}, m_{k_2} \\ m_{k_3}, m_{k_4}}} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(mlj)_{k_1}, n_{r_2}(-mlj)_{k_4}}^{J'} (-1)^{(2j_{k_1}-j_{k_2}-j_{k_3}+2j_{k_4}+M+M'+M'')} \Pi_J^2 \Pi_J \Pi_{J''} \\
& \begin{pmatrix} j_{k_2} & J & j_{k_1} \\ m_{k_2} & -M & m_{k_1} \end{pmatrix} \begin{pmatrix} j_{k_1} & j_{k_4} & J' \\ m_{k_1} & -m_{k_4} & -M' \end{pmatrix} \begin{pmatrix} j_{k_4} & J'' & j_{k_3} \\ m_{k_4} & -M'' & m_{k_3} \end{pmatrix} \begin{pmatrix} j_{k_3} & j_{k_2} & J' \\ m_{k_3} & -m_{k_2} & -M' \end{pmatrix} \\
& \tilde{U}_{n_{r_1}n_{k_1}}^{(ljt)_{k_1}} \tilde{V}_{n_{r_4}n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_3}n_{k_3}}^{(ljt)_{k_3}} \tilde{U}_{n_{r_2}n_{k_4}}^{(ljt)_{k_4}} \quad (28)
\end{aligned}$$

Then using twice (12)

$$\begin{aligned} & \sum_{J'M'} \sum_{\substack{m_{k_1}, m_{k_2} \\ m_{k_3}, m_{k_4}}} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(mlj)_{k_1}, n_{r_2}(-mlj)_{k_4}}^{J'} (-1)^{(2j_{k_1} - j_{k_2} - j_{k_3} + 2j_{k_4} + M + M' + M'')} \Pi_J'^2 \Pi_J \Pi_{J''} \\ & \begin{pmatrix} j_{k_2} & J & j_{k_1} \\ m_{k_2} & -M & m_{k_1} \end{pmatrix} \begin{pmatrix} j_{k_1} & j_{k_4} & J' \\ -m_{k_1} & m_{k_4} & M' \end{pmatrix} \begin{pmatrix} j_{k_4} & J'' & j_{k_3} \\ -m_{k_4} & M'' & -m_{k_3} \end{pmatrix} \begin{pmatrix} j_{k_3} & j_{k_2} & J' \\ m_{k_3} & -m_{k_2} & -M' \end{pmatrix} \\ & (-1)^{(j_{k_1} + j_{k_4} + J')} (-1)^{(j_{k_3} + j_{k_4} + J'')} \tilde{U}_{n_{r_1} n_{k_1}}^{(ljt)_{k_1}} \tilde{V}_{n_{r_4} n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_3} n_{k_3}}^{(ljt)_{k_3}} \tilde{U}_{n_{r_2} n_{k_4}}^{(ljt)_{k_4}} \quad (29) \end{aligned}$$

Thus with (13) taking the sum over m_{k_i} and M' :

$$\begin{aligned} & \sum_{J'} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(lj)_{k_1}, n_{r_2}(lj)_{k_4}}^{J'} (-1)^{(2j_{k_1} - j_{k_3} - j_{k_4} + J)} \Pi_J'^2 (-1)^{J-M} \begin{Bmatrix} j_{k_1} & j_{k_2} & J \\ j_{k_3} & j_{k_4} & J' \end{Bmatrix} \\ & \tilde{U}_{n_{r_1} n_{k_1}}^{(ljt)_{k_1}} \tilde{V}_{n_{r_4} n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_3} n_{k_3}}^{(ljt)_{k_3}} \tilde{U}_{n_{r_2} n_{k_4}}^{(ljt)_{k_4}} \delta_J^{J''} \delta_M^{M''} \quad (30) \end{aligned}$$

Term Five

For term (7e), with $M' = m_{k_3} - m_{k_1}$

$$\begin{aligned} & \sum_{J'M'} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(-mlj)_{k_4}, n_{r_2}(mlj)_{k_2}}^{J'} (-1)^{(j_{k_1} + j_{k_3} - M')} \\ & C_{(jm)_{k_1}, (jm)_{k_2}}^{JM} C_{(jm)_{k_3}, (jm)_{k_4}}^{J''M''} C_{(j-m)_{k_4}, (jm)_{k_2}}^{J'M'} C_{(jm)_{k_3}, (j-m)_{k_1}}^{J'M'} \tilde{V}_{n_{r_4} n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2} n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_3} n_{k_3}}^{(ljt)_{k_3}} \tilde{U}_{n_{r_1} n_{k_4}}^{(ljt)_{k_4}} \quad (31) \end{aligned}$$

Utilizing (10)

$$\begin{aligned} & \sum_{J'M'} \sum_{\substack{m_{k_1}, m_{k_2} \\ m_{k_3}, m_{k_4}}} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(mlj)_{k_1}, n_{r_2}(-mlj)_{k_4}}^{J'} (-1)^{(j_{k_1} - 2j_{k_2} + 3j_{k_3} + 2j_{k_4} + M + M' + M'')} \Pi_J'^2 \Pi_J \Pi_{J''} \\ & \begin{pmatrix} j_{k_2} & J & j_{k_1} \\ m_{k_2} & -M & m_{k_1} \end{pmatrix} \begin{pmatrix} j_{k_4} & j_{k_2} & J' \\ -m_{k_4} & m_{k_2} & -M' \end{pmatrix} \begin{pmatrix} j_{k_3} & j_{k_4} & J'' \\ m_{k_3} & m_{k_4} & -M'' \end{pmatrix} \begin{pmatrix} j_{k_3} & j_{k_1} & J' \\ m_{k_3} & -m_{k_1} & -M' \end{pmatrix} \\ & \tilde{V}_{n_{r_4} n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2} n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_3} n_{k_3}}^{(ljt)_{k_3}} \tilde{U}_{n_{r_1} n_{k_4}}^{(ljt)_{k_4}} \quad (32) \end{aligned}$$

Using twice (12) and once (11)

$$\begin{aligned} & \sum_{J'M'} \sum_{\substack{m_{k_1}, m_{k_2} \\ m_{k_3}, m_{k_4}}} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(mlj)_{k_1}, n_{r_2}(-mlj)_{k_4}}^{J'} (-1)^{(j_{k_1} - 2j_{k_2} + 3j_{k_3} + 2j_{k_4} + M + M' + M'')} \Pi_J'^2 \Pi_J \Pi_{J''} \\ & \begin{pmatrix} j_{k_2} & J & j_{k_1} \\ m_{k_2} & -M & m_{k_1} \end{pmatrix} \begin{pmatrix} j_{k_2} & j_{k_4} & J' \\ -m_{k_2} & m_{k_4} & M' \end{pmatrix} \begin{pmatrix} j_{k_4} & J'' & j_{k_3} \\ -m_{k_4} & M'' & -m_{k_3} \end{pmatrix} \begin{pmatrix} j_{k_3} & j_{k_1} & J' \\ m_{k_3} & -m_{k_1} & -M' \end{pmatrix} \\ & (-1)^{(-1)^{(j_{k_3} + j_{k_4} + J'')}} \tilde{V}_{n_{r_4} n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2} n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_3} n_{k_3}}^{(ljt)_{k_3}} \tilde{U}_{n_{r_1} n_{k_4}}^{(ljt)_{k_4}} \quad (33) \end{aligned}$$

Thus with (13) taking the sum over m_{k_i} and M' :

$$\begin{aligned} & \sum_{J'} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{n_{r_1}(lj)_{k_1}, n_{r_2}(lj)_{k_4}}^{J'} (-1)^{(j_{k_1} - j_{k_3} - J')} \Pi_J'^2 (-1)^{J-M} \begin{Bmatrix} j_{k_1} & j_{k_2} & J \\ j_{k_3} & j_{k_4} & J' \end{Bmatrix} \tilde{V}_{n_{r_4} n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2} n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_3} n_{k_3}}^{(ljt)_{k_3}} \tilde{U}_{n_{r_1} n_{k_4}}^{(ljt)_{k_4}} \delta_J^{J''} \delta_{J'}^{J''} \quad (34) \end{aligned}$$

Term Six

The same exact steps used for the first term, gives for term (7f)

$$\sum_{J'M'} \sum_{\substack{m_{k_1}, m_{k_2} \\ m_{k_3}, m_{k_4}}} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{\substack{n_{r_1}(-mlj)_{k_3}, n_{r_2}(-mlj)_{k_4} \\ n_{r_3}(-mlj)_{k_2}, n_{r_4}(-mlj)_{k_1}}}^{J'} (-1)^{(j_{k_1}+j_{k_2}+m_{k_1}+m_{k_2})} \\ C_{(jm)_{k_1}, (jm)_{k_2}}^{JM} C_{(jm)_{k_3}, (jm)_{k_4}}^{J''M''} C_{(j-m)_{k_3}, (j-m)_{k_4}}^{J'M'} C_{(j-m)_{k_2}, (j-m)_{k_1}}^{J'M'} \tilde{U}_{n_{r_1} n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2} n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_4} n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3} n_{k_4}}^{(ljt)_{k_4}} \quad (35)$$

Using twice (9):

$$\sum_{J'M'} \sum_{\substack{m_{k_1}, m_{k_2} \\ m_{k_3}, m_{k_4}}} \sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{\substack{n_{r_1}(-mlj)_{k_3}, n_{r_2}(-mlj)_{k_4} \\ n_{r_3}(-mlj)_{k_2}, n_{r_4}(-mlj)_{k_1}}}^{J'} (-1)^{(j_{k_1}+j_{k_2}+m_{k_1}+m_{k_2})} \\ (-1)^{(j_{k_3}+j_{k_4}-J')} C_{(jm)_{k_1}, (jm)_{k_2}}^{JM} C_{(jm)_{k_3}, (jm)_{k_4}}^{J''M''} C_{(jm)_{k_3}, (jm)_{k_4}}^{J'M'} C_{(jm)_{k_1}, (jm)_{k_2}}^{J'M'} \tilde{U}_{n_{r_1} n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2} n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_4} n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3} n_{k_4}}^{(ljt)_{k_4}} \quad (36)$$

The orthogonality relations (8) gives:

$$\sum_{\substack{n_{r_1}, n_{r_2} \\ n_{r_3}, n_{r_4}}} \Theta_{\substack{n_{r_1}(lj)_{k_3}, n_{r_2}(lj)_{k_4} \\ n_{r_3}(lj)_{k_2}, n_{r_4}(lj)_{k_1}}}^J (-1)^{(j_{k_1}+j_{k_2}+j_{k_3}+j_{k_4})} (-1)^{(J-M)} \tilde{U}_{n_{r_1} n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2} n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_4} n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3} n_{k_4}}^{(ljt)_{k_4}} \delta_J^{J''} \delta_M^{M''} \quad (37)$$

Final result

By factorizing the overall phase $(-1)^{(J-M)}$ and $\delta_J^{J''} \delta_M^{M''}$ one might write (6) in J-coupled scheme as:

$$\begin{aligned}
\check{\Omega}_{k_1 k_2 \bar{k}_3 \bar{k}_4}^{40} = & \sum_{J'} \sum_{n_{r_1} n_{r_2} n_{r_3} n_{r_4}} \left[\Theta_{\substack{n_{r_1} (lj)_{k_1}, n_{r_2} (lj)_{k_2} \\ n_{r_3} (lj)_{k_4}, n_{r_4} (lj)_{k_3}}}^{J'} \tilde{U}_{n_{r_1} n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2} n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_4} n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3} n_{k_4}}^{(ljt)_{k_4}} \right. \\
& - \Theta_{\substack{n_{r_1} (lj)_{k_1}, n_{r_2} (lj)_{k_3} \\ n_{r_3} (lj)_{k_4}, n_{r_4} (lj)_{k_2}}}^{J'} (-1)^{(2j_{k_1})} \Pi_J'^2 (-1)^{J-M} \left\{ \begin{matrix} j_{k_1} & j_{k_2} & J \\ j_{k_4} & j_{k_3} & J' \end{matrix} \right\} \\
& \tilde{U}_{n_{r_1} n_{k_1}}^{(ljt)_{k_1}} \tilde{V}_{n_{r_4} n_{k_2}}^{(ljt)_{k_2}} \tilde{U}_{n_{r_2} n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3} n_{k_4}}^{(ljt)_{k_4}} \\
& - \Theta_{\substack{n_{r_1} (lj)_{k_3}, n_{r_2} (lj)_{k_2} \\ n_{r_3} (lj)_{k_4}, n_{r_4} (lj)_{k_1}}}^{J'} (-1)^{(j_{k_1} + 2j_{k_2} - j_{k_3} + J + J')} \Pi_J'^2 \left\{ \begin{matrix} j_{k_1} & j_{k_2} & J \\ j_{k_3} & j_{k_4} & J' \end{matrix} \right\} \\
& \tilde{V}_{n_{r_4} n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2} n_{k_2}}^{(ljt)_{k_2}} \tilde{U}_{n_{r_1} n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3} n_{k_4}}^{(ljt)_{k_4}} \\
& + \Theta_{\substack{n_{r_1} (lj)_{k_1}, n_{r_2} (lj)_{k_4} \\ n_{r_3} (lj)_{k_3}, n_{r_4} (lj)_{k_2}}}^{J'} (-1)^{(2j_{k_1} - j_{k_3} - j_{k_4} + J)} \Pi_J'^2 \left\{ \begin{matrix} j_{k_1} & j_{k_2} & J \\ j_{k_3} & j_{k_4} & J' \end{matrix} \right\} \\
& \tilde{U}_{n_{r_1} n_{k_1}}^{(ljt)_{k_1}} \tilde{V}_{n_{r_4} n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_3} n_{k_3}}^{(ljt)_{k_3}} \tilde{U}_{n_{r_2} n_{k_4}}^{(ljt)_{k_4}} \\
& + \Theta_{\substack{n_{r_1} (lj)_{k_1}, n_{r_2} (lj)_{k_4} \\ n_{r_3} (lj)_{k_3}, n_{r_4} (lj)_{k_2}}}^{J'} (-1)^{(j_{k_1} - j_{k_3} - J')} \Pi_J'^2 \left\{ \begin{matrix} j_{k_1} & j_{k_2} & J \\ j_{k_3} & j_{k_4} & J' \end{matrix} \right\} \\
& \tilde{V}_{n_{r_4} n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2} n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_3} n_{k_3}}^{(ljt)_{k_3}} \tilde{U}_{n_{r_1} n_{k_4}}^{(ljt)_{k_4}} \\
& \Theta_{\substack{n_{r_1} (lj)_{k_3}, n_{r_2} (lj)_{k_4} \\ n_{r_3} (lj)_{k_2}, n_{r_4} (lj)_{k_1}}}^{J'} (-1)^{(j_{k_1} + j_{k_2} + j_{k_3} + j_{k_4})} \tilde{U}_{n_{r_1} n_{k_1}}^{(ljt)_{k_1}} \tilde{U}_{n_{r_2} n_{k_2}}^{(ljt)_{k_2}} \tilde{V}_{n_{r_4} n_{k_3}}^{(ljt)_{k_3}} \tilde{V}_{n_{r_3} n_{k_4}}^{(ljt)_{k_4}} \\
& \left. (-1)^{(J-M)} \delta_J^{J''} \delta_M^{M''} \right]
\end{aligned} \tag{38}$$