



Multiscale Methods in CDF Homework 5

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1 Exercise 3.10

1. Let us recall the weak form stated in (3.100):

$$\int_{t_n}^{t_{n+1}} (w, \dot{\phi} + u\phi_{,x} - f)_\Omega dt + (w(t_n^+), \phi(t_n^+) - \phi(t_n^-))_\Omega = 0 \quad \forall w \in Q'$$

If we replace the ϕ and w by:

$$\phi = \sum_B N_B^n(x, t) \phi_B^n = \sum_B N_B(x) \phi_B(t^{n+1})$$

$$w = \sum_A N_A^n(x, t) w_A^n = \sum_A N_A(x) w_A(t^{n+1})$$

We have these expressions because the time discretization is constant. That also leads us to $\dot{\phi} = 0$. Therefore we have the equation:

$$\begin{aligned} & \int_{t_n}^{t_{n+1}} \left(\sum_A N_A w_A(t^{n+1}), +u \sum_B N_{B,x} \phi_B(t^{n+1}) - f \right)_\Omega dt + \left(\sum_A N_A w_A(t^{n+1}), \sum_B N_B(\phi_B(t^{n+1}) - \phi_B(t^n)) \right)_\Omega = 0 \\ & \Rightarrow \Delta t \sum_A \sum_B ((N_A w_{A,n+1}, +u N_{B,x} \phi_{B,n+1} - f)_\Omega + (N_A w_{A,n+1}, N_B(\phi_{B,n+1} - \phi_{B,n}))_\Omega) = 0 \quad \forall w_A, n \end{aligned}$$

As that equality has to hold for all w we end up with the relation:

$$\Delta t \sum_{B=A-1}^{A+1} ((N_A, u N_{B,x} \phi_{B,n+1} - f)_\Omega + (N_A, N_B(\phi_{B,n+1} - \phi_{B,n}))_\Omega) = 0 \quad \forall A, n$$

Now let us compute all the coefficients:

$$\int_{\Omega} N_A N_{A-1,x} dx = -\frac{1}{2}$$

$$\int_{\Omega} N_A N_{A,x} dx = 0$$

$$\int_{\Omega} N_A N_{A+1,x} dx = \frac{1}{2}$$

$$\int_{\Omega} N_A N_{A-1} dx = \frac{\Delta x}{6}$$

$$\int_{\Omega} N_A N_A dx = \frac{2\Delta x}{3}$$

$$\int_{\Omega} N_A N_{A+1} dx = \frac{\Delta x}{6}$$

Thus we have the relation:

$$\begin{aligned} & \left(\frac{-\Delta t u}{2} + \frac{\Delta x}{6} \right) \phi_{A-1,n+1} + \frac{2\Delta x}{3} \phi_{A,n+1} + \left(\frac{\Delta t u}{2} + \frac{\Delta x}{6} \right) \phi_{A+1,n+1} \\ & = \frac{\Delta x}{6} \phi_{A-1,n} + \frac{2\Delta x}{3} \phi_{A,n} + \frac{\Delta x}{6} \phi_{A+1,n} + \Delta x \Delta t f \quad \forall A, n \end{aligned} \tag{1}$$

2. Let us recall again the weak formulation we will be working with (cf. 3.10)

$$(w^h, \phi^h, _t)_{\Omega} + B(w^h, \phi^h) = L(w^h)$$

Thus using:

$$\phi = \sum_B N_B(x) \phi_B(t)$$

$$w = \sum_A N_A(x) w_A$$

And with the precedent calculus:

$$\frac{\Delta x}{6} \dot{\phi}_{A-1}(t) + \frac{2\Delta x}{3} \dot{\phi}_A(t) + \frac{\Delta x}{6} \dot{\phi}_{A+1}(t) - \frac{u\Delta x^2}{2} \phi_{A-1}(t) + \frac{u\Delta x^2}{2} \phi_{A+1}(t) = \Delta x f \quad \forall A \quad (2)$$

$$\text{Replacing } \dot{\phi}_{n+1} = \frac{1}{\Delta t} (\phi_{n+1} - \phi_n)$$

$$\begin{aligned} & \left(\frac{-\Delta tu}{2} + \frac{\Delta x}{6} \right) \phi_{A-1,n+1} + \frac{2\Delta x}{3} \phi_{A,n+1} + \left(\frac{\Delta tu}{2} + \frac{\Delta x}{6} \right) \phi_{A+1,n+1} \\ & = \frac{\Delta x}{6} \phi_{A-1,n} + \frac{2\Delta x}{3} \phi_{A,n} + \frac{\Delta x}{6} \phi_{A+1,n} + \Delta x \Delta t f \quad \forall A, n \end{aligned} \quad (3)$$

We have exactly the same equation as in the first question. So that's mean that for a constant interpolation in time the method we pick doesn't matter.

3. We know that on the slanted the interpolation is constant which means that:

$$\int_{\text{slanted}} \phi^h_{,t} + u \phi^h_{,x} ds = \int_{\text{slanted}} \phi^h_{,s} ds = 0$$

Thus with a such mesh as in Figure 3.36 we are integrating over such lines. Therefore the equation becomes:

$$\int_0^L N_A \left(\int_s f ds + N_{A-1}(u_{A-1,n+1} - u_{A-1,n}) + N_A(u_{A,n+1} - u_{A,n}) + N_{A+1}(u_{A+1,n+1} - u_{A+1,n}) \right) dx = 0$$

Furthermore the expression insinde the integrale doesn't depend on s , thus we just have to multiply the term in the integrale by the lenght of the domain which is $\Delta t \sqrt{1 + u^2}$:

$$\int_0^L N_A (\Delta t \sqrt{1 + u^2} f + N_{A-1}(u_{A-1,n+1} - u_{A-1,n}) + N_A(u_{A,n+1} - u_{A,n}) + N_{A+1}(u_{A+1,n+1} - u_{A+1,n})) dx = 0$$

$$\frac{1}{6} u_{A-1,n+1} + \frac{2}{3} u_{A,n+1} + \frac{1}{6} u_{A+1,n+1} = \frac{1}{6} u_{A-1,n} + \frac{2}{3} u_{A,n} + \frac{1}{6} u_{A+1,n} + \Delta t \sqrt{1 + u^2} f$$

Thus noting:

$$\beta = \delta = \frac{1}{6}, \quad \gamma = \frac{2}{3} \quad \text{and} \quad \varepsilon = \Delta t \sqrt{1 + u^2}$$

We know that one solution is given by (3.238). Thus that's mean that we can propagate the solution in the slanted direction without any distortion but just by adding the loading term.

4. Again replacing in (3.100) the shape functions given in 3.75 we have the 2 equations:

$$\begin{aligned} & \int_{t_n}^{t_{n+1}} \int_0^L N_{A,n+1} ((N_{A-1,n+1,t} + u N_{A-1,n+1,x}) \phi_{A-1,n+1} \\ & + (N_{A,n+1,t} + u N_{A,n+1,x}) \phi_{A,n+1} + (N_{A+1,n+1,t} + u N_{A+1,n+1,x}) \phi_{A+1,n+1} \\ & + (N_{A-1,n,t} + u N_{A-1,n,x}) \phi_{A-1,n}^+ + (N_{A,n,t} + u N_{A,n,x}) \phi_{A,n}^+ + (N_{A+1,n,t} + u N_{A+1,n,x}) \phi_{A+1,n}^+ - f) dx dt \\ & = 0 \quad \forall A, n \end{aligned}$$

$$\begin{aligned}
& \int_{t_n}^{t_{n+1}} \int_0^L N_{A,n} ((N_{A-1,n+1,t} + u N_{A-1,n+1,x}) \phi_{A-1,n+1} \\
& \quad + (N_{A,n+1,t} + u N_{A,n+1,x}) \phi_{A,n+1} + (N_{A+1,n+1,t} + u N_{A+1,n+1,x}) \phi_{A+1,n+1} \\
& \quad + (N_{A-1,n,t} + u N_{A-1,n,x}) \phi_{A-1,n}^+ + (N_{A,n,t} + u N_{A,n,x}) \phi_{A,n}^+ + (N_{A+1,n,t} + u N_{A+1,n,x}) \phi_{A+1,n}^+ - f) dx dt \\
& \quad + \int_0^L (N_{A,n}, N_{A-1,n}(\phi_{A-1,n}^- \phi_{A-1,n}) + N_{A,n}(\phi_{A,n}^- \phi_{A,n}) + N_{A+1,n}(\phi_{A+1,n}^- \phi_{A+1,n})) dx \\
& = 0 \quad \forall A, n
\end{aligned}$$

Let us just compute one integral by hand:

$$\begin{aligned}
\int_{t_n}^{t_{n+1}} \int_0^L N_{A,n} (N_{A-1,n+1,t} + u N_{A-1,n+1,x}) dx dt &= \frac{\Delta t \Delta x}{4} \int_{-1}^1 \int_{-1}^1 N_{A,n} \left(\frac{\partial \eta}{\partial t} N_{A-1,n+1,\eta} + u \frac{\partial \varepsilon}{\partial x} N_{A-1,n+1,\varepsilon} \right) d\eta d\varepsilon \\
&= \frac{\Delta t \Delta x}{4} \int_{-1}^1 \int_{-1}^1 N_{A,n} \left(\frac{2}{\Delta t} N_{A-1,n+1,\eta} + u \frac{2}{\Delta x} N_{A-1,n+1,\varepsilon} \right) d\eta d\varepsilon \\
&= \frac{1}{2} \left(-\frac{\Delta x}{6} - u \frac{\Delta x}{3} \right)
\end{aligned}$$

Therefore after a computation of the integrals with **Mathematica** the equations become:

$$\begin{aligned}
& \left(-\frac{\Delta x}{6} - u \frac{\Delta t}{3} \right) \phi_{A-1,n+1} + \left(\frac{2\Delta x}{3} - u \frac{\Delta t}{6} \right) \phi_{A,n+1} + \left(\frac{\Delta x}{6} + u \frac{\Delta t}{6} \right) \phi_{A+1,n+1} \\
& + \left(-\frac{\Delta x}{6} - u \frac{\Delta t}{6} \right) \phi_{A-1,n}^+ + \left(-\frac{2\Delta x}{3} - u \frac{\Delta t}{6} \right) \phi_{A,n}^+ + \left(-\frac{\Delta x}{6} + u \frac{\Delta t}{3} \right) \phi_{A+1,n}^+ - f \Delta x \Delta t = 0 \quad \forall A, n \\
& \left(-\frac{\Delta x}{6} - u \frac{\Delta t}{6} \right) \phi_{A-1,n+1} + \left(-\frac{2\Delta x}{3} - u \frac{\Delta t}{6} \right) \phi_{A,n+1} + \left(-\frac{\Delta x}{6} + u \frac{\Delta t}{3} \right) \phi_{A+1,n+1} \\
& + \left(-\frac{\Delta x}{6} - u \frac{\Delta t}{3} \right) \phi_{A-1,n}^+ + \left(\frac{2\Delta x}{3} - u \frac{\Delta t}{6} \right) \phi_{A,n}^+ + \left(\frac{\Delta x}{6} + u \frac{\Delta t}{6} \right) \phi_{A+1,n}^+ - f \Delta t \Delta x \\
& + \frac{\Delta x}{3} (\phi_{A-1,n}^- \phi_{A-1,n}) + \frac{2\Delta x}{3} (\phi_{A,n}^- \phi_{A,n}) + \frac{\Delta x}{3} (\phi_{A+1,n}^- \phi_{A+1,n}) = 0 \quad \forall A, n
\end{aligned}$$