



Multiscale Methods in CDF Homework 6

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1 Exercise 6.12

(a) We use in this exercise the classical variable in 1D:

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho e \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}$$

Let us write the Navier Stokes equations system:

$$\begin{pmatrix} \rho \\ \rho u \\ \rho e \end{pmatrix}_{,t} + \left(u \begin{pmatrix} \rho \\ \rho u \\ \rho e \end{pmatrix} + p \begin{pmatrix} 0 \\ 1 \\ u \end{pmatrix} \right)_{,x} = \left(\begin{pmatrix} 0 \\ \tau \\ \tau u \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -q \end{pmatrix} \right)_{,x} + \rho \begin{pmatrix} 0 \\ b \\ ub + r \end{pmatrix}$$

Let us try to express all the unknow terms as a function of \mathbf{U}

$$p = (\gamma - 1) \left(\rho e - \frac{1}{2} \rho u^2 \right)$$

$$\tau = (\lambda + 2\mu) u_{,x} = \frac{(\lambda + 2\mu)}{\rho} \rho u_{,x}$$

$$q = \frac{-\kappa}{C_v} (e_{,x} - uu_{,x}) = \frac{-\kappa}{C_v \rho} (\rho e_{,x} - \rho uu_{,x})$$

Thus the N.S. equation can be write as:

$$\mathbf{U}_{,t} + \mathbf{F}_{,x} = (\mathbf{K} \mathbf{U}_{,x})_{,x} + \mathcal{F}$$

with:

$$\mathbf{U}_{,x} = \begin{pmatrix} \rho_{,x} \\ \rho_{,x} u + \rho u_{,x} \\ \rho_{,x} e + \rho e_{,x} \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \rho u \\ \rho u^2 + (\gamma - 1) (\rho e - \frac{1}{2} \rho u^2) \\ \rho eu + u(\gamma - 1) (\rho e - \frac{1}{2} \rho u^2) \end{pmatrix},$$

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{(\lambda + 2\mu)}{\rho} u & \frac{(\lambda + 2\mu)}{\rho} & 0 \\ -\left(\frac{(\lambda + 2\mu)}{\rho} + \frac{\kappa}{C_v \rho} \right) u^2 - \frac{\kappa}{C_v \rho} e & \left(\frac{(\lambda + 2\mu)}{\rho} + \frac{\kappa}{C_v \rho} \right) u & \frac{\kappa}{C_v \rho} \end{bmatrix}, \quad \mathcal{F} = \rho \begin{pmatrix} 0 \\ b \\ ub + r \end{pmatrix}$$

Therefore now we have to transform $\mathbf{F}_{,x}$ into $\mathbf{A} \mathbf{U}_{,x}$:

$$\begin{aligned} \mathbf{F}_{,x} &= \begin{pmatrix} \rho u \\ \rho u^2 + (\gamma - 1) (\rho e - \frac{1}{2} \rho u^2) \\ \rho eu + u(\gamma - 1) (\rho e - \frac{1}{2} \rho u^2) \end{pmatrix}_{,x} \\ &= \begin{pmatrix} \rho_{,x} u + \rho u_{,x} \\ \rho_{,x} u^2 + 2\rho uu_{,x} + (\gamma - 1) (\rho_{,x} e + \rho e_{,x} - \frac{1}{2} \rho_{,x} u^2 - \rho uu_{,x}) \\ \rho_{,x} eu + \rho e_{,x} u + \rho eu_{,x} + u_{,x} (\gamma - 1) (\rho e - \frac{1}{2} \rho u^2) + u(\gamma - 1) (\rho_{,x} e + \rho e_{,x} - \frac{1}{2} \rho_{,x} u^2 - \rho uu_{,x}) \end{pmatrix} \end{aligned}$$

Thus:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} u^2 (\gamma - 3) & u(\gamma - 3) & \gamma - 1 \\ u^3 (\gamma - 1) - eu\gamma & \frac{3}{2} u^2 (1 - \gamma) + e\gamma & u\gamma \end{bmatrix}$$

(b) With **Mathematica** we end up with the eigenvalues:

$$\zeta = \left\{ u, u - \frac{\sqrt{(2e - u^2)(-1 + \gamma)\gamma}}{\sqrt{2}}, u + \frac{\sqrt{(2e - u^2)(-1 + \gamma)\gamma}}{\sqrt{2}} \right\}$$

and the nasty eigenvector:

$$\begin{aligned} \phi &= \left\{ \frac{2}{u^2}, \frac{2}{u}, 1 \right\}, \\ &\left\{ -\frac{2}{u^2(-1 + \gamma) - 2e\gamma + \sqrt{2}u\sqrt{(2e - u^2)(-1 + \gamma)\gamma}}, \right. \\ &\left. \frac{4e(-1 + \gamma)\gamma + \sqrt{2}u(-3 + \gamma)\sqrt{(2e - u^2)(-1 + \gamma)\gamma} - 2u^2(-1 + \gamma^2)}{2eu(-1 + \gamma)\gamma + 2\sqrt{2}u^2(-1 + \gamma)\sqrt{(2e - u^2)(-1 + \gamma)\gamma} - 2\sqrt{2}e\gamma\sqrt{(2e - u^2)(-1 + \gamma)\gamma} - u^3(-1 + \gamma^2)}, 1 \right\}, \\ &\left\{ \frac{2}{2e\gamma + u(u - u\gamma + \sqrt{2}\sqrt{(2e - u^2)(-1 + \gamma)\gamma})}, \right. \\ &\left. \frac{-4e(-1 + \gamma)\gamma + \sqrt{2}u(-3 + \gamma)\sqrt{(2e - u^2)(-1 + \gamma)\gamma} + 2u^2(-1 + \gamma^2)}{-2eu(-1 + \gamma)\gamma + 2\sqrt{2}u^2(-1 + \gamma)\sqrt{(2e - u^2)(-1 + \gamma)\gamma} - 2\sqrt{2}e\gamma\sqrt{(2e - u^2)(-1 + \gamma)\gamma} + u^3(-1 + \gamma^2)}, 1 \right\} \end{aligned}$$

2 Exercise 6.13

1. We use now a different variable in 1D:

$$\mathbf{Y} = \begin{pmatrix} p \\ u \\ \theta \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

Let us find the expression of $\mathbf{U}(\mathbf{Y})$:

$$\rho = \frac{p}{(\gamma - 1)C_v\theta}, \quad \rho u = \frac{up}{(\gamma - 1)C_v\theta}, \quad \rho e = \frac{p}{(\gamma - 1)} \left(1 + \frac{1}{2C_v\theta}u^2 \right)$$

With that we can already define:

$$\hat{\mathbf{F}} = \begin{pmatrix} \frac{up}{(\gamma - 1)C_v\theta} \\ \frac{u^2p}{(\gamma - 1)C_v\theta} + p \\ \frac{pu}{(\gamma - 1)} \left(1 + \frac{1}{2C_v\theta}u^2 \right) + up \end{pmatrix}, \quad \hat{\mathcal{F}} = \frac{p}{(\gamma - 1)C_v\theta} \begin{pmatrix} 0 \\ b \\ ub + r \end{pmatrix}$$

Furthermore we can compute:

$$\hat{\mathbf{A}}_0 = \frac{\partial \mathbf{U}(\mathbf{Y})}{\partial \mathbf{Y}} = \begin{bmatrix} \frac{1}{(\gamma - 1)C_v\theta} & 0 & -\frac{p}{(\gamma - 1)C_v\theta^2} \\ \frac{p}{(\gamma - 1)C_v\theta} & \frac{p}{(\gamma - 1)C_v\theta} & -\frac{p}{(\gamma - 1)C_v\theta^2} \\ \frac{1}{(\gamma - 1)} \left(1 + \frac{1}{2C_v\theta}u^2 \right) & \frac{p}{(\gamma - 1)C_v\theta}u & -\frac{p}{2(\gamma - 1)C_v\theta^2}u^2 \end{bmatrix}$$

Therefore:

$$\begin{aligned}
 \hat{\mathbf{A}} &= \mathbf{A}(\mathbf{Y})\hat{\mathbf{A}}_0 \\
 &= \begin{bmatrix} 0 & 1 & 0 \\ -u^2 + \frac{1}{2}u^2(-1+\gamma) & 2u - u(-1+\gamma) & -1+\gamma \\ u^3(-1+\gamma) - (C_v\theta + \frac{1}{2}u^2)u\gamma & \frac{3}{2}u^2(1-\gamma) + (C_v\theta + \frac{1}{2}u^2)\gamma & u\gamma \end{bmatrix} \\
 &\times \begin{bmatrix} \frac{1}{(\gamma-1)C_v\theta} & 0 & -\frac{p}{(\gamma-1)C_v\theta^2} \\ \frac{1}{(\gamma-1)C_v\theta} & \frac{p}{(\gamma-1)C_v\theta} & -\frac{p}{(\gamma-1)C_v\theta^2} \\ \frac{1}{(\gamma-1)} \left(1 + \frac{1}{2C_v\theta}u^2\right) & \frac{p}{(\gamma-1)C_v\theta}u & -\frac{p}{2(\gamma-1)C_v\theta^2}u^2 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 & 0 \\ \frac{u^3(-2+\gamma)}{2C_v(-1+\gamma)\theta} & -\frac{pu(-3+\gamma)}{C_v(-1+\gamma)\theta} & -\frac{pu}{2C_v\theta^2} \\ \frac{(u^2+2C_v\theta)(u^3(-2+\gamma)-2C_vu\gamma\theta)}{4C_v(-1+\gamma)\theta} & \frac{pu(u^2(-2+3\gamma)+2C_v\gamma\theta)}{4C_v(-1+\gamma)\theta} & \frac{pu^3\gamma}{8C_v\theta^2-8C_v\gamma\theta^2} \end{bmatrix}
 \end{aligned}$$

$$\begin{aligned}
 \hat{\mathbf{K}} &= \mathbf{K}(\mathbf{Y})\hat{\mathbf{A}}_0 \\
 &= \begin{bmatrix} 0 & \frac{(\gamma-1)C_v\theta(\lambda+2\mu)}{p}u & 0 & 0 \\ -\left(\frac{(\gamma-1)C_v\theta(\lambda+2\mu)}{p} + \frac{\kappa(\gamma-1)\theta}{p}\right)u^2 - \frac{(\gamma-1)\theta\kappa}{p}(C_v\theta - \frac{1}{2}u^2) & \left(\frac{(\gamma-1)C_v\theta(\lambda+2\mu)}{p} + \frac{\kappa(\gamma-1)\theta}{p}\right)u & \frac{(\gamma-1)\theta\kappa}{p} & 0 \\ \frac{1}{(\gamma-1)C_v\theta} & 0 & -\frac{p}{(\gamma-1)C_v\theta^2} & 0 \\ \frac{1}{(\gamma-1)C_v\theta} & \frac{p}{(\gamma-1)C_v\theta} & -\frac{p}{(\gamma-1)C_v\theta^2} & 0 \\ \frac{1}{(\gamma-1)} \left(1 + \frac{1}{2C_v\theta}u^2\right) & \frac{p}{(\gamma-1)C_v\theta}u & -\frac{p}{2(\gamma-1)C_v\theta^2}u^2 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda+2\mu & 0 \\ 0 & u(\lambda+2\mu) & \kappa \end{bmatrix}
 \end{aligned}$$

2. As:

$$(\hat{\mathbf{A}} - \hat{\zeta}\hat{\mathbf{A}}_0)\hat{\phi} = (\mathbf{A} - \hat{\zeta}\mathbf{I})\hat{\phi}\hat{\mathbf{A}}_0$$

the eigenvalues are the same and the eigenvectors verifies:

$$\phi = \hat{\phi}\hat{\mathbf{A}}_0$$

3 Exercise 6.14

In this exercise we will assume the Laplace's relation for perfect gas:

$$p = C\rho^\gamma$$

1. We have to satisfy the 3 equations:

$$\begin{aligned}
 \frac{\partial H}{\partial U_1} &= C\rho^{\gamma-1} - \frac{|\mathbf{u}|^2}{2} \\
 \frac{\partial H}{\partial U_2} &= u_1 \\
 \frac{\partial H}{\partial U_3} &= u_2
 \end{aligned}$$

Thus:

$$H = \frac{C}{\gamma}\rho^\gamma - \rho\frac{|\mathbf{u}|^2}{2} + \text{cst}$$

satisfies the relations.

2. Obviously we have:

$$\mathbf{V} = \begin{pmatrix} CU_1^{(\gamma-1)} - \frac{U_2^2 + U_3^2}{2} \\ \frac{U_2}{U_1} \\ \frac{U_3}{U_1} \end{pmatrix}$$

We will use the notation:

$$\rho = \left(\frac{1}{C} \left(V1 + \frac{V2^2 + V3^2}{2} \right) \right)^{\frac{1}{\gamma-1}} = f(\mathbf{V})$$

And thus:

$$\mathbf{U} = \begin{pmatrix} f(\mathbf{V}) \\ f(\mathbf{V})V_2 \\ f(\mathbf{V})V_3 \end{pmatrix}$$

3.

$$\hat{\mathbf{A}}_0 = \frac{\partial \mathbf{U}(\mathbf{V})}{\partial \mathbf{V}} = \begin{bmatrix} \frac{(\rho^{\gamma-1})^{\frac{1}{\gamma-1}-1}}{C(\gamma-1)} & \frac{V_2(\rho^{\gamma-1})^{\frac{1}{\gamma-1}-1}}{C(\gamma-1)} & \frac{V_3(\rho^{\gamma-1})^{\frac{1}{\gamma-1}-1}}{C(\gamma-1)} \\ \frac{V_2(\rho^{\gamma-1})^{\frac{1}{\gamma-1}-1}}{C(\gamma-1)} & (\rho^{\gamma-1})^{\frac{1}{\gamma-1}} \left(\frac{V_2^2}{(V_1 + \frac{1}{2}(V_2^2 + V_3^2))(\gamma-1)} + 1 \right) & \frac{V_2V_3(\rho^{\gamma-1})^{\frac{1}{\gamma-1}-1}}{C(\gamma-1)} \\ \frac{V_3(\rho^{\gamma-1})^{\frac{1}{\gamma-1}-1}}{C(\gamma-1)} & \frac{V_2V_3(\rho^{\gamma-1})^{\frac{1}{\gamma-1}-1}}{C(\gamma-1)} & \rho \left(\frac{V_3^2}{(V_1 + \frac{1}{2}(V_2^2 + V_3^2))(\gamma-1)} + 1 \right) \end{bmatrix}$$

As

$$\tilde{\mathbf{F}}_1 = \begin{pmatrix} V_2 f(\mathbf{V}) \\ V_2 f(\mathbf{V})V_2 + Cf(\mathbf{V})^\gamma \\ V_2 f(\mathbf{V})V_3 \end{pmatrix}$$

$$\tilde{\mathbf{A}}_1 = \frac{\partial \tilde{\mathbf{F}}_1}{\mathbf{V}} = \begin{bmatrix} \frac{V_2(\rho^{\gamma-1})^{\frac{1}{\gamma-1}-1}}{C(\gamma-1)} & \rho \left(\frac{V_2^2}{(V_1 + \frac{1}{2}(V_2^2 + V_3^2))(\gamma-1)} + 1 \right) & \frac{V_2V_3(\rho^{\gamma-1})^{\frac{1}{\gamma-1}-1}}{C(\gamma-1)} \\ \frac{2(V_2^2(\rho + C\gamma\rho^\gamma))}{(V_2^2 + V_3^2 + 2V_1)(\gamma-1)} & \frac{2^{1-\gamma}V_2 \left(\frac{V_2^2 + V_3^2 + 2V_1}{C} \right)^{\frac{1}{\gamma-1}-1} (3(V_2^2 + V_3^2 + 2V_1))\gamma - 2(V_3^2 + 2V_1)}{C(\gamma-1)} & \frac{2V_3(V_2^2(\rho + C\gamma\rho^\gamma))}{(V_2^2 + V_3^2 + 2V_1)(\gamma-1)} \\ \frac{V_2V_3(\rho^{\gamma-1})^{\frac{1}{\gamma-1}-1}}{C(\gamma-1)} & V_3 \rho \left(\frac{V_2^2}{(V_1 + \frac{1}{2}(V_2^2 + V_3^2))(\gamma-1)} + 1 \right) & V_2 \left(\rho \left(\frac{V_3^2}{(V_1 + \frac{1}{2}(V_2^2 + V_3^2))(\gamma-1)} + 1 \right) \right) \end{bmatrix}$$

and so on for the other $\tilde{\mathbf{A}}_2$

$$\tilde{\mathbf{A}}_2 = \frac{\partial \tilde{\mathbf{F}}_2}{\mathbf{V}} = \begin{bmatrix} \frac{V_3\rho^{\frac{1}{\gamma-1}+\gamma}}{C(\gamma-1)} & \frac{V_2V_3\rho^{\frac{1}{\gamma-1}+\gamma}}{C(\gamma-1)} & \rho \left(\frac{V_3^2}{(V_1 + \frac{1}{2}(V_2^2 + V_3^2))(\gamma-1)} + 1 \right) \\ \frac{V_2V_3\rho^{\frac{1}{\gamma-1}+\gamma}}{C(\gamma-1)} & V_3 \rho \left(\frac{V_2^2}{(V_1 + \frac{1}{2}(V_2^2 + V_3^2))(\gamma-1)} + 1 \right) & V_2 \rho \left(\frac{V_3^2}{(V_1 + \frac{1}{2}(V_2^2 + V_3^2))(\gamma-1)} + 1 \right) \\ \frac{2(V_3^2\rho + C\gamma\rho^\gamma)}{(V_2^2 + V_3^2 + 2V_1)(\gamma-1)} & \frac{2V_2(V_3^2\rho + C\gamma\rho^\gamma)}{(V_2^2 + V_3^2 + 2V_1)(\gamma-1)} & \frac{2^{1-\gamma}V_3 \left(\frac{V_2^2 + V_3^2 + 2V_1}{C} \right)^{\frac{1}{\gamma-1}-1} (3(V_2^2 + V_3^2 + 2V_1))\gamma - 2(V_2^2 + 2V_1)}{C(\gamma-1)} \end{bmatrix}$$

Then the expression of $\tilde{\mathbf{K}}$ came directly from the expression of the viscous term which is directly expressed in terms of $\mathbf{V}_{,x}$

$$\tilde{\mathbf{K}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\lambda + 2\mu) & 0 & 0 & 0 & \lambda \\ 0 & 0 & \mu & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & \mu & 0 \\ 0 & 0 & \lambda & 0 & 0 & (\lambda + 2\mu) \end{bmatrix}$$

It is also easy to find the expression of \mathcal{F} :

$$\mathcal{F} = f(\mathbf{V}) \begin{pmatrix} 0 \\ b_1 \\ b_2 \end{pmatrix}$$

4. $\tilde{\mathbf{K}}$ and $\tilde{\mathbf{A}}_0$ are symmetric but not definite in every case (we would have to assume conditions on λ and the derivative of ρ) and the $\tilde{\mathbf{A}}_i$'s are not even symmetric.

In my point of view as we can express p/ρ in terms of \mathbf{V} we shouldn't have any trouble fixing ρ to constant. Furthermore (the calculus are not developed because I'm out of Advil for this Homework and it will be a pain for you to have to go through it. But we should end up with the classical equation for NS incompressible.