

# Recent progress on the Boltzmann equation without angular cutoff

Oral Defense

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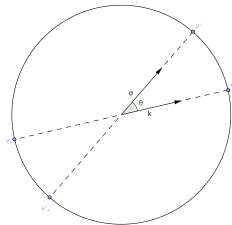
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with

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma$$

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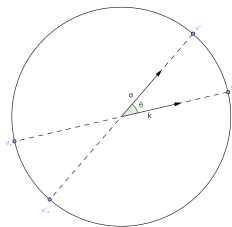
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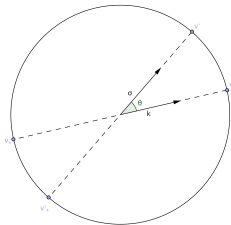
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Non-cutoff assumption

$$\int_{S^{N-1}} B(z, \sigma) d\sigma = +\infty$$

$$B(z, \sigma) \geq \Phi(|z|) b(\cos \theta)$$

with  $\Phi(|z|) = |z|^\gamma$  and

$$\sin^{N-2} \theta b(\cos \theta) \underset{\theta \rightarrow 0^+}{\sim} \frac{K}{\theta^{1+\nu}}$$

# Entropy Dissipation and Long-Range Interactions

Main result

$$\|\sqrt{f}\|_{H^{\nu/2}(|v|<R)}^2 \leq C_{g,R} \left( D(g, f) + \|g\|_{L^1_2} \|f\|_{L^1_2} \right)$$

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- non-cutoff ( $\Phi$  continuous, positive except in 0,  $0 \leq \nu < 2$ )

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- generalize

## Cancellation lemma

$$D(g, f) = \int Bg_*(f - f')dv dv_* d\sigma + \int Bg_* \left( f \log \frac{f}{f'} - f + f' \right) dv dv_* d\sigma$$

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Theorem (Cancellation lemma)

$$S(z) = |S^{N-2}| \int_0^{\frac{\pi}{2}} \sin^{N-2} \theta \left( \frac{B\left(\frac{|z|}{\cos \frac{\theta}{2}}, \cos \theta\right)}{\cos^N \frac{\theta}{2}} - B(|z|, \cos \theta) \right) d\theta$$

We have:

1.  $|S(z)| \leq C_N (\Lambda(|z|) + |z|\Lambda'(|z|))$
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### Corollary

There exists  $C$  depending only on  $N$  and  $B$  such that

$$\left| \int Bg_*(f' - f)dv dv_* d\sigma \right| \leq C \|g\|_{L^1_2} \|f\|_{L^1_2}$$



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### Proof.

Use of the change of variables  $v \rightarrow v'$  forth and backward.  $\square$

Use of *Fourier Analysis* for the second term, easy case :

$$\Phi = 1$$

$$\int Bg_* \left( f \log \frac{f}{f'} - f + f' \right) dv dv_* d\sigma \geq \int \Phi bg_* (F' - F)^2 dv dv_* d\sigma$$

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Fourier transform  $\implies$  with  $\xi^- = \frac{\xi - |\xi|\sigma}{2}$ ,

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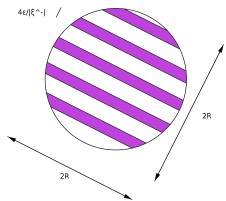
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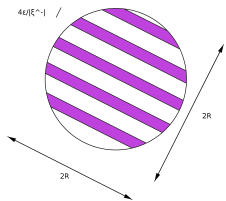
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# Truncature : generalization

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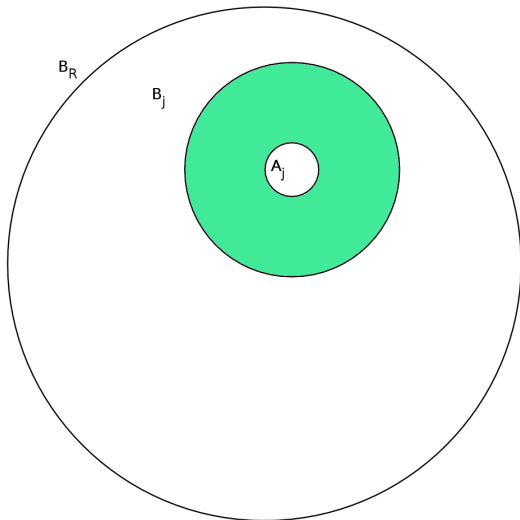
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## *On the Boltzmann Equation for Long-Range Interactions*

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Existence of weak solutions, appearance of strong compactness.

## Definition (weak solution)

$f$ : renormalized solution to the Boltzmann equation with defect measure

$f \in C(\mathbb{R}^+; \mathcal{D}'(\mathbb{R}_x^N \times \mathbb{R}_v^N)) \cap L^\infty(\mathbb{R}^+; L^1((1 + |v|^2 + |x|^2)dx dv))$   
 $\forall \beta \in C^2(\mathbb{R}^+, \mathbb{R}^+)$  non-linear such that :

$$\beta(0) = 0 \qquad 0 < \beta'(f) \leq \frac{C}{1+f} \qquad \beta''(f) < 0$$

$$\implies \begin{cases} \frac{\partial \beta(f)}{\partial t} + v \cdot \nabla_x \beta(f) \geq \beta'(f) Q(f, f) \\ \forall t \geq 0, \int_{\mathbb{R}^N \times \mathbb{R}^N} f(t, x, v) dx dv = \int_{\mathbb{R}^N \times \mathbb{R}^N} f(0, x, v) dx dv \end{cases}$$

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AT MOST BORDERLINE  
SINGULARITY.

$B(z, \sigma) = \frac{\beta_0(k.\sigma)}{|z|^N} + B_1(z, \sigma)$  with

- $\int_{S^{N-1}} \beta_0(k.\sigma)(1 - k.\sigma)d\sigma < +\infty$
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AT MOST BORDERLINE

SINGULARITY.

$B(z, \sigma) = \frac{\beta_0(k.\sigma)}{|z|^N} + B_1(z, \sigma)$  with

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- $\int_{S^{N-1}} \lim_{n \rightarrow \infty} b^n(k, \sigma) d\sigma = \infty$

## Idea of the proof

$\beta'(f)Q(f, f)$  formally cut into three parts :

- something  $\times \int B(f'_* - f_*)dv_*d\sigma \implies$  use of the *cancellation lemma*
- $Q(f, \beta(f)) \implies$  apparition of  $\mathcal{T}$
- $-\int Bf'_*\Gamma(f, f')dv_*d\sigma$  which convergence is a consequence of the convergence of the first two terms

$$\Gamma(f, f') := \beta(f') - \beta(f) - \beta'(f)(f' - f) \leq 0$$

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$$\begin{aligned} S_\varepsilon(|z|) &\propto \frac{1}{|z|^N} \int_0^{\frac{\pi}{2}} \sin^{N-2} \theta \left( \mathbf{1}_{|z| \geq \varepsilon \cos \frac{\theta}{2}} - \mathbf{1}_{|z| \geq \varepsilon} \right) \beta_0(\cos \theta) d\theta \\ &\propto \frac{1}{\varepsilon^N} \frac{1}{(|z|/\varepsilon)^N} \int_{2 \cos^{-1} \frac{|z|}{\varepsilon}}^{\frac{\pi}{2}} \sin^{N-2} \theta \beta_0(\cos \theta) d\theta \mathbf{1}_{\frac{|z|}{\varepsilon} \leq 1} \\ &\propto \frac{1}{\varepsilon^N} J \left( \frac{|z|}{\varepsilon} \right) \end{aligned}$$



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We get  $S_0 \propto \delta_0$

## Idea of the proof of strong compactity appearance

1. get strong compactity results in  $L^1_{(\text{loc})}$  for  $\int \beta(f^n)\phi(v)dv$ ,  
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6. Conclude: check that  $f$  is a RSDM



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## Theorem

$B(z, \sigma) = \Phi(|z|)b(\cos \theta)$  with  $\Phi(|z|) = |z|^\gamma$  and similar assumptions as before:  $\gamma > -3$  and  $\nu \in ]0; 2[$ .

$$C_1 |||(\mathbf{I} - \mathbf{P})g|||^2 \leq (\mathcal{L}g, g)_{L^2(\mathbb{R}^3)} \leq C_2 |||g|||^2$$

## About the non-isotropic norm

$$C_3 \left( \|g\|_{H^{\frac{\nu}{2}}(\mathbb{R}_\nu^3)}^2 + \|g\|_{L^2(\mathbb{R}_\nu^3)}^2 \right) \leq \|g\|^2 \leq C_4 \|g\|_{H^{\frac{\nu}{2}}(\mathbb{R}_\nu^3)}^2$$

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Show in particular that  $J_2^\Phi \sim \|g\|_{L^2_{\frac{\nu+\gamma}{2}}}^2 \sim J_2^{\tilde{\Phi}}$  and

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$\tilde{\Phi}(|z|) = \langle |z| \rangle^\gamma = (1 + |z|^2)^{\frac{\gamma}{2}} \implies$  get rid off the singularity in 0.

# Conclusion

Thank you !

Any questions ?