



Multiscale Methods in CDF Homework 7

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1 Exercise 6.1

Let us compute the $\mathbf{A}_i, \forall i$:

$$\mathbf{F}_i = \begin{pmatrix} U_{i+1} \\ \left(\frac{U_{i+1}}{U_1} \mathbf{U}' + (\gamma - 1)(U_4 + \frac{1}{2U_1}(U_2^2 + U_3^2))\delta i \right) \\ \frac{U_{i+1}U_4}{U_1} + \frac{U_{i+1}}{U_1}(\gamma - 1)(U_4 + \frac{1}{2U_1}(U_2^2 + U_3^2)) \end{pmatrix}$$

Thus for instance:

$$\mathbf{A}_1 = \begin{bmatrix} 0 & \frac{1}{U_1} & 0 & 0 \\ \frac{U_2^2(-3 + \gamma) + U_3^2(-1 + \gamma)}{2U_1^2} & -\frac{U_2(-3 + \gamma)}{U_1} & \frac{U_3 - U_3\gamma}{U_1} & -1 + \gamma \\ -\frac{U_2U_3}{U_1^2} & \frac{U_3}{U_1} & \frac{U_2}{U_1} & 0 \\ \frac{U_2(U_2^2(-1 + \gamma) + U_3^2(-1 + \gamma) - U_1U_4\gamma)}{U_1^3} & -\frac{3U_2^2(-1 + \gamma) + U_3^2(-1 + \gamma) - 2U_1U_4\gamma}{2U_1^2} & -\frac{U_2U_3(-1 + \gamma)}{U_1^2} & \frac{U_2\gamma}{U_1} \end{bmatrix}$$

Which is not symmetric. As the other one is just a matter of interverting U_3 and U_2 and the lines 2 and 3 it is neither symmetric.

To prove that \mathbf{K} is not symmetric nor positive definit it is enough to show that \mathbf{K}_{11} is not.

$$\mathbf{F}_1^{\text{visc}} + \mathbf{F}_1^{\text{heat}} =$$

$$\begin{pmatrix} 0 \\ \frac{\lambda + 2\mu}{U_1}(U_{2,1} - U_{1,1}u_1) + \frac{\lambda}{U_1}(U_{3,2} - U_{1,2}u_2) \\ \frac{\mu}{U_1}(U_{2,2} + U_{3,1} - U_{1,1}u_2 - U_{1,2}u_1) \\ \frac{(\lambda + 2\mu)U_2}{U_1^2}(U_{2,1} - U_{1,1}u_1) + \frac{(\lambda)U_2}{U_1^2}(U_{3,2} - U_{1,2}u_2) + \frac{\mu U_3}{U_1^2}(U_{2,2} + U_{3,1} - U_{1,1}u_2 - U_{1,2}u_1) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{-\kappa}{C_v U_1}(U_{4,1} - U_{1,1}e - U_{2,1}u_1 - U_{3,1}u_2 + U_{1,1}(u_2^2 + u_1^2)) \end{pmatrix}$$

Thus:

$$\mathbf{K}_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{\lambda + 2\mu}{U_1}u_1 & \frac{\lambda + 2\mu}{U_1} & 0 & 0 \\ -\frac{\mu}{U_1}u_2 & \frac{\mu}{U_1} & 0 & 0 \\ \frac{\kappa}{C_v U_1}(e - (u_2^2 + u_1^2)) - \frac{(\lambda + 2\mu)U_2}{U_1^2}(u_1) - \frac{\mu U_3}{U_1^2}(u_2) & \frac{\kappa}{C_v U_1}u_1 + \frac{(\lambda + 2\mu)U_2}{U_1^2} & \frac{\kappa}{C_v U_1}u_2 + \frac{\mu U_3}{U_1^2}U_{3,1} & \frac{-\kappa}{C_v U_1} \end{bmatrix}$$

Which is obviously not symetric and neither definite positive because of the first line

2 Exercise 6.2

$$\tilde{\mathbf{A}}_0 = \frac{\partial \mathbf{U}}{\mathbf{V}}(\mathbf{V}) = \mathbf{U}^* \frac{\partial(\rho\iota)^T}{\mathbf{V}} + \rho\iota \frac{\partial \mathbf{U}^*}{\mathbf{V}} \quad \text{with } \mathbf{U}^* = \frac{\mathbf{U}}{\rho\iota}$$

Recall:

$$\rho\iota = \rho_0\iota_0 \left(\frac{-1}{\iota_0 V_5} \right)^{\frac{\alpha}{\alpha-1}} \exp \left(\frac{-\gamma - V_1 + \frac{V_2^2 + V_3^2 + V_4^2}{2V_5}}{\gamma - 1} \right)$$

Thus

$$\frac{\partial(\rho\iota)}{\mathbf{V}} = \left(\begin{array}{c} -\frac{e^{\frac{V_2^2 + V_3^2 + V_4^2 - 2V_1V_5 + 2V_5\gamma}{2V_5 - 2V_5\gamma}} \iota_0 (\iota_0 V_5)^{-\frac{\gamma}{-1+\gamma}} \rho_0}{-1 + \gamma} \\ \frac{e^{\frac{V_2^2 + V_3^2 + V_4^2 - 2V_1V_5 + 2V_5\gamma}{2V_5 - 2V_5\gamma}} \iota_0 V_2 (\iota_0 V_5)^{-\frac{\gamma}{-1+\gamma}} \rho_0}{V_5(-1 + \gamma)} \\ \frac{e^{\frac{V_2^2 + V_3^2 + V_4^2 - 2V_1V_5 + 2V_5\gamma}{2V_5 - 2V_5\gamma}} \iota_0 V_3 (\iota_0 V_5)^{-\frac{\gamma}{-1+\gamma}} \rho_0}{V_5(-1 + \gamma)} \\ \frac{e^{\frac{V_2^2 + V_3^2 + V_4^2 - 2V_1V_5 + 2V_5\gamma}{2V_5 - 2V_5\gamma}} \iota_0 V_4 (\iota_0 V_5)^{-\frac{\gamma}{-1+\gamma}} \rho_0}{V_5(-1 + \gamma)} \\ -\frac{e^{\frac{V_2^2 + V_3^2 + V_4^2 - 2V_1V_5 + 2V_5\gamma}{2V_5 - 2V_5\gamma}} \iota_0 (\iota_0 V_5)^{-\frac{\gamma}{-1+\gamma}} (V_2^2 + V_3^2 + V_4^2 - 2V_5\gamma) \rho_0}{2V_5^2(-1 + \gamma)} \end{array} \right) = \left(\begin{array}{c} \frac{E}{EV_2} \\ \frac{V_5}{EV_3} \\ \frac{V_5}{EV_4} \\ -\frac{E(V_2^2 + V_3^2 + V_4^2 - 2V_5\gamma)}{2V_5^2} \end{array} \right)$$

With

$$E = \frac{e^{\frac{V_2^2 + V_3^2 + V_4^2 - 2V_1V_5 + 2V_5\gamma}{2V_5 - 2V_5\gamma}} \iota_0 (\iota_0 V_5)^{-\frac{\gamma}{-1+\gamma}} \rho_0}{-1 + \gamma}$$

And

$$\frac{\partial \mathbf{U}^*}{\mathbf{V}} = \begin{bmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{-V_2}{V_5} & \frac{-V_3}{V_5} & \frac{-V_4}{V_5} & \frac{V_2^2 + V_3^2 + V_4^2}{2V_5^2} \end{bmatrix}$$

The matrix is too big to be written fully but we know that to be symmetric we need to verify the conditions:

$$\left\{ \begin{array}{lcl} \frac{\partial \rho\iota}{\partial V_1} V_2 & = & -\frac{\partial \rho\iota}{\partial V_2} V_5 \\ \frac{\partial \rho\iota}{\partial V_1} V_3 & = & -\frac{\partial \rho\iota}{\partial V_3} V_5 \\ \frac{\partial \rho\iota}{\partial V_1} V_4 & = & -\frac{\partial \rho\iota}{\partial V_4} V_5 \\ \frac{\partial \rho\iota}{\partial V_1} \left(1 - \frac{V_2^2 + V_3^2 + V_4^2}{2V_5} \right) & = & -\frac{\partial \rho\iota}{\partial V_5} V_5 - \rho\iota \\ \frac{\partial \rho\iota}{\partial V_2} V_3 & = & \frac{\partial \rho\iota}{\partial V_3} V_2 \\ \frac{\partial \rho\iota}{\partial V_2} V_4 & = & -\frac{\partial \rho\iota}{\partial V_4} V_2 \\ \frac{\partial \rho\iota}{\partial V_3} V_4 & = & -\frac{\partial \rho\iota}{\partial V_4} V_3 \\ \frac{\partial \rho\iota}{\partial V_2} \left(1 - \frac{V_2^2 + V_3^2 + V_4^2}{2V_5} \right) - \frac{V_2}{V_5} \rho\iota & = & \frac{\partial \rho\iota}{\partial V_5} V_2 \\ \frac{\partial \rho\iota}{\partial V_3} \left(1 - \frac{V_2^2 + V_3^2 + V_4^2}{2V_5} \right) - \frac{V_3}{V_5} \rho\iota & = & \frac{\partial \rho\iota}{\partial V_5} V_3 \\ \frac{\partial \rho\iota}{\partial V_4} \left(1 - \frac{V_2^2 + V_3^2 + V_4^2}{2V_5} \right) - \frac{V_4}{V_5} \rho\iota & = & \frac{\partial \rho\iota}{\partial V_5} V_4 \end{array} \right.$$

$$\tilde{\mathbf{A}}_0 = \begin{bmatrix} EV_5 & -EV_2 & -EV_3 & -EV_4 & A \\ -EV_2 & \frac{EV_2^2}{V_5} + \rho\iota & \frac{EV_2V_3}{V_5} & \frac{EV_2V_4}{V_5} & B \\ -EV_3 & \frac{EV_2V_3}{V_5} & \frac{EV_3^2}{V_5} + \rho\iota & \frac{EV_3V_4}{V_5} & C \\ -EV_4 & \frac{EV_2V_4}{V_5} & \frac{EV_3V_4}{V_5} & \frac{EV_4^2}{V_5} + \rho\iota & D \\ A & B & C & D & F \end{bmatrix}$$

$$\begin{aligned}
A &= \frac{E(V_2^2 + V_3^2 + V_4^2 - 2V_5)}{2V_5} \\
B &= EV_2\gamma - \frac{EV_2(V_2^2 + V_3^2 + V_4^2)}{2V_5^2} \\
c &= EV_3\gamma - \frac{EV_3(V_2^2 + V_3^2 + V_4^2)}{2V_5^2} \\
D &= EV_4\gamma - \frac{EV_4(V_2^2 + V_3^2 + V_4^2)}{2V_5^2} \\
F &= \frac{E(V_2^2 + V_3^2 + V_4^2 - 2V_5)(V_2^2 + V_3^2 + V_4^2 - 2V_5^2\gamma)}{4V_5^3} - \frac{e^{\frac{V_2^2 + V_3^2 + V_4^2 - 2V_1V_5 + 2V_5\gamma}{2V_5 - 2V_5\gamma}}\iota_0(V_2^2 + V_3^2 + V_4^2)(\iota_0V_5)^{-\frac{\gamma}{\gamma-1}\rho_0}}{2V_5}
\end{aligned}$$

3 Exercise 6.3

4 Exercise 6.5

$$\mathbf{F}_i = \begin{pmatrix} U_{i+1} \\ (\frac{U_{i+1}}{U_1}U_2 + (\gamma-1)(U_5 + \frac{1}{2U_1}(U_2^2 + U_3^2 + U_4^2))\delta i \\ (\frac{U_{i+1}}{U_1}U_3 + (\gamma-1)(U_5 + \frac{1}{2U_1}(U_2^2 + U_3^2 + U_4^2))\delta i \\ (\frac{U_{i+1}}{U_1}U_4 + (\gamma-1)(U_5 + \frac{1}{2U_1}(U_2^2 + U_3^2 + U_4^2))\delta i \\ \frac{U_{i+1}U_5}{U_1} + \frac{U_{i+1}}{U_1}(\gamma-1)(U_5 + \frac{1}{2U_1}(U_2^2 + U_3^2 + U_4^2)) \end{pmatrix}$$

$$\tilde{\mathbf{F}}_i = \begin{pmatrix} (\rho\iota V_{i+1}) \\ (-\frac{(\rho\iota V_{i+1})}{\rho\iota V_5}(\rho\iota V_2) + (\gamma-1)(\rho\iota(1 - \frac{V_2^2 + V_3^2 + V_4^2}{2V_5}) - \frac{1}{2\rho\iota V_5}((\rho\iota V_2)^2 + (\rho\iota V_3)^2 + (\rho\iota V_4)^2))\delta i \\ (-\frac{(\rho\iota V_{i+1})}{\rho\iota V_5}(\rho\iota V_3) + (\gamma-1)(\rho\iota(1 - \frac{V_2^2 + V_3^2 + V_4^2}{2V_5}) - \frac{1}{2\rho\iota V_5}((\rho\iota V_2)^2 + (\rho\iota V_3)^2 + (\rho\iota V_4)^2))\delta i \\ (-\frac{(\rho\iota V_{i+1})}{\rho\iota V_5}(\rho\iota V_4) + (\gamma-1)(\rho\iota(1 - \frac{V_2^2 + V_3^2 + V_4^2}{2V_5}) - \frac{1}{2\rho\iota V_5}((\rho\iota V_2)^2 + (\rho\iota V_3)^2 + (\rho\iota V_4)^2))\delta i \\ -\frac{(\rho\iota V_{i+1})\rho\iota(1 - \frac{V_2^2 + V_3^2 + V_4^2}{2V_5})}{\rho\iota V_5} - \frac{(\rho\iota V_{i+1})}{\rho\iota V_5}(\gamma-1)(\rho\iota(1 - \frac{V_2^2 + V_3^2 + V_4^2}{2V_5}) - \frac{1}{2\rho\iota V_5}((\rho\iota V_2)^2 + (\rho\iota V_3)^2 + (\rho\iota V_4)^2)) \end{pmatrix}$$

$$\tilde{\mathbf{F}}_i = \begin{pmatrix} V_{i+1}\iota\rho \\ -\frac{V_{i+1}V_2\iota\rho}{V_5} + \iota\rho(\gamma-1)\delta_i \\ -\frac{V_{i+1}V_3\iota\rho}{V_5} + \iota\rho(\gamma-1)\delta_i \\ -\frac{V_{i+1}V_4\iota\rho}{V_5} + \iota\rho(\gamma-1)\delta_i \\ \frac{V_{i+1}\iota\rho(V_2^2 + V_3^2 + V_4^2 - 2V_5\gamma)}{2V_5^2} \end{pmatrix}$$