

8th Lecture : Graphs and orders MPRI 2013-2014

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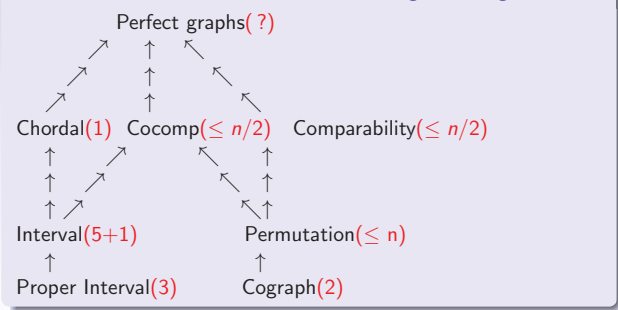
Sophie Germain, 12 novembre 2013

Table des Matières

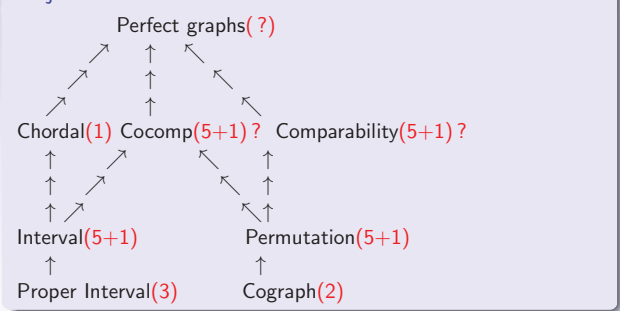
- More on graph classes
- Basic notions for partial orders
- Definitions related to partial orders
- Dilworth theorem
- Applications

A hierarchy of some hereditary classes of graphs

Minimum number of LBFS used in a recognition algorithm



Conjectures

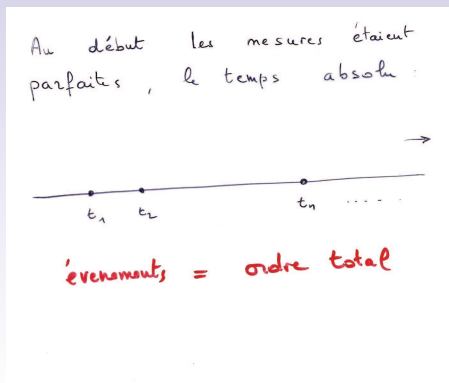


This number of LBFS needed to recognize some class of graphs is a kind of measure of its structural complexity. But LBFS is also used as a preprocessing for the following problems :

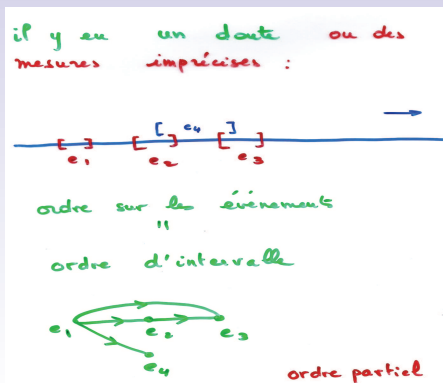
- ▶ Modular decomposition
- ▶ Transitive orientation of comparability graphs
- ▶ Recognition of circular arc graphs
- ▶ ...

Modelisations using partial orders are very important in applications : distributed systems, logics, physics, biology ...

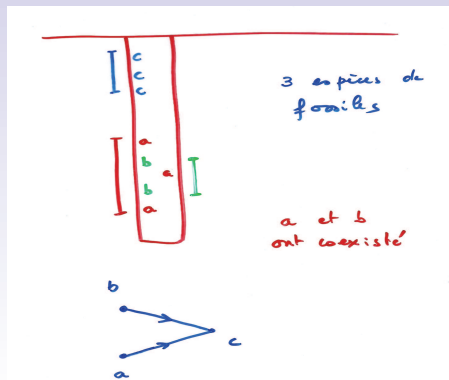
Examples



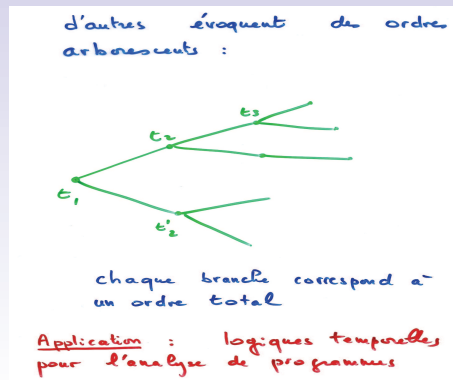
Examples



Examples



Examples



If the data is not accurate enough, the natural generalization is to consider paths on a rooted directed tree, which leads to directed path graphs.
 But we could also deal with subtrees but then we found another subclass of chordal graphs.
 In some particular cases we can only use the size of the intersection of intervals this leads to tolerance graphs and orders.

3 simple paradigms to solve an algorithmic problem

- ▶ Play with the definition of the problem (ex : compute factoring permutation instead of modular decomposition tree)
- ▶ Play with the representations (ex : use various geometric representations of particular classes of graphs)
- ▶ Play with dualities, if any (ex : G versus \overline{G} , or in case of planarity use the dual)
- ▶ Of course this often is clear, only afterwards!

Graphs and orders

Definitions

$P = (X, \leq)$ is an order if the relation \leq is reflexive, transitive and antisymmetric.
 $Q = (X, \leq)$ is a quasi-order if the relation \leq is reflexive and transitive.
 If either $x \leq_P y$ or $y \leq_P x$, then x, y are said to be comparable, else they are incomparable (denoted by $x \parallel y$).
 We say that y covers x denoted by $x <_P y$, if there no $z \in X$, $z \neq x, y$ such that $x \leq_P z \leq_P y$.

Definitions

An antichain (resp. chain) is a set of pairwise incomparable (resp. comparable) elements.
 N.B. an antichain (resp. a chain) is an independant set (resp. a directed path),
 A linear extension τ of P is a total order compatible with P , i.e. $x \leq_P y$ implies $x \leq_\tau y$.

A partial order can be represented by an acyclic directed graph $G = (X, U)$ satisfying $x \leq_P y$ iff \exists a path from x to y in G .
 If $xy \in U$ and it exists a path of length ≥ 2 from x to y in G , then xy is called a transitivity arc, else it is a covering arc.
 Among the lattice of these acyclic graphs, they are two extremal ones :
 $G^t = (X, U^t)$ the transitive closure of P , for which $x \leq_P y$ iff $x, y \in U^t$.
 $G_r = (X, U_r)$ the transitive reduction of P , having no transitivity arcs. Other names : Hasse diagram, directed covering graph.
 G^t has all possible transitivity arcs, while G_r has none.

Unfortunately these two representations of an order have not the same "complexity" or size. Since for a chain order, $G^t = (X, U^t)$ can be quadratic in size of $G_r = (X, U_r)$.

The transitive closure-reduction gap

It is not known if it is possible to extract $G^t = (X, U^t)$ or $G_r = (X, U_r)$ from a given representation G of P in linear time.

Best known algorithms require either $O(n.m)$ or $O(n^\alpha)$, with $\alpha > 2$. Best complexity for boolean matrix multiplication.

Graal for algorithms on partial orders

A linear-time algorithm accepting any given representation G of P .

Other representations : planar diagrams.

Undirected graphs associated with an order

Comparability graphs

An undirected graph $G = (X, E)$ is a comparability graph if it can be directed as a partial order (more precisely as the graph of the transitive closure)

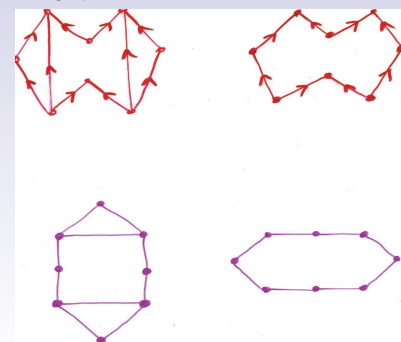
Covering graph

An undirected graph $G = (X, E)$ is a covering graph if it can be directed as the directed covering graph of a partial order (more precisely as the graph of the transitive reduction of the partial order)

4 different graphs

Transitive graph

Directed covering graph



Comparability graph

Covering graph

These notions are very different

- ▶ It is polynomial to recognize a comparability graph, while it is NP-hard for a covering graph (Nesestril, Rödl 89).
- ▶ There exists a nice characterization of comparability graphs, while there exist only necessary conditions for covering graphs.

Examples of comparability graphs

- ▶ Tree are comparability graphs, but C_5 a cycle with 5 vertices is not a comparability graph.
- ▶ Interval graphs are complement of comparability graphs (cocomparability graphs).
- ▶ Chordal are not always comparability graphs (cf. the 3-sun or its complement)

Counterexamples

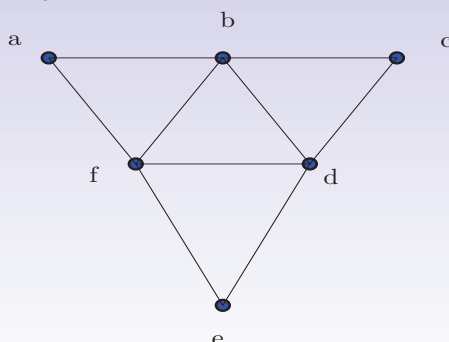


FIGURE: A chordal graph which is not a comparability graph

The duality between graphs and orders

Cocomparability graphs

The complement of a comparability graph is called a cocomparability graph. There is a kind of duality between a cocomparability graph and the partial orders associated with its complement.

Examples

- cographs and series-parallel orders
- Interval graphs and interval orders
- Permutation graphs and permutation orders also called 2-dimensional orders.
- Trapezoid graphs and trapezoid orders
- ...

Using this duality can help to solve a problem (many examples of that).

Dilworth theorem

Dilworth 1950

For every finite order P , The maximum size of an antichain (denoted by $width(P)$) is equal to the minimum size of a chain partition of P (denoted by $\theta(P)$).

Consequences

Comparability and cocomparability graphs are perfect.

Proof

It is well known that G is perfect iff \overline{G} is perfect. Dilworth's theorem just says that $cocomp(P)$ is perfect and therefore also $comp(P)$.

Another min-max polynomial theorem similar to max flow min cut. Computation of $width(P)$ can be done in $O(n^{5/2})$ using a maximum matching algorithm.

Erdős, Szekeres 1935

From every sequence of $pq + 1$ integers one can always extract a decreasing subsequence of size $p + 1$ or an increasing one of size $q + 1$.

Application for computing a maximum independent set in a comparability graph G

1. Transitively orient G as a partial order P
2. Compute a minimal path partition of P via a matching algorithm on a bipartite graph
3. Extract an independent set from this set of paths