

8ème Cours: Treewidth et al.

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Schedule

Motivations for graph decompositions
 Treewidth
 Duality : brambles and treewidth
 Graph Minors
 Big theorems on Graph Minors
 Other width parameters
 Cliquewidth
 Branch decomposition and connectivity functions
 Coping with branchwidth
 Relationships between treewidth and branchwidth
 Rankwidth
 Some relationships between these widths

Is it worth to work on graph decompositions ?

- ▶ Apply divide and conquer techniques to solve hard problems on difficult instances.
- ▶ Find efficient encoding to store graphs
- ▶ Find structural properties (comparability graphs, tournaments ...)
- ▶ **Byproduct.** Many efficient algorithms are based on a decomposition technique.
 Example : ear decomposition for planarity testing.

Measure the distance from a given graph to a tree

Easy cases

Some graphs are like trees : chordal graphs, cographs and some variations of P_4 -free graphs ...

General case

One can associated a tree to a graph : modular decomposition tree, split decomposition tree ...

Bad news

Most graphs are prime.

Using Google Scholar on 12 november 2012

- ▶ Graph Decomposition 902 000 answers
- ▶ Treewidth 10 000 answers
- ▶ NP-complete 160 000 answers
 NP-hard 173 000 answers

k-trees

Recursive Definition

A clique with $k+1$ vertices is a k -tree. If G is a k -tree, then we obtain another k -tree G' by adding a new vertex and joining this vertex to a k -clique of G .

Clearly a k -tree is a chordal graph.

1-trees = ordinary trees

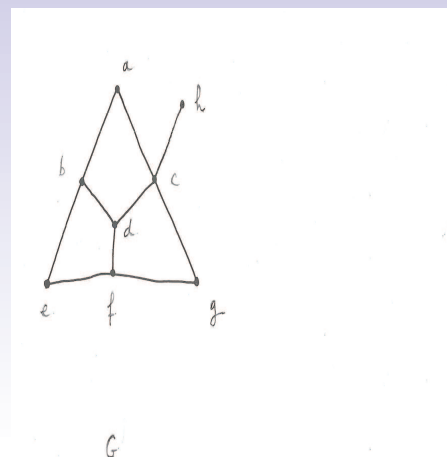
2-trees = series-parallel networks

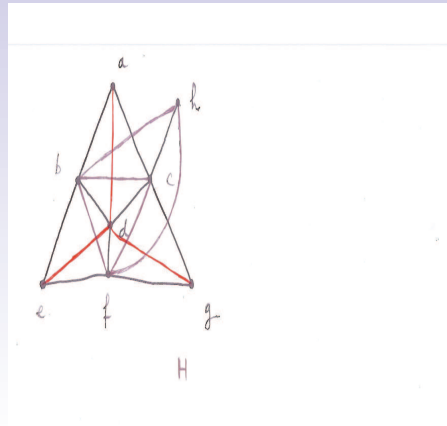
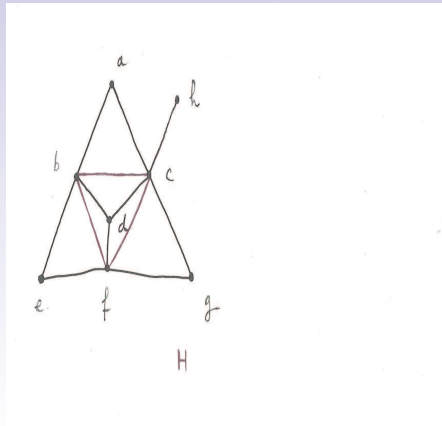
Partial k-trees

A partial k -tree is a partial subgraph of a k -tree (deleting edges and vertices)

Definition

$Treewidth(G) =$ least integer k s.t. G is a partial k -tree.





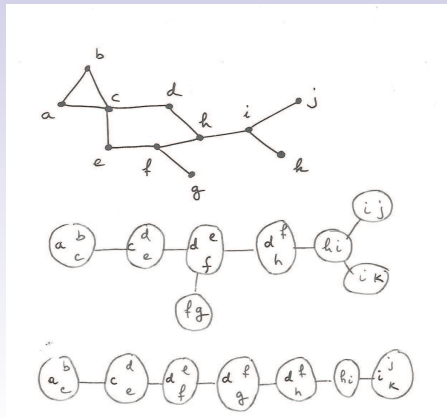
Tree decomposition

$G = (V, E)$ has a tree decomposition $D = (S, T)$
 S is a collection of subsets of V called **bags**, T a tree whose vertices are elements of S such that :

- (0) The union of elements in S is V
- (i) $\forall e \in E, \exists i \in I$ with $e \in G(S_i)$.
- (ii) $\forall x \in V$, the elements of S containing x form a subtree of T .

Definition

$$Decomp(G) = \text{Min}_D(\text{Max}_{S_i \in S} \{|S_i| - 1\})$$



Equivalences

1. $\forall G, \text{Treewidth}(G) = Decomp(G)$
2. $\text{Treewidth}(G) = \text{Min}_H \text{triangulation of } G \{\omega(H) - 1\}$
3. Computing treewidth is NP-hard.

Other definitions of treewidth in terms of cop-robber games, using graph grammars
 But it turns out that this parameter is a fundamental parameter for graph theory.

Some examples

1. G is a tree iff $\text{treewidth}(G) = 1$
2. $\text{treewidth}(K_n) = n - 1$
3. If G is a cycle then $\text{treewidth}(G) = 2$. (It can be seen as two chains in parallel, i.e. a series-parallel graph)
4. $\text{treewidth}(K_{n,m}) = \min(n, m)$
5. $\text{treewidth}(G_{n,m}) = \min(n, m)$, the lower bound is hard to obtain !
6. $\text{treewidth}(G)$ (resp. pathwidth) measures the distance from G to a tree (resp. to a chain)

Easy properties

If G is chordal then $\text{treewidth}(G) = \omega(G) - 1$
 $\text{treewidth}(G) = k$ iff G can be decomposed using only separators of size less than k .

Fundamental lemma

Let ab an edge of T some tree decomposition of G and T_1, T_2 be the two connected components of $T - ab$, then $V_a \cap V_b$ is a separator between $V_1 - V_2$ and $V_2 - V_1$, where $V_1 = \cup_{i \in T_1} V_i$ and $V_2 = \cup_{j \in T_2} V_j$.

Proof of the lemma

Démonstration.

Let ab be an edge of a tree decomposition T of G . $T - ab$ is disconnected into T_1 and T_2 , two subtrees of T .

Let $V_1 = \cup_{t \in T_1} V_t$ and $V_2 = \cup_{t \in T_2} V_t$. If $V_a \cap V_b$ is not a separator, then it exists $u \in V_1 - V_2$ and $v \in V_2 - V_1$ and $uv \in E$. But then in which bag can the edge uv belongs to? Since using property (i) of tree decomposition each edge must belong to some bag. This cannot be in T_1 , neither in T_2 , a contradiction. \square

For the previous lemma, we only use the definition of any tree-decomposition, not an optimal one. It also explains the use of property (i) in the definition of tree decomposition.

Computations of treewidth

- ▶ There exists polynomial approximation algorithms
- ▶ For every fixed k , it exists a linear algorithm to check whether $Treewidth(G) \leq k$ Bodlaender 1992. (Big constant for the linearity).
- ▶ Find an efficient algorithm for small values 3, 4, 5... is still a research problem

Real applications of treewidth

1. Graphs associated with programs have bounded treewidth 4,5, 6 depending on the programming language, Thorup 1997.
2. Constraint Satisfaction Problem, Feder's Theorem
3. Can also be defined on hypergraphs which yields applications for Data bases via Acyclic Hypergraphs
4. Good Heuristic for the Traveling Salesman Problem, V. Chvátal

Some applications of treewidth

- ▶ Pacman from V. Limouzy
Playing with Pacman is equivalent to compute the treewidth of a special graph (3 or 4)
- ▶ Treewidth of some Internet subgraph Laurent Viennot
For a network of routers N , $80 \leq treewidth(N) \leq 160$

Other definitions of treewidth in terms of cop-robber games, using graph grammars
But it turns out that this parameter is a fundamental parameter for graph theory and has many theoretical applications.

1. $Treewidth(G) = \text{Min}_H \text{chordal completion of } G\{\omega(H) - 1\}$
2. $MinFillin(G) = \text{Min}_{H=(V,F)} \text{chordal completion of } G\{|F|\}$
3. $Pathwidth(G) = \text{Min}_H \text{interval completion of } G\{\omega(H) - 1\}$
4. $MinIntervalcompletion(G) = \text{Min}_{H=(V,F)} \text{interval completion of } G\{|F|\}$
5. Theorem Arnborg, Corneil, Proskurowski, 1987 : These 4 problems are NP-hard.

Divide and Conquer Approach

E. Proskurowski

When he introduced partial k-trees as a generalisation of trees preserving dynamic programming he aimed at polynomial algorithms for bounded tree-graphs (i.e. the size of the bags is bounded).

Generic Divide and Conquer Algorithm

Solve the problem for a leaf and then recurse

Metaconsequence

For most graph parameters Π , there exists an exact algorithm in $O(2^{O(\text{treewidth}(G))})$.

Balanced separator

Every graph G , $|G| \geq k + 4$, with $\text{treewidth}(G) = k$ admits a separator S of size $k + 1$ such that $G - S$ is partitioned into A, B , with no edge between A and B and :

$$\frac{1}{3}(n - k - 1) \leq |A|, |B| \leq \frac{2}{3}(n - k - 1)$$

The study of the interplay between logics and combinatorial structures yields knowledge on complexity theories

B. Courcelle studying graph rewriting systems or graph grammars obtained :

Meta-Theorem

Any graph problem that can be expressed with a formula of the Monadic Second Order logic (MSO), if G has bounded treewidth then it exists a linear algorithm to solve this problem on G .

Why Meta

A unique theorem for a whole class of problems on a class of graphs.

Monadic Second Order Logic

For graphs :

x_1, \dots, x_n variables

X_i subset of vertices

Atoms :

$E(x, y)$ true iff xy is an edge of G

$X(x)$ true if $x \in X$

Classical logical connectors (equality, implication, negation ...) to make formulas

Quantification over variables and subsets of vertices are allowed.

Examples :

- ▶ $\phi : \exists x_1 \dots \exists x_k \forall y \bigwedge_{1 \leq i \leq k} ((x_i = y) \vee (E(x_i, y)))$
 $\phi = G$ has a dominating set of size k .
- ▶ Idem $\psi_k = G$ has a k -coloration but here we need second order logic.
 $\exists X_1, \dots, \exists X_k, \forall x \exists \bigvee x \in X_j, \forall i, \forall x \forall y \in X_i \text{ not } E(x, y)$
- ▶ The number of quantifiers is an indication of the complexity of the formula (i.e. the problem)
We do not need to use quantifiers on subsets of edges (via le biparti d'incidence)
- ▶ MSO does not correspond to a complexity class : it contains also NP-complete problems such as 3-colorability.

Not all graph problems can be expressed in MSO. Isomorphism for example or computing the permanent of a matrix.
 MSO_1 vertices variables
 MSO_2 also edge variables
(Nearly the same logics for our purpose).

Linear ?

Linear but with a giant constant :

$$(2^{2^{\dots}})^h$$

h the size of this exponential tower which depends on the MSOL formula.

Grohe, Frick 2005

h is unbounded unless $P = NP$.

Recent Courcelle's work try to avoid the construction of the automaton (which can be of huge size), using lazy evaluation techniques from verification or game theory.

A Duality Theorem

Brambles

$G = (V, E)$; $\mathcal{B} = \{X_i | X_i \subseteq V\}$ such that :

$G(X_i)$ connected and $\forall i, j, G(X_i \cup X_j)$ connected.

A transversal of a bramble is a set $\tau \subseteq V$ such that $\forall i, \tau \cap X_i \neq \emptyset$

$$bn(G) = \text{Max}_{\tau \text{ transversal of } \mathcal{B}} (\min |\tau|)$$

Duality Theorem Roberston Seymour 86

For every graph G , $\text{treewidth}(G) = bn(G) - 1$

$$bn(G) \leq \text{treewidth}(G) + 1$$

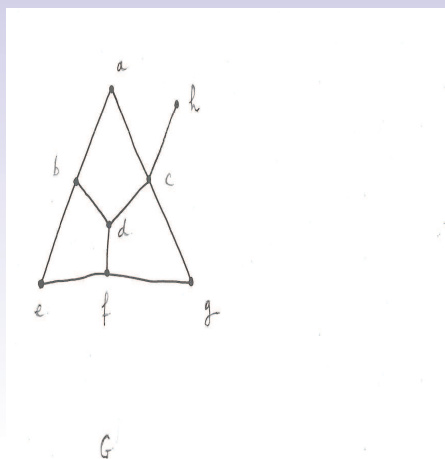
Démonstration.

Let us consider an optimal tree decomposition T and a bramble \mathcal{B} . Every $X_i \in \mathcal{B}$ corresponds to a subtree T_{X_i} in T , using connectivity (i).

Furthermore condition (ii) implies $\forall i, j, T_{X_i} \cap T_{X_j} \neq \emptyset$

Using Helly property on these subtrees, they all have a common vertex t . The bag associated with t meets every element of the bramble and is a transversal for \mathcal{B} .

The size of t is $\leq \text{treewidth}(G) + 1$. □



Lower bound 1

Consider the family $\{\{d\}, \{a, b\}, \{e, f\}, \{g, c\}\}$ which satisfies the properties of a bramble .

With a transversal $\{d, b, c, f\}$ of minimum of size 4.

Therefore using bramble theorem $\text{treewidth}(G) \geq 3$

Therefore this example has exactly treewidth 3

This Min(Max) Max(Min) theorem gives evidence for treewidth computations as for example for the grid.

Consider the bramble made up with all the crosses it has a transversal of size $\min(n, m)$,

therefore $\text{treewidth}(G_{n,m}) \geq \min(n, m) - 1$

Bramble again

- For a connected graph G , the set of all connected subgraphs with $\lceil \frac{n}{2} \rceil$ vertices is a bramble of G .