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# Assignment 5

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1. (a) If we assume that there is a single source and frequency we have the coupled equations:

$$\min_m J(m) = \frac{1}{2} \int_{\Omega} (u - u^{\text{obs}})^2 dx + \frac{\alpha}{2} \int_{\Omega} \nabla m \cdot \nabla m dx \quad (1)$$

$$-\Delta u(x) - k^2 m(x) u(x) = f(x) \quad \text{in } \Omega \quad (2)$$

$$u = 0 \quad \text{on } \partial\Omega \quad (3)$$

Let us first compute the weak form of (2):

Find  $u \in H_0^1$  such that:

$$\begin{aligned} & \int_{\Omega} (\Delta u + k^2 m u + f) p dx = 0 \quad \forall p \in H_0^1 \\ & \Rightarrow \int_{\Omega} (-\nabla u \nabla p + k^2 m u p + f p) dx + \int_{\partial\Omega} p \nabla u \cdot n = 0 \quad \forall p \in H_0^1 \end{aligned}$$

Thus we can compute the Lagrangian:

$$L(u, m, p) = \frac{1}{2} \int_{\Omega} (u - u^{\text{obs}})^2 dx + \frac{\alpha}{2} \int_{\Omega} \nabla m \cdot \nabla m dx + \int_{\Omega} (-\nabla u \nabla p + k^2 m u p + f p) dx$$

We need to compute its three derivatives to be able to compute fully the gradient of  $J$ :

- $\partial_p L = 0$

$$\frac{d}{d\varepsilon} L(u, p + \varepsilon \hat{p}, m)|_{\varepsilon=0} = 0 \Rightarrow \int_{\Omega} -\nabla u \nabla \hat{p} + k^2 m \hat{p} + \hat{p} f dx = 0 \quad \forall \hat{p} \in H_0^1$$

- $\partial_u L = 0$

$$\frac{d}{d\varepsilon} L(u + \varepsilon \hat{u}, p, m)|_{\varepsilon=0} = 0 \Rightarrow \int_{\Omega} (\hat{u}(u - u^{\text{obs}}) - \nabla p \nabla \hat{u} + k^2 m \hat{u}) dx = 0 \quad \forall \hat{u} \in H_0^1$$

- $\partial_m L = 0$

$$\frac{d}{d\varepsilon} L(u, p, m + \varepsilon \hat{m})|_{\varepsilon=0} = 0 \Rightarrow \int_{\Omega} \alpha m \hat{m} + k^2 p m \hat{m} dx = 0 \quad \forall \hat{m} \in H^1$$

Therefore, solving the first equation for  $p$ , the second for  $u$  and injecting in the last one we end up with the gradient of  $J$ .

(b) With the inverse convention as in the notes:

- $-\partial_{pp} L = 0$

$$\begin{aligned} & -\partial_{pu} L \\ & = \int_{\Omega} -\nabla \tilde{u} \nabla \hat{p} + k^2 \tilde{u} m \hat{p} dx \end{aligned}$$

$$\begin{aligned} & -\partial_{pm} L \\ & = \int_{\Omega} k^2 u \tilde{m} \hat{p} dx \end{aligned}$$

Incremental state equation:

$$\int_{\Omega} -\nabla \tilde{u} \nabla \hat{p} + k^2 \tilde{u} m \hat{p} + k^2 u \tilde{m} \hat{p} dx = 0 \quad \forall \hat{p}$$

- $-\partial_{up} L$

$$= \int_{\Omega} -\nabla \tilde{p} \nabla \hat{u} + k^2 \tilde{p} m \hat{u} dx$$

$$\begin{aligned}
& - \partial_{uu} L \\
& \quad = \int_{\Omega} \hat{u} \tilde{u} dx \\
& - \partial_{um} L \\
& \quad = \int_{\Omega} -\nabla p \nabla \hat{u} + k^2 p \tilde{m} \hat{u} dx
\end{aligned}$$

Incremental adjoint equation:

$$\int_{\Omega} \hat{u} \tilde{u} - \nabla p \nabla \hat{u} + k^2 p \tilde{m} \hat{u} - \nabla p \nabla \hat{u} + k^2 p \tilde{m} \hat{u} dx = 0 \quad \forall \hat{u}$$

- $- \partial_{mp} L$
- $= \int_{\Omega} k^2 \tilde{p} u \hat{m} dx$
- $- \partial_{mu} L$
- $= \int_{\Omega} k^2 p \tilde{u} \hat{m} dx$
- $- \partial_{mm} L$
- $= \int_{\Omega} \alpha \tilde{m} \hat{m} dx$

Incremental control equation:

$$\int_{\Omega} k^2 \tilde{p} u \hat{m} + k^2 p \tilde{u} \hat{m} + \alpha \tilde{m} \hat{m} dx = -g \quad \forall \hat{m}$$

(c) With the same method as in (a) we can compute the Lagragian for arbitrary  $i$  and  $j$ :

$$L(\mathbf{u}, m, \mathbf{p}) = \frac{1}{2} \sum_i \sum_j \int_{\Omega} (u_{ij} - u_{ij}^{\text{obs}})^2 dx + \frac{\alpha}{2} \int_{\Omega} \nabla m \cdot \nabla m dx + \int_{\Omega} (-\nabla u_{ij} \nabla p_{ij} + k_i^2 u_{ij} m p_{ij} + f_j p_{ij}) dx$$

Thus  $\forall i, j :$

- $\partial_{p_{ij}} L = 0$
- $\frac{d}{d\varepsilon} L(u_{ij}, p_{ij} + \varepsilon \hat{p}_{ij}, m)|_{\varepsilon=0} = 0 \Rightarrow \int_{\Omega} -\nabla u_{ij} \nabla \hat{p}_{ij} + k_i^2 u_{ij} m \hat{p}_{ij} + \hat{p}_{ij} f_j dx = 0 \quad \forall \hat{p}_{ij} \in H_0^1$
- $\partial_{u_{ij}} L = 0$
- $\frac{d}{d\varepsilon} L(u_{ij} + \varepsilon \hat{u}_{ij}, p_{ij}, m)|_{\varepsilon=0} = 0 \Rightarrow \int_{\Omega} (\hat{u}_{ij} (u_{ij} - u_{ij}^{\text{obs}}) - \nabla p_{ij} \nabla \hat{u}_{ij} + k_i^2 p m \hat{u}_{ij}) dx = 0 \quad \forall \hat{u}_{ij} \in H_0^1$
- $\partial_m L = 0$
- $\frac{d}{d\varepsilon} L(u_{ij}, p_{ij}, m + \varepsilon \hat{m})|_{\varepsilon=0} = \int_{\Omega} \alpha m \hat{m} + k_i^2 p_{ij} u_{ij} \hat{m} dx (= 0) \quad \forall \hat{m} \in H^1$

Thus we need to solve  $i \times j$  state equations and as many adjoint to compute one gradient.

2. We want to recover the parameter  $m$  name `a_true`: For all this part we will use  $\gamma = 1e^{-9}$ . It is not the best choice but we would have to compare the results for different  $\gamma$  and we already did that in the others homeworks. Thus we will just focus on the implementation of the method.

(a) The knew weak form of the state equation is :

$$\int_{\Omega} m \nabla u \nabla \hat{u} + \hat{\mathbf{u}} \mathbf{v} \nabla \mathbf{u} - \hat{u} f dx$$

Therefore we can write the Lagrangian:

$$L(u, m, p) = \frac{1}{2} \int_{\Omega} (u - u^{\text{obs}})^2 dx + \frac{\gamma}{2} \int_{\Omega} \nabla m \cdot \nabla m dx - \int_{\Omega} (m \nabla u \nabla p + v p \nabla u - f p) dx$$

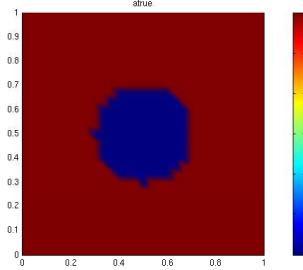


Figure 1: parameter wa want to recover

We have already compute several times the derivatives of the Lagrangian in class and homeworks thus I will just focus on the changes in the code. Thus the expressions given to COMSOL become with:

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 $u = u,$   $p = p,$   $u^{\text{obs}} = ud,$   $\tilde{p} = u_{\text{test}},$   $\tilde{u} = p_{\text{test}},$   $m = a,$   $\tilde{m} = \text{grad\_test},$   $g = \text{grad}$ 

fem.equ.expr.goal = '-(atrue * (udx * udx_test + udy * udy_test)+30*ud_test*udx
- f * ud_test)';
fem.equ.expr.state = '-(a * (ux * ux_test + uy * uy_test)+30*u_test*ux
- f * u_test)';
fem.equ.expr.adjoint = '-(a * (px * px_test + py * py_test)+30*px_test*p
- (ud - u) * p_test)';
fem.equ.expr.control = [ '(grad * grad_test - gamma * (ax * gradx_test + ay * grady_test)
- (px * ux + py * uy) * grad_test)' ];

```

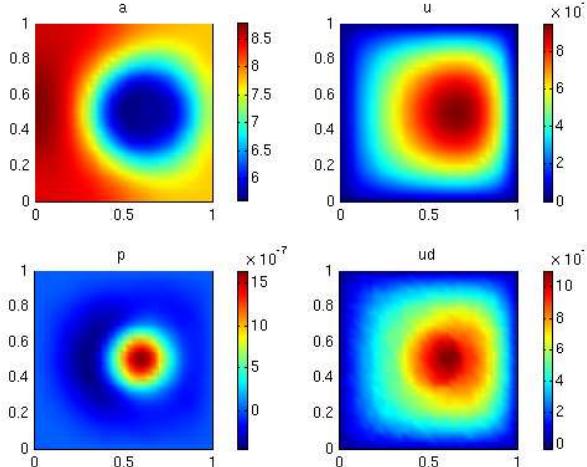


Figure 2: TN reconstitution for 1000 iteration

(b) The minimization term becomes:

$$\frac{1}{2} \int_{\Omega} (u - u^{\text{obs}})^2 dx + \frac{\gamma}{2} \int_{\Omega} (\nabla m \cdot \nabla m + \varepsilon)^{\frac{1}{2}} dx$$

Therefore we can write the Lagrangian:

$$L(u, m, p) = \frac{1}{2} \int_{\Omega} (u - u^{\text{obs}})^2 dx + \frac{\gamma}{2} \int_{\Omega} (\nabla m \cdot \nabla m + \varepsilon)^{\frac{1}{2}} dx - \int_{\Omega} (m \nabla u \nabla p + vp \nabla u - fp) dx$$

Obviously, the only term which will change is the control term in  $\gamma$ , from the homework 3 we know that it is:

$$\nabla m \nabla \hat{m} \rightarrow \frac{\nabla m \nabla \hat{m}}{(\nabla m \nabla m + \varepsilon)^{\frac{1}{2}}}$$

Thus the equation becomes:

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fem.equ.expr.control = ['
    (grad * grad_test - gamma * (ax * gradx_test + ay * grady_test) /
    realsqrt(ax*ax+ay*ay+0.1),
     -(px * ux + py * uy) * grad_test)];
```

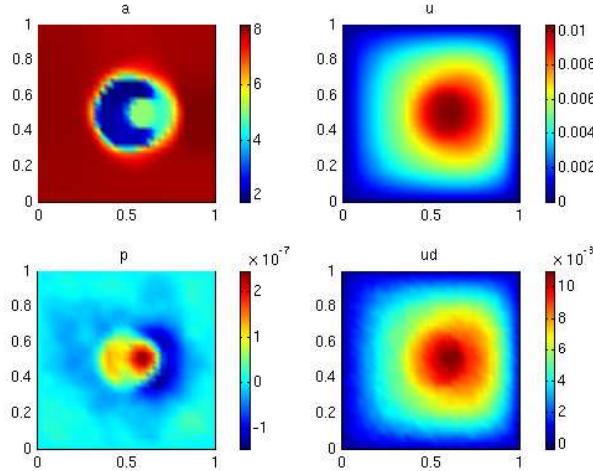


Figure 3: TV reconstitution for 1000 iteration

#### Conclusion:

As expected, the TN method smooths too much the parameter (as there is a big gradient for  $r = 2$ ), thus TV is more appropriate. Furthermore one can see that the advection affects the recovery of the parameter.