# Recent progress on the Boltzmann equation without angular cutoff

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**Abstract**

In this report, I will present the three papers I read during my Master 1 internship at Tsinghua University (Beijing) from march to august 2013. Thus, I will introduce some recent results about the Boltzmann equation without angular cutoff. The papers where chosen by my internship supervisor He LingBing.

In particular, I worked on the Boltzmann collision operator itself in a first paper *Entropy dissipation and long-range interactions* by R. Alexandre, L. Desvillettes, C. Villani and B. Wennberg (2000). In this paper, some general results are given on the collision operator. I will present them. Once we will know more about this operator, we will be able to get to the point: showing the existence of solutions under some assumptions.

Then, I focused on the existence of weak solutions in a second paper: *On the Boltzmann equation for long-range interactions* by R. Alexandre and C. Villani (2002). Weak is not to be taken in its classical sense. They define the notion of renormalised solution to the Boltzmann equation with defect mesure. The idea is to take a sequence $(B^n)$ of cross sections that converge (in some sense) to the targeted cross section $B$. Calling $f^n$ a weak solution for cross section $B^n$, they show some appearance of strong compacity of $(f^n)$ and that the limit is a weak solution to the Boltzmann equation.

And, to finish, I worked on a norm, called non-isotropic by the authors, which was useful to them in order to get the existence of classical solutions. The paper I worked on is: *The Boltzmann equation without angular cutoff in the whole space: I, Global existence for soft potential*, part 2 *Non-isotropic norm and estimates of linearized collision operators* by R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu and T. Yang (2011). They present the new norm, compare it to wheighted Sobolev norms, and use it to bound the linearized collision operator which is obtained by looking for a solution close to the static solution:

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}$$

The equation I'm working on, the Boltzmann equation, is a partial diffential equation. It can be written, with unknown $f$, as:

$$\frac{\partial f}{\partial t} + \nabla_x f.v = Q(f, f)$$

whith

$$Q(g, f) = \int_{\mathbb{R}^N} \int_{S^{N-1}} B(v - v_*, \sigma)(g'_* f' - g_* f) \mathrm{d}\sigma \mathrm{d}v_*$$

$Q$ is the Boltzmann collision operator and I used the notation: $g'_* = g(v'_*)$, $f' = f(v')$ and $g_* = g(v_*)$ with

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma$$
$$v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma$$

the precollisional velocities.

The Boltzmann equation modelizes the collision between particles in a gaz.

It is assumed that the cross section $B$ depends only on $|v - v_*|$ and the scalar product $k.\sigma$ where $k = \frac{v - v_*}{|v - v_*|}$ and is supported in $k.\sigma \geq 0$. The non-cutoff assumption consists in taking

$$\int_{S^{N-1}} B(z, \sigma) \mathrm{d}\sigma = +\infty$$

As my work consisted only in reading (and understanding) those papers, I present them one by one, insisting on the points I found hard to understand.

The main idea is to use the work done earlier for the cutoff case by getting rid of the irregularities due to the cross section. So, I insist on some results, such as the cancellation lemma or the truncation, which were useful for that matter.

One of the important tools they all used is the Fourier transform.

I will introduce those tools during the study of the first paper. For the two last papers, I will go faster on the computations, as to me, the used tools are more important than the computation itself. Moreover, the computations can be found in the papers.

# Contents

# Introduction

I have been working on the Boltzmann equation without angular cutoff under the supervision of He LingBing from the departement of mathematical sciences of Tsinghua University (Beijing) for about five months.

Figure 1: Yuanming Yuan : the ruins in the old summer palace, Haidian district, Beijing

Beijing is an gigantic city. Paris is a village compared to it. I was living close to the old summer palace in Haidian District. The summer palace was destroyed by the french and british armies during the Second Opium War. It's all ruins now. Though, the chinese built a new one after the war. When the pollution was not too high, I could see it from my window on the 22nd floor of a building that must have been built in the 70s and that was really looking like a wreck. I would spend fifteen minutes to get to the nearest subway station by bike or bus and more than one hour to go to the forbidden city by subway. If I wanted to go to Beijing Trade Center (Guomao) where all the skyscrapers are very modern and beautiful, I would spend more than one hour in the subway. This is just to give you an idea of how big the city is.

As every chinese city I've been to, Beijing is animated 24 hours a day. If you wait for a taxi at 4am to go to the airport, you will find one in less than 10 minutes. And it's cheap! You would pay the same price for a cab in France in euros as you do in Beijing in RMB (1€≈ 8RMB). Life in China is very comfortable. And you feel safe all the time, day or night. Yet, I have to acknowledge that speaking chinese made my life there easier than it is for any foreigner, as most of my foreign friends would often call me to be their interpret.

Tsinghua University is one of the two best universities in mainland China. Every chinese student dreams of getting into Tsinghua. It's an honor for me to have been working there and I know my experience in Tsinghua will make a difference when I want to get a job in China.

Tsinghua University has the biggest campus of all chinese universities and is located in the North-West of Beijing. Actually, the campus is so big that every student needs a bike to cross it. I was amazed by the beauty of this campus which is inside the city but full of parks where the students can hang out. I got the chance to see it under the snow, it was really marvelous. Unfortunately, I didn't take a picture.

Most of the students live in dormitories inside the campus, and I could have chosen to. But, I was afraid of sharing my room with seven other students and chose to live outside. I was fifteen minutes away by bike from the main gate of the university and had fifteen more minutes biking to get to my supervisor's office. Actually, they did not have an office for me in the departement and I was working outside. I used to meet with He LingBing once a week and work in some cafetarias or at home.

Not having a fixed place and time to work can have its pros and cons. Because I was free to travel, visit China whenever I would want to, I found it hard to get to work and have a regular schedule.


Figure 2: Me in front of the old gate of Tsinghua University, Haidian District, Beijing

He LingBing had me work on three papers about the Boltzmann equation without angular cutoff. When I got to Tsinghua, I knew nothing about this equation. The point of my work was to understand and sum up those papers. As they are very specific, I spent quite a lot of time understanding results that would appear obvious to the authors or anyone that would have been working on this topic for years.

The Boltzmann equation is a partial differential equation from statistical physics:

$$\frac{\partial f}{\partial t} + \nabla_x f.v = Q(f, f) \tag{1}$$

where

$$Q(g, f) = \int_{\mathbb{R}^N} \int_{S^{N-1}} B(v - v_*, \sigma)(g'_* f' - g_* f) \mathrm{d}\sigma \mathrm{d}v_* \tag{2}$$

is the Boltzmann collision operator.

$g'_* = g(v'_*)$, $f' = f(v')$ and $g_* = g(v_*)$ with

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma$$
$$v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma$$

We assume that $B$ depends only on $|v - v_*|$ and the scalar product $k.\sigma$ where $k = \frac{v - v_*}{|v - v_*|}$ and is supported in $k.\sigma \geq 0$.

The non-cutoff assumption consists in taking $\int_{S^{N-1}} B(z, \sigma)\mathrm{d}\sigma = +\infty$.

Most results for the cut-off case dates from the late 80's. Most of the recent work has been done for the non-cutoff case which appears to be more interesting for physics.

The three papers I have worked on are:

- *Entropy dissipation and long-range interactions* by R. Alexandre, L. Desvillettes, C. Villani and B. Wennberg (2000)
- *On the Boltzmann equation for long-range interactions* by R. Alexandre and C. Villani (2002)
- *The Boltzmann equation without angular cutoff in the whole space: I, Global existence for soft potential*, part 2 *Non-isotropic norm and estimates of linearized collision operators* by R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu and T. Yang (2011)

The first paper gives some general results on the Boltzmann collision operator in the non-cutoff case, some of which are used in the second one to show the existence of weak solutions to the Boltzmann equation without angular cutoff. The third paper is about the existence of classical solutions. The part I studied introduces a new norm and compares it to wheighed Sobolev norms. This new norm will be used later in the paper to get the targeted result.

# Part I

# About the paper *Entropy dissipation and long-range interactions*

This article deals with the study of Boltzmann's collision operator.

## 1 Definitions and aim

**Definition 1.1** (Boltzmann's collision operator)**.** Is called Boltzmann's collision operator every operator $Q : L^1(\mathbb{R}^N) \times L^1(\mathbb{R}^N) \to L^1(\mathbb{R}^N)$ with $N \geq 2$ et

$$Q(g,f)(v) = \int_{\mathbb{R}^N} \int_{S^{N-1}} B(v - v_*, \sigma)(g(v'_*)f(v') - g(v_*)f(v))\mathrm{d}\sigma\mathrm{d}v_*$$

where $v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma$ et $v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma$. $S^{N-1}$ is unity sphere of $\mathbb{R}^N$. Moreover, $B(z,\sigma)$ is non-negative-valued and depends only of $|z|$ and $\frac{z}{|z|}.\sigma$ where . is the scalar product. $B(z,\sigma) = 0$ when this scalar product is negative.

In reality, $f$ and $g$ are non-negative-valued.
Let us name $\theta$ the angle $(v - v_*, \sigma)$: $\frac{v-v_*}{|v-v_*|}.\sigma = \cos\theta$. Note that $\theta$ actually lives between 0 and $\frac{\pi}{2}$.
In order to be more clear, we state $g'_* = g(v'_*)$, and so on... as a consequence,

$$Q(g,f) = \int B(g'_* f' - g_* f)\mathrm{d}\sigma\mathrm{d}v_* \tag{3}$$

Let also $k = \dfrac{v - v_*}{|v - v_*|}$.

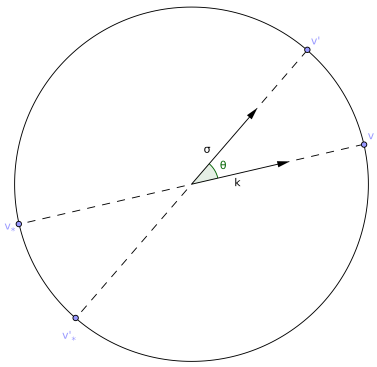


Figure 3: The variables

### 1.1 Assumptions on $B$

We assume that $B$ satisfies the following property:

$$B(z,\sigma) \geq \Phi(|z|)b(k.\sigma) \tag{4}$$

$\Phi$ is the kinetic part of $B$, it is continuous and positive, except in 0 where it may be null.
$b$ is the angular part, it satisfies the following singularity assumption for a $\nu > 0$:

$$\sin^{N-2}\theta\, b(\cos(\theta)) \underset{\theta\to 0}{\sim} \frac{K}{\theta^{1+\nu}} \tag{5}$$

We assume $\nu < 2$, otherwise it has been proven that $Q$ defined by 3 doesn't make sense.
Without loss of generality, $b$ is supported on $\cos\theta \geq 0$.

### 1.2 Other tools

Let

$$D(g,f) = -\int_{\mathbb{R}^N} Q(g,f)\log(f)\mathrm{d}v \tag{6}$$

be the generalized entropy dissipation functional.

Let also: $\Lambda(|z|) = \int_{S^{N-1}} B(z,\sigma)(1-k.\sigma)\mathrm{d}\sigma$ and $\Lambda'(|z|) = \int_{S^{N-1}} B'(z,\sigma)(1-k.\sigma)\mathrm{d}\sigma$ with $B'(z,\sigma) = \sup_{1<\lambda\leq\sqrt{2}} \frac{|B(\lambda z,\sigma)-B(z,\sigma)|}{(\lambda-1)|z|}$.

And, let $\|f\|_{L^1_\alpha} = \int_{\mathbb{R}^N} f(v)(1+|v|)^\alpha \mathrm{d}v$ and $\|f\|_{L\log L} = \int_{\mathbb{R}^N} f\log(1+f)\mathrm{d}v$.

Moreover, we assume that there exists $C_0$ such that

$$\Lambda(|z|) + |z|\Lambda'(|z|) \leq C_0(1+|z|)^2 \tag{7}$$

### 1.3 Aim

This article gives a proof of the following result: under the assumptions 4, 5 and 7, for all $R > 0$, there exists a constant $C_{g,R}$ dependending only on $b$, $\|g\|_{L^1_1}$, $\|g\|_{L\log L}$, $R$ and $\Phi$ such that:

$$\|\sqrt{f}\|^2_{H^{\gamma/2}(|v|<R)} \leq C_{g,R}\left(D(g,f) + \|g\|_{L^1_2}\|f\|_{L^1_2}\right) \tag{8}$$

## 2 Two useful changes of variables

### 2.1 Pre-post-collisional change of variables

The idea is to inverse the role of $(v, v_*)$ and $(v', v'_*)$. The change of variables we do is the following: $(v, v_*, \sigma) \to (v', v'_*, k)$. It is involutive. Thus it's jacobian has 1 as absolute value. See fig. 3.

A first use of it on the first term in the expression of $D(g,f)$ 6 gives:

$$D^-(g,f) = \int B(v-v_*,\sigma)g'_*f'\log f\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma = \int B(v-v_*,\sigma)g'_*f'\log f\mathrm{d}v'\mathrm{d}v'_*\mathrm{d}k$$

But, $B(v-v_*,\sigma) = B(v'-v'_*,k)$ as $|v-v_*| = |v'-v'_*|$ and $\frac{v-v_*}{|v-v_*|}.\sigma = k.\sigma = k.\frac{v'-v'_*}{|v'-v'_*|}$.

In addition to that, we notice that $v = \frac{v'+v'_*}{2} + \frac{v'-v'_*}{2}k$. It's the same formula as for $v'$ in function of $v$, $v_*$ and $\sigma$.

Therefore,

$$\begin{aligned} D^-(g,f) &= \int B(v'-v'_*,k)g(v'_*)f(v')\log f\left(\frac{v'+v'_*}{2} + \frac{v'-v'_*}{2}k\right)\mathrm{d}k\mathrm{d}v'_*\mathrm{d}v' \\ &= \int B(v-v_*,\sigma)g(v_*)f(v)\log f\left(\frac{v+v_*}{2} + \frac{v-v_*}{2}\sigma\right)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v \\ &= \int Bg_*f\log f'\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v \end{aligned}$$

As a consequence, we obtain a new formula for $D$:

$$D(g,f) = \int Bg_*f\log\frac{f}{f'}\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \tag{9}$$

And, we can write:

$$D(g,f) = \int Bg_*\left(f\log\frac{f}{f'} - f + f'\right)\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma + \int Bg_*(f-f')\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \tag{10}$$

We will deal with the two terms of this equality one by one. So, we set:

$$D_1(g,f) = \int Bg_*\left(f\log\frac{f}{f'} - f + f'\right)\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \tag{11}$$

$$D_2(g,f) = \int Bg_*(f-f')\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \tag{12}$$

## 2.2 $v \to v'$ change of variables

To deal with both of them, we will use an other change of variables, more complicated, which consist in replacing $v$ by $v'$ and leaving $v_*$ and $\sigma$ as before.

Remind the formula for $v'$:

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma \tag{13}$$

Let us compute the jacobian of this change of variables $\dfrac{\partial v'}{\partial v}$.

$$\begin{aligned}
\frac{\partial v'}{\partial v} &= \frac{1}{2}\left(I + \frac{\partial}{\partial v}\sqrt{|v - v_*|^2}\sigma\right) \\
&= \frac{1}{2}\left(I + \mathrm{grad}_v(\sqrt{|v - v_*|^2}) \times \sigma^{\mathbf{t}}\right) \\
&= \frac{1}{2}\left(I + \frac{1}{|v - v_*|}(v - v_*) \times \sigma^{\mathbf{t}}\right) \\
&= \frac{1}{2}\left(I + k \times \sigma^{\mathbf{t}}\right)
\end{aligned}$$

Thus, we have $\left|\dfrac{\partial v'}{\partial v}\right| = \dfrac{1}{2^N}|I + k \times \sigma^{\mathbf{t}}| = \dfrac{1}{2^N}|I + \sigma^{\mathbf{t}} \times k| = \dfrac{1}{2^N}(1 + k.\sigma)$.

But, $k.\sigma = \cos\theta$ entails $1 + k.\sigma = 2\cos^2\frac{\theta}{2}$.

We define $k' = \dfrac{v' - v_*}{|v' - v_*|}$, and we notice that $(\sigma, k) = 2(\sigma, k')$ (angles at the centre and on the circle).

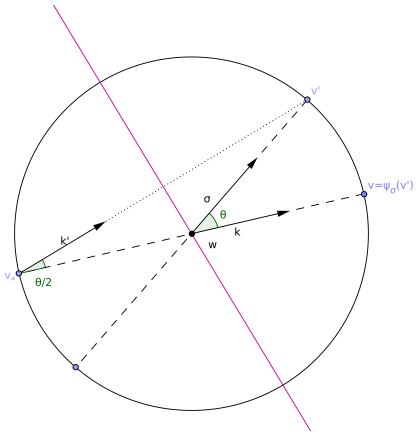


Figure 4: $v \to v'$ change of variables

Given that $\theta \in \left]0, \frac{\pi}{2}\right[$, it entails $\frac{\theta}{2} \in \left]-\frac{\pi}{4}, \frac{\pi}{4}\right[$. We get:

$$\left|\frac{\partial v}{\partial v'}\right| = \frac{(k'.\sigma)^2}{2^{N-1}} \geq \frac{1}{2^N} \tag{14}$$

As a consequence, this change of variables is valid.

We denote $\psi_\sigma : v' \mapsto v$ the inverse transformation. It is bijective! In the article, they give a geometrical definition of $\psi_\sigma$: " Draw the hyperplane which bisects the segment $[v_*, v']$ (hyperplan), let $w$ be the intersection of this hyperplane with the line through $v'$, parallel to $\sigma$, then $v$ is the velocity which is symetric to $v_*$ with respect to $w$. "

Let us check that this affirmation is accurate.

We work in the plane $\mathcal{P} = (v_*, v, \sigma)$. Given its definition, $v' \in \mathcal{P}$. We can deduce $w \in \mathcal{P}$, then $\psi_\sigma(v') \in \mathcal{P}$.

Remind that $w = \frac{\psi_\sigma(v') + v_*}{2}$. We have $v' = w + |v' - w|\sigma$. But, $|v' - w| = |v_* - w| = \frac{|\psi_\sigma(v') - v_*|}{2}$. Therefore,

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma = \frac{\psi_\sigma(v') + v_*}{2} + \frac{|\psi_\sigma(v') - v_*|}{2}\sigma$$

$v \mapsto v'$ being bijective, we can conclude $v = \psi_\sigma(v')$.

We notice the following: $k'.\sigma = k.k' = \dfrac{|v' - v_*|}{|\psi_\sigma(v') - v_*|}$. This formula is the same as:

$$|v_* - \psi_\sigma(v)| = \frac{|v - v_*|}{k.\sigma} \tag{15}$$

Use for those formula will be seen later.

# 3 Cancellation lemma: $D_2$

This part deals with $D_2$ defined by 12: $D_2 = \int Bg_*(f - f')\mathrm{d}v\mathrm{d}v'\mathrm{d}\sigma$. For $\theta$ next to 0, $B$ goes to infinity, still, $f$ and $f'$ as nearly the same value. This will allow us to minore the term $D_2$.

**Theorem 3.1** (Cancellation lemma)**.** *Let*

$$S(z) = |S^{N-2}| \int_0^{\frac{\pi}{2}} \sin^{N-2}\theta \left( \frac{1}{\cos^N \frac{\theta}{2}} B\left( \frac{|z|}{\cos\frac{\theta}{2}}, \cos\theta \right) - B(|z|, \cos\theta) \right) \mathrm{d}\theta \tag{16}$$

$\star$ *is the convolution product. We have:*

1. $|S(z)| \leq C_N (\Lambda(|z|) + |z|\Lambda'(|z|))$
2. $\int B(f' - f)\mathrm{d}v\mathrm{d}\sigma = (f \star S)(v_*)$

We first do the change of variables (Part 2.2) $v \to v'$ in $\int Bf'\mathrm{d}v\mathrm{d}\sigma$:

$$\begin{aligned}
\int Bf'\mathrm{d}v\mathrm{d}\sigma &= \int_{k'.\sigma \geq \frac{1}{\sqrt{2}}} B(\psi_\sigma(v') - v_*, \sigma) f(v') \frac{2^{N-1}}{(k'.\sigma)^2} \mathrm{d}v'\mathrm{d}\sigma \\
&= \int_{k.\sigma \geq \frac{1}{\sqrt{2}}} B(\psi_\sigma(v) - v_*, \sigma) f(v') \frac{2^{N-1}}{(k.\sigma)^2} \mathrm{d}v\mathrm{d}\sigma
\end{aligned}$$

But,

$$\begin{aligned}
B(\psi_\sigma(v) - v_*, \sigma) &= B\left( |\psi_\sigma(v) - v_*|, \sigma.\frac{\psi_\sigma(v) - v_*}{|\psi_\sigma(v) - v_*|} \right) \\
&= B(|\psi_\sigma(v) - v_*|, \cos(2\theta)) \\
&= B(|\psi_\sigma(v) - v_*|, 2\cos^2\theta - 1) \\
&= B(|\psi_\sigma(v) - v_*|, 2(k.\sigma)^2 - 1)
\end{aligned}$$

Therefore,

$$\int B(f' - f)\mathrm{d}v\mathrm{d}\sigma = \int_{k.\sigma \geq \frac{1}{\sqrt{2}}} \frac{2^{N-1}}{(k.\sigma)^2} B(|v_* - \psi_\sigma(v)|, 1 - (k.\sigma)^2) f\mathrm{d}v\mathrm{d}\sigma - \int_{k.\sigma \leq 0} B(|v - v_*|, k.\sigma) f\mathrm{d}v\mathrm{d}\sigma$$

In the first term, we use the formula 16:

$$\begin{aligned}
\int_{k.\sigma \geq \frac{1}{\sqrt{2}}} \frac{2^{N-1}}{(k.\sigma)^2} B(|v_* - \psi_\sigma(v)|, 2(k.\sigma)^2 - 1) f\mathrm{d}v\mathrm{d}\sigma &= \int_{k.\sigma \geq \frac{1}{\sqrt{2}}} \frac{2^{N-1}}{(k.\sigma)^2} B\left( \frac{|v - v_*|}{k.\sigma}, 2(k.\sigma)^2 - 1 \right) f\mathrm{d}v\mathrm{d}\sigma \\
&= |S^{N-2}| \int_0^{\frac{\pi}{4}} \sin^{N-2}\theta \frac{2^{N-1}}{cos^2\theta} B\left( \frac{|v - v_*|}{\cos\theta}, \cos(2\theta) \right) f\mathrm{d}\theta\mathrm{d}v \\
&= |S^{N-2}| \int_0^{\frac{\pi}{2}} \sin^{N-2}(\theta/2) \frac{2^{N-2}}{cos^2(\theta/2)} B\left( \frac{|v - v_*|}{\cos(\theta/2)}, \cos\theta \right) f\mathrm{d}\theta\mathrm{d}v \\
&= |S^{N-2}| \int_0^{\frac{\pi}{2}} \frac{\sin^{N-2}\theta}{cos^N(\theta/2)} B\left( \frac{|v - v_*|}{\cos(\theta/2)}, \cos\theta \right) f\mathrm{d}\theta\mathrm{d}v
\end{aligned}$$

Then, we obtain $\int B(f' - f)\mathrm{d}v\mathrm{d}\sigma = \int_0^{\frac{\pi}{2}} \sin^{N-2}\theta \left( \frac{1}{cos^N(\theta/2)} B\left( \frac{|v - v_*|}{\cos(\theta/2)}, \cos\theta \right) - B(|v - v_*|, \cos\theta) \right) f\mathrm{d}v\mathrm{d}\theta$, which is the expected formula.

Still remains the majoration of $|S(|z|)|$. To do that, we use the formulas of part 1.2 and the fact that $\cos\frac{\theta}{2} \geq \frac{1}{\sqrt{2}}$. In addition to that, we use the majoration $1-\cos^N\alpha \leq N(1-\cos\alpha)$ for $\alpha \in \left]0;\frac{\pi}{4}\right[$.

$$\begin{aligned}
|S(|z|)| &\leq |S^{N-2}| \int_0^{\frac{pi}{2}} \frac{\sin^{N-2}\theta}{\cos^N\left(\frac{\theta}{2}\right)} \left| B\left(\frac{|z|}{\cos\left(\frac{\theta}{2}\right)}\right) - B(|z,\cos\theta)\right| \mathrm{d}\theta + |S^{N-2}| \int_0^{\frac{\pi}{2}} \sin^{N-2}\theta \left(\frac{1}{cos^N\left(\frac{\theta}{2}\right)}\right) B(|z|,\cos\theta)\mathrm{d}\theta \\
&\leq |S^{N-2}| 2^{\frac{N}{2}} \left[\int_0^{\frac{\pi}{2}} \sin^{N-2}\left(1-\cos\frac{\theta}{2}\right)|z|B'(|z|,\cos\theta)\mathrm{d}\theta + \int_0^{\frac{\pi}{2}} \sin^{N-2}\left(1-cos^N\frac{\theta}{2}\right) B(|z|,\cos\theta)\mathrm{d}\theta\right] \\
&\leq 2^{\frac{N}{2}} N\left(|z|\Lambda'(|z|) + \Lambda(|z|)\right)
\end{aligned}$$

And the theorem is proven!
This entails the following result:

**Corollary 3.1.** *There exists C depending only on N and B such that:*

$$\left|\int Bg_*(f'-f)\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma\right| \leq C\|g\|_{L^1_2}\|f\|_{L^1_2} \tag{17}$$

Actually, we have

$$\begin{aligned}
\left|\int Bg_*(f'-f)\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma\right| &= \left|\int (f\star S)(v_*)g(v_*)\mathrm{d}v_*\right| \\
&= \left|\int\int f(v)S(v-v_*)\mathrm{d}v g(v_*)\mathrm{d}v\right| \\
&\leq \int f(v)g(v_*)|S(v-v_*)|\mathrm{d}v\mathrm{d}v_* \\
&\leq C_N \int f(v)g(v_*)\left(|v-v_*|\Lambda'(|v-v_*|) + \Lambda(|v-v_*|)\right)\mathrm{d}v\mathrm{d}v_* \\
&\leq C_N C_0 \int fg_*(1+|v-v_*|)^2\mathrm{d}v\mathrm{d}v_*
\end{aligned}$$

thanks to the assumption 7.
Still,

$$(1+|v|)(1+|v_*|) \geq 1+|v|+|v_*| = 1+|(v-v_*)+v_*)|+|v_*| \geq 1+|v-v_*|-|v_*|+|v_*| = 1+|v-v_*| \tag{18}$$

Then,

$$\begin{aligned}
\left|\int Bg_*(f'-f)\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma\right| &\leq C\int f(v)(1+|v|)^2\, g(v_*)(1+|v_*|)^2\mathrm{d}v\mathrm{d}v_* \\
&\leq C\|g\|_{L^1_2}\|f\|_{L^1_2}
\end{aligned}$$

# 4 Fourier analysis and truncature: $D_1$

Here, we deal with $D_1$ defined by 11: $D_1(g,f) = \int Bg_*\left(f\log\frac{f}{f'} - f + f'\right)\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma$.
Let us first consider a simple case.
Let $F = \sqrt{f}$.

Let $h(x) = \log x - 1 + \frac{1}{x} - \left(\frac{1}{\sqrt{x}}-1\right)^2 = \log x - 2 + \frac{2}{\sqrt{x}}$. We have $h'(x) = \frac{1}{x} + \frac{1}{\sqrt{x^3}} = \frac{1}{\sqrt{x^3}}\left(\sqrt{x}-1\right)$. Then $h'$ is null in 1 and only there, it is negative before, positive after. This entails that $h$ is minimum in 1. But, $h(1) = 0$. Then, $\forall x > 0, \log(x) - 1 + \frac{1}{x} \geq \left(\frac{1}{\sqrt{x}}-1\right)$. With $x = \frac{f}{f'}$, this entails:

$$D_1 \geq \int Bg_*(F'-F)^2\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \geq \int \Phi(|v-v_*|)b(k.\sigma)g_*(F'-F)^2\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \tag{19}$$

So, we set $M = \int \Phi(|v-v_*|)b(k.\sigma)g_*(F'-F)^2\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma$ and it's this expression that we will try to minore.

## 4.1 First case: $\Phi = 1$, Fourier analysis

As a first case, we assume that $\Phi(|z|) = 1$. Then

$$M = \int b(k.\sigma) g_* (F' - F)^2 \mathrm{d}v \mathrm{d}v_* \mathrm{d}\sigma = \int b(k.\sigma) g_* (F'^2 - 2FF' + F^2) \mathrm{d}v \mathrm{d}v_* \mathrm{d}\sigma$$

We cut $M$ into three parts: $M = M_1 - 2M_2 + M_3$ where:

$$M_1 = \int b(k.\sigma) g_* F'^2 \mathrm{d}v \mathrm{d}v_* \mathrm{d}\sigma \tag{20}$$

$$M_2 = \int b(k.\sigma) g_* F F' \mathrm{d}v \mathrm{d}v_* \mathrm{d}\sigma \tag{21}$$

$$M_3 = \int b(k.\sigma) g_* F^2 \mathrm{d}v \mathrm{d}v_* \mathrm{d}\sigma \tag{22}$$

### 4.1.1 $M_2$

We denote $\hat{F}(\xi) = \mathcal{F}F(\xi) = \int_{\mathbb{R}^N} \mathrm{e}^{-\mathrm{i}v.\xi} F(v) \mathrm{d}v$ the Fourier transform of $F \in L^2(\mathbb{R}^N)$.

Let $Q^+(g, F)(v) = \int b(k.\sigma) g'_* F' \mathrm{d}v_* \mathrm{d}\sigma$.

We obtain $M_2 = \int Q^+(g, F) F \mathrm{d}v$ by using the pre-post-collisional change of variables (part 2.1). $F, Q^+(g, F) \in L^2$ so we can use Parseval's identity:

$$M_2 = \frac{1}{(2\pi)^N} \int \mathcal{F}(Q^+(g, F)) \overline{\hat{F}} \mathrm{d}\xi$$

Remains to compute $\mathcal{F}(Q^+(g, F))$.

Let $\varphi$ be a test function.

In $\int Q^+(g, F)\varphi \mathrm{d}v = \int b(k.\sigma) g'_* F' \varphi \mathrm{d}v \mathrm{d}v_* \mathrm{d}\sigma$, we use the pre-post-collisional introduced in part 2.1.

It entails $\int Q^+(g, F)\varphi \mathrm{d}v = \int b(k.\sigma) g_* F \varphi' \mathrm{d}v \mathrm{d}v_* \mathrm{d}\sigma$. Given that $\varphi(v) = \mathrm{e}^{-\mathrm{i}v.\xi}$, we obtain:

$$\begin{aligned}
\mathcal{F}(Q^+(g, F))(\xi) &= \int b(k.\sigma) g_* f \mathrm{e}^{\mathrm{i}v'.\xi} \mathrm{d}v \\
&= \int b(k.\sigma) g_* f \mathrm{e}^{-\mathrm{i}\frac{v+v_*}{2}.\xi} \mathrm{e}^{-\mathrm{i}\frac{|v-v_*|}{2}\sigma.\xi} \mathrm{d}v \mathrm{d}v_* \mathrm{d}\sigma \\
&= \int \left( \int b(k.\sigma) \mathrm{e}^{-\mathrm{i}|v-v_*|\frac{|\xi|}{2}\frac{\xi}{|\xi|}.\sigma} \mathrm{d}\sigma \right) g_* f \mathrm{e}^{-\mathrm{i}\frac{v+v_*}{2}.\xi} \mathrm{d}v \mathrm{d}v_*
\end{aligned}$$

Let $H$ be the hyperplan that bisects $\frac{\xi}{|\xi|}$ et $k$. Let $s$ be the symetry wit respect to $H$, it is an isometry of $S^{N-1}$. Now, we do the change of variables $\sigma \mapsto \tau = s(\sigma)$:

$$\int b(k.\sigma) \mathrm{e}^{-\mathrm{i}|v-v_*|\frac{|\xi|}{2}\frac{\xi}{|\xi|}.\sigma} \mathrm{d}\sigma = \int b(k.s(\tau)) \mathrm{e}^{-\mathrm{i}|v-v_*|\frac{|\xi|}{2}\frac{\xi}{|\xi|}.s(\tau)} \mathrm{d}\tau = \int b\left(\frac{\xi}{|\xi|}.\tau\right) \mathrm{e}^{-\mathrm{i}|v-v_*|\frac{|\xi|}{2}k.\tau} \mathrm{d}\tau$$

Then,

$$\begin{aligned}
\mathcal{F}(Q^+(g, F))(\xi) &= \int g_* F b\left(\frac{\xi}{|\xi|}.\sigma\right) \mathrm{e}^{-\mathrm{i}\left(\xi.\frac{v+v_*}{2} + |\xi|\sigma.\frac{v-v_*}{2}\right)} \mathrm{d}v \mathrm{d}v_* \mathrm{d}\sigma \\
&= \int g_* F b\left(\frac{\xi}{|\xi|}.\sigma\right) \mathrm{e}^{-\mathrm{i}v.\xi^+} \mathrm{e}^{-\mathrm{i}v_*.\xi^-} \mathrm{d}v \mathrm{d}v_* \mathrm{d}\sigma
\end{aligned}$$

with $\xi^+ = \frac{\xi + |\xi|\sigma}{2}$ and $\xi^- = \frac{\xi - |\xi|\sigma}{2}$.

Thus,

$$\begin{aligned}
\mathcal{F}(Q^+(g, F))(\xi) &= \frac{1}{(2\pi)^N} \int \hat{(}g)(\eta_*) \mathrm{e}^{\mathrm{i}\eta_*.v_*} \hat{F}(\eta) \mathrm{e}^{\mathrm{i}\eta.v} b\left(\frac{\xi}{|\xi|}.\sigma\right) \mathrm{e}^{-\mathrm{i}v.\xi^+} \mathrm{e}^{-\mathrm{i}v_*.\xi^-} \mathrm{d}\eta_* \mathrm{d}\eta \mathrm{d}v \mathrm{d}v_* \mathrm{d}\sigma \\
&= \frac{1}{(2\pi)^N} \int \hat{g}(\eta_*) \hat{F}(\eta) b\left(\frac{\xi}{|\xi|}.\sigma\right) \int \mathrm{e}^{\mathrm{i}v.(\eta - \xi^+)} \mathrm{d}v \int \mathrm{e}^{\mathrm{i}v_*.(\eta_* - \xi^-)} \mathrm{d}v_* \mathrm{d}\eta \mathrm{d}\eta_* \mathrm{d}\sigma \\
&= \int \hat{g}(\eta_*) \hat{F}(\eta) b\left(\frac{\xi}{|\xi|}.\sigma\right) \delta_\eta^{\xi^+} \delta_{\eta_*}^{\xi^-} \mathrm{d}\eta_* \mathrm{d}\eta \mathrm{d}\sigma
\end{aligned}$$

So,

$$\mathcal{F}(Q^+(g,F))(\xi) = \int \hat{g}(\xi^-)\hat{F}(\xi^+)b\left(\frac{\xi}{|\xi|}.\sigma\right)\mathrm{d}\sigma \tag{23}$$

This entails:

$$M_2 = \frac{1}{(2\pi)^N}\int b\left(\frac{\xi}{|\xi|}.\sigma\right)\hat{g}(\xi^-)\hat{F}(\xi^+)\overline{\hat{F}}(\xi)\mathrm{d}\xi\mathrm{d}\sigma = \int b\left(\frac{\xi}{|\xi|}.\sigma\right)\overline{\hat{g}}(\xi^-)\overline{\hat{F}}(\xi^+)\hat{F}(\xi)\mathrm{d}\xi\mathrm{d}\sigma \tag{24}$$

#### 4.1.2 $M_3$

$\int b(k.\sigma)\mathrm{d}\sigma$ does not depend on the unit vector $k$.

So, we write:

$$\begin{aligned} M_3 &= \int g_*\mathrm{d}v_* \int F^2\mathrm{d}v \int b(k.\sigma)\mathrm{d}\sigma \\ &= \hat{g}(0)\int F^2 \int b\left(\frac{\xi}{|\xi}.\sigma\right)\mathrm{d}\sigma \end{aligned}$$

To conclude,

$$M_3 = \frac{1}{(2\pi)^N}\hat{g}(0)\int b\left(\frac{\xi}{|\xi|}.\sigma\right)|\hat{F}|^2(\xi)\mathrm{d}\sigma\mathrm{d}\xi \tag{25}$$

#### 4.1.3 $M_1$

Let us first do the change of variables $v \mapsto q = v - v_*$. $v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma = \frac{q+|q|\sigma}{2} + v_*$. This entails:

$$M_1 = \int g(v_*)b\left(\frac{q}{|q|}.\sigma\right)F\left(v_* + \frac{q+|q|\sigma}{2}\right)\mathrm{d}q\mathrm{d}v_*\mathrm{d}\sigma$$

Now, we use the change of variables $q \mapsto v' = \frac{q+|q|\sigma}{2}$ which is actually the change of variables introduced in part 2.2 with $v_* = 0$.

$$M_1 = \int g_* b\left(2\left(\frac{v'}{|v'|}.\sigma\right)^2 - 1\right)\frac{2^{N-1}}{\left(\frac{v'}{|v'|}.\sigma\right)^2}F(v_*+v')^2\mathrm{d}v'\mathrm{d}v_*\mathrm{d}\sigma$$

As before, $\int b\left(2\left(\frac{v'}{|v'|}.\sigma\right)^2 - 1\right)\mathrm{d}\sigma$ does not depend on $\frac{v'}{|v'|}$ unit vector. In addition, $\int F(v_*+v')^2\mathrm{d}v'$ does not depend on $v_*$ and is equal to $\int F(v')^2\mathrm{d}v' = \frac{1}{(2\pi)^N}\int\left|\hat{F}\right|^2(\xi)\mathrm{d}\xi$. The only factor in the integrand that depend on $v_*$ is actually $g_*$ and we have $\int g_*\mathrm{d}v_* = \hat{g}(0)$. This entails:

$$M_1 = \frac{1}{(2\pi)^N}\hat{g}(0)\int b\left(2\left(\frac{\xi}{|\xi|}.\sigma\right)^2 - 1\right)\left|\hat{F}\right|^2(\xi)\mathrm{d}\xi\mathrm{d}\sigma$$

We use again $q \mapsto v'$ in the opposite sense. It gives:

$$M_1 = \frac{1}{(2\pi)^N}\hat{g}(0)\int b\left(\frac{\xi}{|\xi|}.\sigma\right)\left|\hat{F}\left(\frac{\xi+|\xi|\sigma}{2}\right)\right|^2\mathrm{d}\sigma\mathrm{d}\xi \tag{26}$$

#### 4.1.4 Conclusion

From the formula 26, 24 and 25, we get:

$$M = \frac{1}{(2\pi)^N}\int b\left(\frac{\xi}{|\xi|}\right)\left[\hat{g}(0)\left|\hat{F}(\xi)\right|^2 + \hat{g}(0)\left|\hat{F}(\xi^+)\right|^2 - \hat{g}(\xi^-)\hat{F}(\xi^+)\overline{\hat{F}}(\xi) - \overline{\hat{g}}(\xi^-)\overline{\hat{F}}(\xi^+)\hat{F}(\xi)\right]\mathrm{d}\xi\mathrm{d}\sigma \tag{27}$$

Still, $\forall \varepsilon > 0, \hat{g}(\xi^-)\hat{F}(\xi^+)\overline{\hat{F}}(\xi) + \overline{\hat{g}}(\xi^-)\overline{\hat{F}}(\xi^+)\hat{F}(\xi) \leq \varepsilon \left|\hat{F}(\xi)\right|^2 + \frac{1}{\varepsilon}\left|\hat{g}(\xi^-)\hat{F}(\xi^+)\right|^2$. As a consequence, with $\varepsilon = |\hat{g}(\xi^-)|$,

$$\begin{aligned} M &\geq \frac{1}{(2\pi)^N}\int b\left(\frac{\xi}{|\xi|}\right)\left[\hat{g}(0)\left|\hat{F}(\xi)\right|^2 + \hat{g}(0)\left|\hat{F}(\xi^+)\right|^2 - \left|\hat{g}(\xi^-)\right|\left|\hat{F}(\xi)\right|^2 - \left|\hat{g}(\xi^-)\right|\left|\hat{F}(\xi^+)\right|^2\right]\mathrm{d}\xi\mathrm{d}\sigma \\ &\geq \frac{1}{(2\pi)^N}\int b\left(\frac{\xi}{|\xi|}\right)\left(\hat{g}(0) - \left|\hat{g}(\xi^-)\right|\right)\left(\left|\hat{F}(\xi)\right|^2 + \left|\hat{F}(\xi^+)\right|^2\right)\mathrm{d}\xi\mathrm{d}\sigma \\ M &\geq \frac{1}{(2\pi)^N}\int \left|\hat{F}(\xi)\right|^2\left(\int b\left(\frac{\xi}{|\xi|}.\sigma\right)\left(\hat{g}(0) - \left|\hat{g}(\xi^-)\right|\right)\mathrm{d}\sigma\right)\mathrm{d}\xi \end{aligned}$$

We want to minore the factor into parenthesis. To do so, we first focus on the factor $\hat{g}(0) - |\hat{g}(\xi^-)|$. Fixing $\xi^-$, there exists $\alpha \in \mathbb{R}$ such that $g(\xi^-) = |g(\xi^-)|\,\mathrm{e}^{\mathrm{i}\alpha}$. Then $\hat{g}(0) - |\hat{g}(\xi^-)| = \int g(v)\mathrm{d}v - \int g(v)\mathrm{e}^{-\mathrm{i}(v.\xi^- + \alpha)}\mathrm{d}v = \int g(v)\left(1 - \cos(v.\xi^- + \alpha)\right)$ as it's a real number. Then, we get, for $r > 0$ and $\varepsilon > 0$:

$$\begin{aligned} \hat{g}(0) - \left|\hat{g}(\xi^-)\right| &= 2\int g(v)\sin^2\left(\frac{v.\xi^- + \alpha}{2}\right)\mathrm{d}v \\ &\geq 2\int_{\{|v|\leq r, \forall p\in\mathbb{Z}, |v.\xi^- + \alpha - 2p\pi| \geq 2\varepsilon\}} g(v)\sin^2\left(\frac{v.\xi^- + \alpha}{2}\right)\mathrm{d}v \\ &\geq 2\sin^2\varepsilon\int_{\{|v|\leq r, \forall p\in\mathbb{Z}, |v.\xi^- + \alpha - 2p\pi| \geq 2\varepsilon\}} g(v)\mathrm{d}v \\ &\geq 2\sin^2\varepsilon\left(\int_{\mathbb{R}} g(v)\mathrm{d}v - \int_{\{|v|>r\}} g(v)\mathrm{d}v - \int_{\{|v|\leq r, \exists p\in\mathbb{Z}, |v.\xi^- + \alpha - 2p\pi| < 2\varepsilon\}} g(v)\mathrm{d}v\right) \end{aligned}$$

Still, $\int_{\{|v|>r\}} g(v)\mathrm{d}v \leq \int_{\{|v|>r\}} g(v)\left(\frac{1+|v|}{r}\right)\mathrm{d}v$ because for $|v| > r$, $\frac{1+|v|}{r} > 1$. As a consequence,

$$\int_{\{|v|>r\}} g(v)\mathrm{d}v < \frac{1}{r}\int_{\mathbb{R}} g(v)(1+|v|) = \frac{\|g\|_{L^1_1(\mathbb{R}^N)}}{r}$$

This entails:

$$\hat{g}(0) - \left|\hat{g}(\xi^-)\right| \geq 2\sin^2\varepsilon\left(\|g\|_{L^1_1(\mathbb{R}^N)} - \frac{\|g\|_{L^1_1(\mathbb{R}^N)}}{r} - \int_{\{|v|\leq r, \exists p\in\mathbb{Z}, |v.\xi^- + \alpha - 2p\pi| < 2\varepsilon\}} g(v)\mathrm{d}v\right)$$

Let $A = \{|v| \leq r, \exists p \in \mathbb{Z}, |v.\xi^- + \alpha - 2p\pi| < 2\varepsilon\}$.

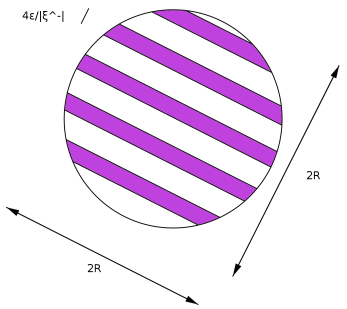


Figure 5: $A$

It is clear on the figure 5 that $A$ est made of less than $1 + \frac{r|\xi^-|}{\pi}$ colums of width $4\frac{\varepsilon}{|\xi^-|}$ and of surface less than $(2r)^{N-1}$. Then $|A| \leq \frac{4\varepsilon}{|\xi^-|}(2r)^{N-1}\left(1 + \frac{r|\xi^-|}{\pi}\right)$.

Therefore,

$$\hat{g}(0) - \left|\hat{g}(\xi^-)\right| \geq 2\sin^2\varepsilon\left(\|g\|_{L^1_1(\mathbb{R}^N)} - \frac{\|g\|_{L^1_1(\mathbb{R}^N)}}{r} - \sup_{|A|\leq\frac{4\varepsilon}{|\xi^-|}(2r)^{N-1}\left(1+\frac{r|\xi^-|}{\pi}\right)}\int_A g(v)\mathrm{d}v\right) \tag{28}$$

**First case:** $|\xi^-| \geq 1$ Here, we assume that $|\xi^-| \geq 1$.

Then, $\frac{4\varepsilon}{|\xi^-|}(2r)^{N-1}\left(1+\frac{r|\xi^-|}{\pi}\right) = 4\varepsilon(2r)^{N-1}\left(\frac{1}{|\xi^-|}+\frac{r}{\pi}\right) \leq 4\varepsilon(2r)^{N-1}\left(1+\frac{r}{\pi}\right)$.

Let $C'_g = 2\sin^2\varepsilon\left(\|g\|_{L^1_1(\mathbb{R}^N)} - \frac{\|g\|_{L^1_1(\mathbb{R}^N)}}{r} - \sup_{|A|\leq 4\varepsilon(2r)^{N-1}\left(1+\frac{r}{\pi}\right)}\int_A g(v)\mathrm{d}v\right)$. First, we choose $r$ such that $\frac{\|g\|_{L^1_1(\mathbb{R}^N)}}{r} \leq \frac{1}{3}\|g\|_{L^1(\mathbb{R}^N)}$. Then, we can adjust $\varepsilon$ such that $\sup_{|A|\leq 4\varepsilon(2r)^{N-1}\left(1+\frac{r}{\pi}\right)}\int_A g(v)\mathrm{d}v \leq \frac{1}{3}\|g\|_{L^1(\mathbb{R}^N)}$. With those choices, $C'_g > 0$ et on a $\hat{g}(0) - |\hat{g}(\xi^-)| \geq c_g$.

**Second case:** $|\xi^-| \leq 1$ Here, we assume that $|\xi^-| \leq 1$.

Let $\delta = \frac{\varepsilon}{|\xi^-|}$. We rewrite 28 in function of $\delta$ instead of $\varepsilon$.

$$\begin{aligned}
\hat{g}(0) - \left|\hat{g}(\xi^-)\right| &\geq 2\sin^2(|\xi^-|\delta)\left(\|g\|_{L^1_1(\mathbb{R}^N)} - \frac{\|g\|_{L^1_1(\mathbb{R}^N)}}{r} - \sup_{|A|\leq 4\delta(2r)^{N-1}\left(1+\frac{r|\xi^-|}{\pi}\right)}\int_A g(v)\mathrm{d}v\right)\\
&\geq 2|\xi^-|^2\delta^2\frac{\sin^2(|\xi^-|\delta)}{|\xi^-|^2\delta^2}\left(\|g\|_{L^1_1(\mathbb{R}^N)} - \frac{\|g\|_{L^1_1(\mathbb{R}^N)}}{r} - \sup_{|A|\leq 4\delta(2r)^{N-1}\left(1+\frac{r}{\pi}\right)}\int_A g(v)\mathrm{d}v\right)\\
&\geq 2|\xi^-|^2\delta^2\inf_{|\xi|\leq 1}\left|\mathrm{sinc}^2(\delta|\xi|)\right|\left(\|g\|_{L^1_1(\mathbb{R}^N)} - \frac{\|g\|_{L^1_1(\mathbb{R}^N)}}{r} - \sup_{|A|\leq 4\delta(2r)^{N-1}\left(1+\frac{r}{\pi}\right)}\int_A g(v)\mathrm{d}v\right)
\end{aligned}$$

We choose $r$ as in the first case. Then, we choose $\delta < \pi$ as we did for $\varepsilon$ in the first case.

To conclude, $\hat{g}(0) - |\hat{g}(\xi^-)| \geq |\xi^-|^2 c'_g$, where $c'_g > 0$ depends only on $N$ and $g$.

**Conclusion** There exists $C'_g > 0$ depending only on $N$ and $g$ such that:

$$\hat{g}(0) - \left|\hat{g}(\xi^-)\right| \geq C'_g\min(1, |\xi^-|^2) \tag{29}$$

Therefore, $\int b\left(\frac{\xi}{|\xi|}.\sigma\right)(\hat{g}(0) - |\hat{g}(\xi^-)|)\,\mathrm{d}\sigma \geq C'_g\int b\left(\frac{\xi}{|\xi|}.\sigma\right)\min(1, |\xi^-|^2)\mathrm{d}\sigma$. Or, $|\xi^-|^2 = \frac{|\xi|^2}{2}\left(1 - \frac{\xi}{|\xi|}.\sigma\right)$. Then,

$$\begin{aligned}
\int b\left(\frac{\xi}{|\xi|}.\sigma\right)\left(\hat{g}(0) - \left|\hat{g}(\xi^-)\right|\right)\mathrm{d}\sigma &\geq C'_g|S^{N-2}|\int_0^{\frac{\pi}{2}}\sin^{N-2}\theta b(\cos\theta)\min\left(\frac{|\xi|^2}{2}(1-\cos\theta), 1\right)\mathrm{d}\theta\\
&\geq C'_g\frac{K}{2}|S^{N-2}|\int_0^{\frac{\alpha}{|\xi|}}\min\left(\frac{|\xi|^2}{2}(1-\cos\theta), 1\right)\frac{\mathrm{d}\theta}{\theta^{1+\nu}}\\
&\geq C'_g\frac{K}{2}|S^{N-2}||\xi|^\nu\int_0^{\alpha}\min\left(\frac{|\xi|^2}{2}(1-\cos\tau), 1\right)\frac{\mathrm{d}\tau}{\tau^{1+\nu}}
\end{aligned}$$

for an $\alpha > 0$, small, and given the formula 5.

This entails the existence of a constant $C_g$ depending only on $N$, $b$ and $g$ such that:

$$\int b\left(\frac{\xi}{|\xi|}.\sigma\right)\left(\hat{g}(0) - \left|\hat{g}(\xi^-)\right|\right)\mathrm{d}\sigma \geq C_g|\xi|^\nu \tag{30}$$

And so, there exists $C$ depending only on $N$, $b$ and $g$ such that:

$$M \geq C\|\sqrt{f}\|^2_{H^{\frac{\nu}{2}}(\mathbb{R}^N)} \tag{31}$$

In particular, $D_1 \geq C\|\sqrt{f}\|_{H^{\frac{\nu}{2}}(\mathbb{R}^N)}$.

Then, as $D(g,f) = D_1(g,f) + D_2(g,f)$, we have $D(g,f) - D_2(g,f) \geq C\|\sqrt{f}\|_{H^{\frac{\nu}{2}}}(\mathbb{R}^N)$. And, given 17,

$$D(g,f) + C_1\|g\|_{L^1_2}\|f\|_{L^1_2} \geq C\|\sqrt{f}\|_{H^{\frac{\nu}{2}}(\mathbb{R}^N)} \tag{32}$$

We proved what we wanted to.

## 4.2 Second case: $\Phi(0) \neq 0$

### 4.2.1 Truncation

Let $A$ and $B$ be two parts of $\mathbb{R}^N$. We denote by $\chi_A$ and $\chi_B$ the mollified ($C^\infty$) characteristic functions of $A$ and $B$. When $A \cap B = \emptyset$, we shall take $\mathrm{Supp}A \cap \mathrm{Supp}B = \emptyset$.

We wish to replace $f$ and $g$ by $f\chi_A$ and $g\chi_B$ respectively.

First of all, $g_*(F'-F)^2 \geq g_*\chi_{B*}(F'-F)^2\chi_A^2$. We also notice that $(F'\chi'_A - F\chi_A)^2 \leq 2F'^2(\chi'_A-\chi_A)+2(F-F')^2\chi_A^2$.

As a consequence,

$$\begin{aligned}\Phi(|v-v_*|)b(k.\sigma)g_*(F-F')^2 &\geq \Phi(|v-v_*|)b(k.\sigma)g_*\chi_{B*}(F-F')^2\chi_A^2 \\ &\geq \frac{1}{2}\min(1,\Phi(|v-v_*|))b(k.\sigma)g_*\chi_{B*}(F'\chi'_A - F\chi_A)^2 - b(k.\sigma)g_*\chi_{B*}F'^2(\chi'_A-\chi_A)^2\end{aligned}$$

But, $(\chi'_A-\chi_A)^2 \leq \|\nabla\chi_A\|^2_{L^\infty}|v-v'|^2 \leq C_A\max(1,\mathrm{diam}A)^{-2}|v-v_*|^2\sin^2\frac{\theta}{2}$ with a suitable choice for $\chi_A$.

Then,

$$\begin{aligned}\int g_*\chi_{B*}b(k.\sigma)f'(\chi'_A-\chi_A)^2\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma &\leq C_A\int g_*b(\cos\theta)f'|v-v_*|^2\sin^2\frac{\theta}{2}\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \\ &\leq C_A\int g_*b(\cos\theta)f'\frac{|v'-v_*|^2}{\cos^2\frac{\theta}{2}}\sin^2\frac{\theta}{2}\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \\ &\leq C_A\int_{|\theta|\leq\frac{\pi}{2}} g_*b(\cos\theta)f'\frac{|v'-v_*|^2}{\cos^2\frac{\theta}{2}}\sin^2\frac{\theta}{2}\frac{2^{N-1}}{\cos^2\frac{\theta}{2}}\mathrm{d}v'\mathrm{d}v_*\mathrm{d}\sigma\end{aligned}$$

Noticing that $\frac{\sin^2\frac{\theta}{2}}{\cos^4\frac{\theta}{2}} \leq 1$ given $|\theta| \leq \frac{\pi}{2}$, and using 18, we conclude :

$$\int g_*\chi_{B*}b(k.\sigma)f'(\chi'_A-\chi_A)^2\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \leq C'_A\|g\|_{L^1_2}\|f\|_{L^1_2} \tag{33}$$

Thus, we have:

$$\int\Phi(|v-v_*|)b(k.\sigma)g_*(F-F')^2\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma+C'_A\|g\|_{L^1_2}\|f\|_{L^1_2} \geq \frac{1}{2}\int\min(1,\Phi(|v-v_*|))b(k.\sigma)g_*\chi_{B*}(F'\chi'_A-F\chi_A)^2\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \tag{34}$$

### 4.2.2 Use of troncature in this case

In this case, we chose $A = B = B_R$ the $\mathbb{R}^N$-ball centered in 0 of radius $R$. The integrande in the left-hand side of inequality 34 is not null when $\chi_{B*} \neq 0$ and $\chi_A$ or $\chi'_A$ is not null that is $|v_*| \leq R$ and $\min(|v|,|v'|) \leq R$. Given that $|v'-v_*| \leq |v-v_*| \leq \sqrt{2}|v'-v_*|$, this integrande is null when $|v-v_*| > 8R^2$.

This entails,

$$\int\Phi(|v-v_*|)b(k.\sigma)g_*(F-F')^2\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma+C'_A\|g\|_{L^1_2}\|f\|_{L^1_2} \geq \frac{1}{2}\min_{|z|\leq 2\sqrt{2}R}(\Phi(|z|),1)\int b(k.\sigma)g_*\chi_{B_R*}(F'\chi'_{B_R}-F\chi_{B_R})^2\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \tag{35}$$

We recognise $M$ associated to $g\chi_{B_R}$ and $f\chi_{B_R}$ on the left-hand side, which gives thanks to inequality 31 :

$$\begin{aligned}\int\Phi(|v-v_*|)b(k.\sigma)g_*(F-F')^2\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma + C'_A\|g\|_{L^1_2}\|f\|_{L^1_2} &\geq \frac{1}{2}\min_{|z|\leq 2\sqrt{2}R}(\Phi(|z|),1)C\|\sqrt{f\chi_{B_R}}\|^2_{H^{\frac{\gamma}{2}}} \\ &\geq C'\|\sqrt{f}\|^2_{H^{\frac{\gamma}{2}}(B_R)}\end{aligned}$$

That is $D_1 + C'_A\|g\|_{L^1_2}\|f\|_{L^1_2} \geq C'\|\sqrt{f}\|^2_{H^{\frac{\gamma}{2}}(B_R)}$. Remembering the cancellation lemma, in particular inequality 17 we conclude

$$\|\sqrt{f}\|^2_{H^{\gamma/2}(|v|<R)} \leq C_{g,R}\left(D(g,f)+\|g\|_{L^1_2}\|f\|_{L^1_2}\right)$$

We have proven the theorem in this case. Remains the study of the general case when $\Phi(0)$ can be 0.

### 4.3 General case : $\Phi(0)$ may be null !

The idea is to use the results of part 4.2.1 with a well-chosen set of $A_j, B_j$ such that $\Phi$ would not be null.

Let $r_0 > 0$ be a small radius. As $B_R$ is compact, there exist a finite set $J$ such that $B_R \subset \bigcup_{j\in J} B\left(v_j, \frac{r_0}{4}\right)$ with $v_j \in B_R, \forall j \in J$. We set $A_j = B_R \cap B\left(v_j, \frac{r_0}{4}\right)$ and $B_j = B_R \cap \overline{B(v_j, r_0)}$.

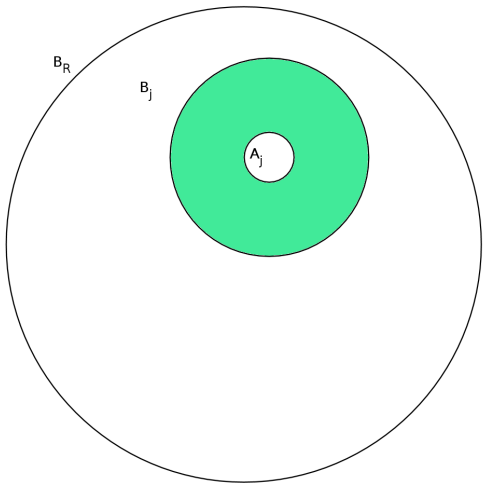


Figure 6: $A_j$ and $B_j$

Let us first notice that

$$\| \sqrt{f}\|^2_{H^{\frac{\nu}{2}}(B_R)} \leq \sum_{j\in J} \|F\chi_{A_j}\|^2_{H^{\frac{\nu}{2}}} \tag{36}$$

Reasoning as before, given that

$$\begin{cases} v_* \in B_j, v \in A_j \implies |v - v_*| > \frac{r_0}{2} \\ v_* \in B_j, v' \in A_j \implies |v - v_*| > \frac{r_0}{2} \end{cases}$$

we obtain:

$$\begin{aligned} 2\int \Phi(|v - v_*|)b(k.\sigma)g_*(F' - F)^2 \mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma & \\ +C'_{A_j}\|g\|_{L^1_2}\|f\|_{L^1_2} \geq \min_{\frac{r_0}{2}<|z|<2\sqrt{2}R}(1, \Phi(|z|)) & \int b(k.\sigma)g_*\chi_{B_j*}(F'\chi'_{A_j} - F\chi_{A_j})^2 \mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \\ \geq C_{g_*\chi_{B_j*}}\| \sqrt{f}\chi_{A_j}\|^2_{H^{\frac{nu}{2}}} & \end{aligned}$$

This inequality combined with inequality 36 hences the conclusion:

$$\| \sqrt{f}\|^2_{H^{\nu/2}(|v|<R)} \leq C_{g,R}\left(D(g, f) + \|g\|_{L^1_2}\|f\|_{L^1_2}\right)$$

And the theorem is proven in all cases!

# Part II

# About the paper *On the Boltzmann Equation for Long-Range Interactions*

This work comes after the work of DiPerna and Lions about the compactness of a set of weak solutions of the Boltzmann equation with angular cutoff called renormalized solutions. Here Alexandre and Villani go further. They define the notion of renormalized solution with a defect measure (RSDM) which is weaker than the one defined by DiPerna and Lions and show compactness results with less assumptions than DiPerna and Lions did. In particular, their result holds for the non-cutoff case.

## 5 Definitions, assumptions and aim

As before, we shall write $f(t,x,v)$ and we work in dimension $N$.

$H(f) = \int_{\mathbb{R}^N_x \times \mathbb{R}^N_v} f \log f \mathrm{d}x\mathrm{d}v$ denotes the entropy.

And they define the nonnegative entropy dissipation

$$D(f) = \frac{1}{4} \int_{\mathbb{R}^N_v \times \mathbb{R}^N_{v_*} \times S^{N-1}_\sigma} B(v - v_*, \sigma)(f' f'_* - f f_*) \log \frac{f' f'_*}{f f_*} \mathrm{d}v \mathrm{d}v_* \mathrm{d}\sigma$$

### 5.1 Notion of RSDM

**Definition 5.1.** Let $f \in C(\mathbb{R}^+; \mathcal{D}'(\mathbb{R}^N_x \times \mathbb{R}^N_v)) \cap L^\infty(\mathbb{R}^+; L^1((1+|v|^2+|x|^2)\mathrm{d}x\mathrm{d}v))$. We say that $f$ is a renormalized solution to the Boltzmann equation with defect measure (RSDM) when for all $\beta \in C^2(\mathbb{R}^+, \mathbb{R}^+)$ non-linear such that :

$$\beta(0) = 0, 0 < \beta'(f) \leq \frac{C}{1+f}, \beta''(f) < 0 \tag{37}$$

the following inequality 38 in the sense of distributions and the mass conservation assumption 39 hold.

$$\frac{\partial \beta(f)}{\partial t} + v.\nabla_x \beta(f) \geq \beta'(f) Q(f,f) \tag{38}$$

$$\forall t \geq 0, \int_{\mathbb{R}^N \times \mathbb{R}^N} f(t,x,v)\mathrm{d}x\mathrm{d}v = \int_{\mathbb{R}^N \times \mathbb{R}^N} f(0,x,v)\mathrm{d}x\mathrm{d}v \tag{39}$$

For more lisibility, we set $D_f = C(\mathbb{R}^+; \mathcal{D}'(\mathbb{R}^N_x \times \mathbb{R}^N_v)) \cap L^\infty(\mathbb{R}^+; L^1((1+|v|^2+|x|^2)\mathrm{d}x\mathrm{d}v))$.

*Remark* 5.1. The "defect measure" corresponds to the difference between the two sides of the inequality 38.

A notion of weak solutions has been defined here, because the inequality 38 holds in the sense of distributions.

### 5.2 Assumptions on the cross-section $B$

As in the previous paper, we assume that $B(v - v_*, \sigma)$ depends only on $\cos\theta = \frac{v-v_*}{|v-v_*|}.\sigma$ and is nonnegative, a.e. finite and supported in $\theta \in [0\,;\frac{\pi}{2}]$.

As before, we denote $k := \frac{v-v_*}{|v-v_*|}$

They do three assumptions on the behavior of $B$.

#### 5.2.1 At most borderline singularity

$B$ is assumed to be of the following shape :

$$B(z,\sigma) = \frac{\beta_0(k.\sigma)}{|z|^N} + B_1(z,\sigma) \tag{40}$$

where $\beta_0$ and $B_1$ are nonnegative measurable functions.

We also assume that

$$\mu_0 := \int_{S^{N-1}} \beta_0(k.\sigma)(1 - k.\sigma)\mathrm{d}\sigma < +\infty \tag{41}$$

$$M_1(|z|) := \int_{S^{N-1}} B_1(z,\sigma)(1-k.\sigma)\mathrm{d}\sigma \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \tag{42}$$

$$M_1'(|z|) := \int_{S^{N-1}} B_1'(z,\sigma)(1-k.\sigma)\mathrm{d}\sigma \in L^1_{\mathrm{loc}}(\mathbb{R}^N) \tag{43}$$

where $B_1' := \sup_{1<\lambda\leq\sqrt{2}} \frac{|B_1(\lambda z,\sigma)-B_1(z,\sigma)|}{(\lambda-1)|z|}$.

### 5.2.2 Behavior at infinity

For $\alpha \in [0\,;2]$, let $M^\alpha(|z|) := \int_{S^{N-1}} B(z,\sigma)(1-k.\sigma)^{\frac{\alpha}{2}}\mathrm{d}\sigma$. It is easy to see, as we did for the previous paper, that $M^\alpha(|z|)$ does not depend on the direction of $z$ but only of its norm.

We assume that for one $\alpha$,

$$M^\alpha(|z|) = \underset{|z|\to\infty}{o}(|z|^{2-\alpha}) \tag{44}$$

Defining $M'$ similarily to $M_1'$ in 43 with $B$ instead of $B_1$, we aslo assume that

$$|z|M'(|z|) = \underset{|z|\to\infty}{o}(|z|^2) \tag{45}$$

*Remark* 5.2. $M$, $M_1$ and their primed versions play the same role as $\Lambda$ in previous paper.

### 5.2.3 Angular singularity condition

As in the previous paper, they make the assumption that $B(z,\sigma) \geq \Phi_0(|z|)b_0(k.\sigma)$ where $\Phi_0$ is a continuous function that may be null only in $|z| = 0$.

We make the non-cutoff assumption, that is

$$\int_{S^{N-1}} b_0(k.\sigma)\mathrm{d}\sigma = +\infty \tag{46}$$

so that they can use the results from the previous paper.

## 5.3 Definitions for sequences of RSDM

As the aim is to show a compactness result, we will be working with sequences of RSDM for cross-section $(B^n)$.

### 5.3.1 $B^n$ approximates $B$

Remind the result from previous paper called cancellation lemma. We will show in section 6.1 that we have a similar result for a cross-section $B$ that satisfies the assumptions of section 5.2.1: $\int B(f_*' - f_*)\mathrm{d}v_*\mathrm{d}\sigma = f \star_v S$ a.e. where $S$ is a distribution. We denote $S^n$ the $S$-quantity associated to $B^n$.

For cross-section $B$ and fixed $v_*$, let

$$\mathcal{T} : \phi \mapsto \int_{S^{N-1}} B(\phi' - \phi)\mathrm{d}\sigma \tag{47}$$

Similarily, for $B^n$, we set $\mathcal{T}^n : \phi \mapsto \int_{S^{N-1}} B^n(\phi' - \phi)\mathrm{d}\sigma$.

**Definition 5.2.** We say that $B^n$ approximates $B$ ($B^n \to B$) when

- $\forall n$, $B^n$ satisfies the at most borderline singularity assumption (section 5.2.1),
- $S^n \rightharpoonup S$ locally in the weak-measure sense,
- $\mathcal{T}^n \rightharpoonup \mathcal{T}$,
- $B^n \to B$ a.e. and,
- we have the assumption of behaviour at infinity (section 5.2.2) uniformly in $n$.

### 5.3.2 Overall angular singularity condition

It is assumed that there exist $\Phi_0$ and $b_0^n$ as in section 5.2.3 such that

- $\forall n, B^n(z,\sigma) \geq \Phi_0(|z|) b_0^n(k.\sigma)$ and,
- $\int_{S^{N-1}} \varliminf_{n\to\infty} b_0^n(k.\sigma)\mathrm{d}\sigma = +\infty$.

## 5.4 Aim: a result of compactness and existence of weak solution

They prove the following theorem of stability and appearance of strong compactness.

**Theorem 5.1.** *Under assumptions of parts 5.2.1, 5.2.2 and 5.2.3, for $(f_n)$ a sequence of RSDM satisfying*

$$\sup_{n\in\mathbb{N}} \sup_{T\in[0;T]} \int_{\mathbb{R}^N\times\mathbb{R}^N} f^n(1+|v|^2+|x|^2+\log f^n)\mathrm{d}x\mathrm{d}v < +\infty \tag{48}$$

*which basically corresponds to the finite mass, energy, entropy assumptions, and the assumption on entropy dissipation*

$$\int_0^T \int_{\mathbb{R}_x^N} D(f^n(t,x,.))\mathrm{d}x\mathrm{d}T \leq H(f^n(0,.,.)) - H(f^n(T,.,.)) \tag{49}$$

*such that $f^n \rightharpoonup f$ weakly in $L^P([0\,;T], L^1(\mathbb{R}_x^N\times\mathbb{R}_v^N))$ for one $p\in[1\,;+\infty[$. Then,*

1. *$f$ is a RSDM,*
2. *$f^n \to f$ strongly.*

This theorem leads to the following corrollary which states the existence of RSDM to the Boltzmann equation.

**Corollary 5.1.** *Under the same assumptions as theorem 5.1, let $f_0$ be an initial datum such that*

$$\int_{\mathbb{R}_x^N\times\mathbb{R}_v^N} f_0(1+|v|^2+|x|^2+\log f_0)\mathrm{d}x\mathrm{d}v < +\infty \tag{50}$$

*Then there exist $f$ a RSDM with initial datum $f(0,.,.) = f_0$. Moreover, this solution satisfies $\forall t\in[0\,;T]$ :*

$$\int_{\mathbb{R}^{2N}} f(t,x,v)\mathrm{d}x\mathrm{d}v = \int_{\mathbb{R}^{2N}} f_0\mathrm{d}x\mathrm{d}v$$

$$\int_{\mathbb{R}^{2N}} f(t,x,v)v\mathrm{d}x\mathrm{d}v = \int_{\mathbb{R}^{2N}} f_0 v\mathrm{d}x\mathrm{d}v$$

$$\int_{\mathbb{R}^{2N}} f(t,x,v)\frac{|v|^2}{2}\mathrm{d}x\mathrm{d}v = \int_{\mathbb{R}^{2N}} f_0\frac{|v|^2}{2}\mathrm{d}x\mathrm{d}v$$

$$H(f(T,.,.)) + \int_0^T \int_{\mathbb{R}^N} D(f(t,x,.))\mathrm{d}x\mathrm{d}t \leq H(f_0)$$

Actually, theorem 5.1 is a consequence of the more general result:

**Theorem 5.2.** *Let $B$ be a cross section satisfying assumptions of part 5.2 and $(B^n)$ be a sequence of cross sections such that $B^n \to B$ and that satisfies assumption of part 5.3.2.*

*Let $(f^n)$ be a sequence of RSDM with cross section $(B^n)$ such that:*

1. $\sup_{n\in\mathbb{N}} \sup_{t\in[0;T]} \int f^n(1+|v|^2+|x|^2+\log f^n)\mathrm{d}x\mathrm{d}v < +\infty$,
2. $\forall n\in\mathbb{N}, \int_0^T \int D^n(f^n(t,x,\cdot)) \leq H^n(f_0^n) - H^n(f^n(T,\cdot,\cdot))$,
3. *$f^n \to f$ in $w-L^P([0\,;T], L^1(\mathbb{R}_x^N\times\mathbb{R}_v^N))$ for all $1\leq p<+\infty$.*

*Then,*

1. *$f^n \to f$ strongly,*

2. $f$ *is a RSDM with cross section* $B$,
3. $\forall t \in [0\,;T]$, $\int f(t,x,v)\mathrm{d}x\mathrm{d}v = \int f_0(x,v)\mathrm{d}x\mathrm{d}v$ *(mass conservation),*
4. $\forall t \in [0\,;T]$, $\int f(t,x,v)v\mathrm{d}x\mathrm{d}v = \int f_0(x,v)v\mathrm{d}x\mathrm{d}v$ *(momentum conservation),*
5. $\forall t \in [0\,;T]$, $\int f(t,x,v)\frac{|v|^2}{2}\mathrm{d}x\mathrm{d}v \leq \int f_0(x,v)\frac{|v|^2}{2}\mathrm{d}x\mathrm{d}v$ *(energy dissipation),*
6. $H(f(T,\cdot,\cdot)) + \int_0^T \int D(f(t,x,\cdot))\mathrm{d}x\mathrm{d}t \leq H(f_0)$ *(entropy inequality).*

# 6 An useful decomposition of $\beta'(f)Q(f,f)$

Lets consider $\beta$ satisfying 37. They introduce

$$\Gamma(f,f') := \beta(f') - \beta(f) - \beta'(f)(f'-f) \tag{51}$$

Using Taylor's formula, we know that there exists $\xi \in ]f\,;f'[$ or $]f'\,;f[$ such that $\Gamma(f,f') = \frac{1}{2}(f'-f)^2\beta''(\xi) < 0$ for $f \neq f'$.

As suggered by inequation 38, lets compute $\beta'(f)Q(f,f)$.

$$\begin{aligned}
\beta'(f)Q(f,f) &= \int B\beta'(f)(f'f'_* - ff_*)\mathrm{d}v_*\mathrm{d}\sigma \\
&= \int B[\beta'(f)f'_*(f'-f) + \beta'(f)f(f'_*-f_*)]\mathrm{d}v_*\mathrm{d}\sigma \\
&= \int B[f'_*(\beta(f') - \beta(f) - \Gamma(f,f')) + \beta'(f)f(f'_*-f_*)]\mathrm{d}v_*\mathrm{d}\sigma \\
&= \int B(f'_*-f_*)(f\beta'(f)-\beta(f)) + B(\beta(f)(f'_*-f_*) + f'_*(\beta(f')-\beta(f)))\mathrm{d}v_*\mathrm{d}\sigma - \int Bf'_*\Gamma(f,f')\mathrm{d}v_*\mathrm{d}\sigma \\
&= \int B(f\beta'(f)-\beta(f))(f'_*-f_*)\mathrm{d}v_*\mathrm{d}\sigma + \int B(f'_*\beta(f') - f_*\beta(f))\mathrm{d}v_*\mathrm{d}\sigma - \int Bf'_*\Gamma(f,f')\mathrm{d}v_*\mathrm{d}\sigma \\
&= R_1 + R_2 + R_3
\end{aligned}$$

For the moment, it is not proven that $R_1$, $R_2$ and $R_3$ make sense. This is the point of sections 6.1, 6.2 and 6.3 respectively. It will be shown that they make sense for

$$f \in \mathcal{D}'([0\,;T]\times\mathbb{R}^N_x\times\mathbb{R}^N_v) \tag{52}$$

and such that

$$\sup_{t\in[0;T]}\int f(t,x,v)(1+|v|^2)\mathrm{d}x\mathrm{d}v < +\infty \tag{53}$$

## 6.1 Integrability of $R_1$: use of the cancellation lemma

Recall $R_1 = (f\beta'(f)-\beta(f))\int B(f'_*-f_*)\mathrm{d}v_*\mathrm{d}\sigma$. Thus, we have to prove that $\int B(f'_*-f_*)\mathrm{d}v_*\mathrm{d}\sigma$ converges.

We want to apply the cancellation lemma from previous paper and get : $\int B(f'_*-f_*)\mathrm{d}v_*\mathrm{d}\sigma = f\star_v S$ almost everywhere with

$$S(|z|) = |S^{N-2}|\int_0^{\frac{\pi}{2}} \sin^{N-2}\theta\left(\frac{1}{\cos^N\frac{\theta}{2}}B\left(\frac{|z|}{\cos\frac{\theta}{2}},\cos\theta\right) - B(|z|,\cos\theta)\right)\mathrm{d}\theta \tag{54}$$

We consider that $B$ satisfy assumption of part 5.2.1: $B(z,\sigma) = \frac{\beta_0(k.\sigma)}{|z|^N} + B_1(z,\sigma)$. With obvious notations, we have $S(z) = S_0(z) + S_1(z)$. Thus, we need to compute $S_0$ and show that the formula given in first paper does make sense for $B_0(z,\sigma) = \frac{\beta_0(\cos\theta)}{|z|^N}$: $S_0(z) = |S^{N-2}|\int_0^{\frac{\pi i}{2}} \sin^{N-2}\theta\left[\frac{1}{\cos^N\frac{\theta}{2}}B_0\left(\frac{|z|}{\cos^N\frac{\theta}{2}},\cos\theta\right) - B_0(|z|,\cos\theta)\right]\mathrm{d}\theta$. Actually, $S_0 = \lim_{\varepsilon\to 0^+} S_\varepsilon$ for $S_\varepsilon$ obtained with the same formula using $B_\varepsilon(z,\sigma) := B_0(z,\sigma)\mathbf{1}_{|z|\geq\varepsilon}$ instead of $B_0(z,\sigma)$.

$$
\begin{aligned}
S_\varepsilon(z) &= |S^{N-2}| \int_0^{\frac{\pi}{2}} \sin^{N-2}\theta \left[ \frac{1}{\cos^N \frac{\theta}{2}} \beta_0(\cos\theta) \frac{\cos^N \frac{\theta}{2}}{|z|^N} \mathbf{1}_{\frac{|z|}{\cos\frac{\theta}{2}} \geq \varepsilon} - \beta_0(\cos\theta) \frac{1}{|z|^N} \mathbf{1}_{|z|\geq\varepsilon} \right] \mathrm{d}\theta \\
&= \frac{|S^{N-2}|}{|z|^N} \int_0^{\frac{\pi}{2}} \sin^{N-2}\theta \beta_0(\cos\theta) (\mathbf{1}_{|z|\geq\varepsilon\cos\frac{\theta}{2}} - \mathbf{1}_{|z|\geq\varepsilon}) \mathrm{d}\theta \\
&= \frac{|S^{N-2}|}{|z|^N} \int_0^{\frac{\pi}{2}} \sin^{N-2}\theta \beta_0(\cos\theta) \mathbf{1}_{\varepsilon\cos\frac{\theta}{2}\leq|z|<\varepsilon} \mathrm{d}\theta \\
&= \frac{|S^{N-2}|}{\varepsilon^N} \left(\frac{|z|}{\varepsilon}\right)^{-N} \mathbf{1}_{\frac{|z|}{\varepsilon}<1} \int_{2\cos^{-1}\frac{|z|}{\varepsilon}}^{\frac{\pi}{2}} \sin^{N-2}\theta \beta_0(\cos\theta) \mathrm{d}\theta \\
&= \varepsilon^{-N} J\left(\frac{|z|}{\varepsilon}\right)
\end{aligned}
$$

with obvious notations. From this, we get that $\int_z S_\varepsilon(z)\mathrm{d}z$ does not depend on $\varepsilon$. Moreover, $S_\varepsilon$ is supported in $\mathcal{B}(0,\varepsilon)$. As a consequence, $S_0(z) = \int_z J(z)\mathrm{d}z\delta_0(z)$ where $\delta_0$ is the Dirac distribution in 0.

Yet,

$$
\begin{aligned}
\int J(z)\mathrm{d}z &= |S^{N-2}| \int_z \int_{2\cos^{-1}|z|} \sin^{N-2}\theta \beta_0(\cos\theta)\mathrm{d}\theta\mathrm{d}z \\
&= |S^{N-2}||S^{N-1}| \int_0^1 \frac{1}{r} \int_{2\cos^{-1} r}^{\frac{\pi}{2}} \sin^{N-2}\theta \beta_0(\cos\theta)\mathrm{d}\theta\mathrm{d}r \\
&= |S^{N-2}||S^{N-1}| \int_0^{\frac{\pi}{2}} \int_{\cos\frac{\theta}{2}}^1 \frac{\mathrm{d}r}{r} \sin^{N-2}\theta \beta_0(\cos\theta)\mathrm{d}\theta \\
&= |S^{N-2}||S^{N-1}| \int_0^{\frac{\pi}{2}} \ln\left(\cos\frac{\theta}{2}\right) \sin^{N-2}\theta \beta_0(\cos\theta)\mathrm{d}\theta
\end{aligned}
$$

To conclude,

$$
S_0(z) = -|S^{N-2}||S^{N-1}| \int_0^{\frac{\pi}{2}} \ln\left(\cos\frac{\theta}{2}\right) \sin^{N-2}\theta \beta_0(\cos\theta)\mathrm{d}\theta\delta_0(z) = \lambda\delta_0(z) \tag{55}
$$

Still, we need to show that $\lambda \neq \infty$ which is not difficult noticing that $-\log u \leq \frac{1}{u} - 1$:

$$
\begin{aligned}
0 \leq \lambda &\lesssim \int_0^{\frac{\pi}{2}} \left(\frac{1}{\cos\frac{\theta}{2}} - 1\right) \beta_0(\cos\theta) \sin^{N-2}\theta \mathrm{d}\theta \\
&\lesssim \int_0^{\frac{\pi}{2}} \frac{1}{2\cos\frac{\theta}{2}} \left(2 - 2\cos^2\frac{\theta}{2}\right) \beta_0(\cos\theta) \sin^{N-2}\theta \mathrm{d}\theta \\
&\lesssim \int_0^{\frac{\pi}{2}} (1-\cos\theta) \beta_0(\cos\theta) \sin^{N-2}\theta \mathrm{d}\theta \\
&\lesssim \mu_0 < \infty
\end{aligned}
$$

thanks to the at most borderline singularity assumption (equation 41).

Moreover, as in previous paper, we have

$$
|S_1(z)| \leq C_N \left(M_1(|z|) + |z| M_1'(|z|)\right) < +\infty \tag{56}
$$

because of the same assumption (equations 42 and 43).

$f\beta'(f) - \beta(f) = f\beta'(f) - (\beta(f) - \beta(0)) = -f^2\beta''(\xi_f f) > 0$ for some $\xi_f \in ]0\,;1[$. So it's in $L^\infty([0\,;T];L^1(\mathbb{R}^{2N}))$ as $f$ satisfies 52 and $\beta''$ is continuous.

Remains to prove that $f \star_v S$ makes sense. I will do it my own way.

First, let's notice that $M^\alpha$ as defined in section 5.2.2 is such that $M \leq M^\alpha$ where $M = \int B(1 - k.\sigma)\mathrm{d}\sigma$. And, obviously $M_1 \leq M$. We also have $M_1' \leq M'$. Remind that assumption of section 5.2.2 gives us $M^\alpha(|z|) + |z|M'(|z|) = |z|^{2-\alpha}\epsilon_1(|z|) + |z|^2\epsilon_2(|z|) \leq |z|^2\epsilon(|z|)$ where $\epsilon(|z|) \to 0$ when $|z| \to \infty$. We can choose $\epsilon$ decreasing.

Then, for $R > 0$,

$$\begin{aligned}
\|f \star_v S\|_{L^1(\mathbb{R}^N_x \times \mathcal{B}_v(0,R))} &= \|f \star_v (\lambda\delta_0 + S_1)\|_{L^1} \\
&= \|\lambda f + f \star_v S_1\|_{L^1} \\
&\leq \lambda\|f\|_{L^1(\mathbb{R}^{2N})} + \|f \star_v S_1\|_{L^1(\mathbb{R}^N_x \times \mathcal{B}_v(0,R))} \\
&\leq \|f\|_{L^1(\mathbb{R}^{2N})}(\lambda + C_N \sup_{|z|\leq R}(M_1(|z|) + |z|M_1'(|z|))) \\
&\leq \|f\|_{L^1}(\lambda + C_N \sup_{|z|<R}|z|^2\epsilon(|z|)) \\
&\leq \|f\|_{L^1}(\lambda + C_N\epsilon(0)R^2)
\end{aligned}$$

Taking the sup for $T \in [0\,;T]$ on both side of this inequality, we can conclude that $R_1$ is well-defined almost everywhere and is a in $L^\infty([0\,;T]; L^1(\mathbb{R}^N_x \times \mathcal{B}_v(0,R)))$ for any $R > 0$.

## 6.2 Integrability of $R_2$

First, notice that $R_2 = Q(f,\beta(f))$. We will define it as a distribution and then show that it is actually an element of $L^\infty([0\,;T]; L^1(\mathbb{R}^N_x; W^{-2,1}(\mathcal{B}_v(0,R))))$ for any $R < 0$.

Let $\phi$ be a test function. Using the pre-postcollisional change of variables in the first term of $\int R_2\phi(v)\mathrm{d}v = \int B(f'_*\beta(f') - f_*\beta(f))\phi\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v$, we get:

$$\int R_2\phi(v)\mathrm{d}v = \int\left(\int B(\phi' - \phi)\mathrm{d}\sigma\right) f_*\beta(f)\mathrm{d}v_*\mathrm{d}v = \int \mathcal{T}\phi f_*\beta(f)\mathrm{d}v_*\mathrm{d}v$$

where $\mathcal{T}$ is defined by equation 47.

$\mathcal{T}$ is to be seen as a linear operator. We will now show that it's bounded as a linear operator of $W^{2,\infty} \to L^\infty$ when $B$ satisfies assumptions of part 5.2.1.

**Proposition 6.1.** $\forall\phi \in W^{2,\infty}(\mathbb{R}^N_v)$,

$$|\mathcal{T}\phi(v)| \leq \frac{1}{2}\|\phi\|_{W^{2,\infty}}|v - v_*|\left(1 + \frac{|v - v_*|}{2}\right)M(|v - v_*|) \tag{57}$$

*And, for all* $\alpha \in [0\,;2]$, $\forall\phi \in W^{2,\infty}(\mathbb{R}^N_v)$,

$$|\mathcal{T}\phi(v)| \leq 2\|\phi\|_{W^{2,\infty}}(1 + |v - v_*|)^{\alpha/2}M^\alpha(|v - v_*|) \tag{58}$$

The proof of this result is given in the paper. It uses Taylor's decomposition of $\phi' - \phi$ in the formula for $\mathcal{T}\phi$. And the fact that $I = \int B(v - v_*, \sigma)(v' - v)\mathrm{d}\sigma$ does not depend on the direction of $v' - v$ but only of its norm. Thus, we have $I = \int B(v - v_*, \sigma)(v'v.k)k\mathrm{d}\sigma$.

From this proposition, they deduce, when $B$ satisfies assumptions of part 5.2.2 and $f \in D_f$, that $R_2$ is well-defined and is in $L^\infty([0\,;T]; L^1(\mathbb{R}^N_x; W^{-2,1}(\mathcal{B}_v(0,R))))$.

They first bound $\|R_2\|_{W_v^{-2,1}(\mathcal{B}_v(0,R))} = \sup\left\{R_2\phi\mathrm{d}v, \phi \in (W^{-2,1})^* = W^{2,\infty}(\mathcal{B}_v(0,R)), \|phi\|_{W^{2,\infty}} \leq 1\right\}$ and then integrate the obtained bound w.r.t. $x$.

## 6.3 Integrability of $R_3$

They assume that $f$ is a RSDM with cross-sections $B$ satisfying assumptions of parts 5.2.1 and 5.2.2. They show that $R_3 = -\int Bf'_*\Gamma(f,f')\mathrm{d}v_*\mathrm{d}\sigma \in L^1([0\,;T]; L^1(\mathbb{R}^N_x \times \mathcal{B}_v(0,R)))$.

Noticing that the integrand in $R_3$ is nonnegative, they can rewrite the fact that $f$ is a RSDM as:

$$\frac{\partial\beta(f)}{\partial t} + v.\nabla_x\beta(f) \geq R_1 + R_2 + R_3$$

which gives for $\phi(v) \in W^{,\infty}$ a nonnegative compact-supported test function such that $\phi = 1$ in $\mathcal{B}_v(0,R)$ and $\phi = 0$ outside of $\mathcal{B}_v(0,2R)$ and $\|\phi\|_{W^{2\infty}} \leq CR^{-2}$:

$$\int\frac{\partial\beta(f)}{\partial t}\phi(v)\mathrm{d}t\mathrm{d}x\mathrm{d}v + \int v.\nabla_x\beta(f)\phi(v)\mathrm{d}t\mathrm{d}x\mathrm{d}v \geq \int R_1\phi(v)\mathrm{d}t\mathrm{d}x\mathrm{d}v + \int R_2\phi(v)\mathrm{d}t\mathrm{d}x\mathrm{d}v + \int R_3\phi(v)\mathrm{d}t\mathrm{d}x\mathrm{d}v$$

$$\implies \int[\beta(f(.,x,v))]_0^T\phi(v)\mathrm{d}x\mathrm{d}v + 0 \text{ (by integration by parts w.r.t. } v) \geq \text{some bounded things} + \int_0^T\int_{\mathbb{R}^N_x}\int_{\mathcal{B}_v(0,R)} R_3\mathrm{d}v\mathrm{d}x\mathrm{d}t$$

Remains to deal with the first term of this inequality.

$$\begin{aligned}
\int [\beta(f(.,x,v))]_0^T \phi(v)\mathrm{d}x\mathrm{d}v &\leq \int \beta(f(T,x,v))\phi v\mathrm{d}x\mathrm{d}v \\
&\leq \int_{\mathbb{R}_x^N \times \mathcal{B}_v(0,2R)} \beta(f(T,x,v))\mathrm{d}x\mathrm{d}v \\
&\leq C \int_{\mathbb{R}_x^N \times \mathcal{B}_v(0,2R)} f(T,x,v)\mathrm{d}x\mathrm{d}v \qquad \text{thanks to assumption 37 on } \beta \\
&\leq C \int_{\mathbb{R}_x^N \times \mathbb{R}_v^N} f(T,x,v)\mathrm{d}x\mathrm{d}v \\
&\leq C\|f_0\|_{L^1(\mathbb{R}_x^N \times \mathbb{R}_v^N)} \qquad \text{by mass conservation (equation 39)}
\end{aligned}$$

With this decompotisition (that makes sense) in hands, we can now get to the point and show theorem 5.2. I will only give the idea of the proof, which is not entirely given in the paper.

# 7 How to get strong compactness appearance ?

For more clearness, here is the theorem we are about to prove:

Let $B$ be a cross section satisfying assumptions of part 5.2 and $(B^n)$ be a sequence of cross sections such that $B^n \to B$ and that satisfies assumption of part 5.3.2.

Let $(f^n)$ be a sequence of RSDM with cross section $(B^n)$ such that:

1. $\sup\limits_{n\in\mathbb{N}} \sup\limits_{t\in[0;T]} \int f^n(1+|v|^2+|x|^2+\log f^n)\mathrm{d}x\mathrm{d}v < +\infty$,
2. $\forall n \in \mathbb{N}, \int_0^T \int D^n(f^n(t,x,\cdot)) \leq H^n(f_0^n) - H^n(f^n(T,\cdot,\cdot))$,
3. $f^n \to f$ in $w - L^P([0\,;T], L^1(\mathbb{R}_x^N \times \mathbb{R}_v^N))$ for all $1 \leq p < +\infty$.

Then,

1. $f^n \to f$ strongly,
2. $f$ is a RSDM with cross section $B$,
3. $\forall t \in [0\,;T], \int f(t,x,v)\mathrm{d}x\mathrm{d}v = \int f_0(x,v)\mathrm{d}x\mathrm{d}v$ (mass conservation),
4. $\forall t \in [0\,;T], \int f(t,x,v)v\mathrm{d}x\mathrm{d}v = \int f_0(x,v)v\mathrm{d}x\mathrm{d}v$ (momentum conservation),
5. $\forall t \in [0\,;T], \int f(t,x,v)\frac{|v|^2}{2}\mathrm{d}x\mathrm{d}v \leq \int f_0(x,v)\frac{|v|^2}{2}\mathrm{d}x\mathrm{d}v$ (energy dissipation),
6. $H(f(T,\cdot,\cdot)) + \int_0^T \int D(f(t,x,\cdot))\mathrm{d}x\mathrm{d}t \leq H(f_0)$ (entropy inequality).

## 7.1 Strong compactness for convolution products

In this section, I will present *Step 1* of the paper. I won't insist on this part because it's mostly the use of DiPerna and Lions' results. I didn't read their work.

They use the averaging lemmas given by DiPerna and Lions to show that exctracting a subsequence $\int \beta(f^n)\phi(v)\mathrm{d}v \to \int \beta(f)\phi(v)\mathrm{d}v$ strongly with respect to $(t,x)$ for any smooth compact-supported test function $\phi$.

Actually, using also the bounds on $f^n$, they can get strong compactness of $\beta(f^n) \star_v \phi$ in $L^1([0\,;T]\times\mathbb{R}_x^N \times \mathbb{R}_v^N)$.

Using this and other analysis arguments, they obtain $L^1$ strong compacity for $\sqrt{f^n} \star_v \phi$.

To finish, they also show that $\int f^n \mathrm{d}v$ is strongly compact w.r.t. $(t,x)$. The limit can only be $\int f\mathrm{d}v$.

## 7.2 How to get strong compactness for $f^n$?

In this section, I will present *Step 2, 3* and *4* of the paper. The point is to show that $(f^n)$ is strongly compact in $L^1$. To get this result, we will fisrt show that $(\sqrt{f^n})$ is strongly compact in $L^2$ by truncating it.

### 7.2.1 Truncation: velocity smoothness

Here they use the truncation results from previous paper. They denote $f_R^n = f^n \chi_R$ where $\chi_R$ is smooth, supported in $\mathcal{B}_v(0, 3R)$ and such that $\chi_R = 1$ in $\mathcal{B}_v(0, 2R)$.

Let $Z_n(a) = |S^{N-2}| \int_a^{\frac{\pi}{2}} b_0^n(\cos\theta) \sin^{N-2}\theta \mathrm{d}\theta$.

The result they use is the following.

**Theorem 7.1.** *$\forall L, \varepsilon \in ]0\,;+\infty[$, for almost every $(t, x)$ such that:*

- *$H^n(t, x) \leq L$ and*
- *$\rho^n(t, x) \geq \varepsilon$,*

*we have pointwisely with $C(f^n, R, \Phi_0)$ depending only on $R$, $\Phi_0$, $L$ and $\varepsilon$,*

$$\int_{|\xi|\geq 1} |\mathcal{F}\sqrt{f_R^n}|^2 Z_n\left(\frac{1}{|\xi|}\right)\mathrm{d}\xi \leq C(f^n, R, \Phi_0)(D(f^n) + \|f^n\|_{L^1_2}^2) \tag{59}$$

Let us denote $g \in L^2$ the *weak*-limit of $\sqrt{f^n}$. It exists. We now want to prove that the convergence is actually *strong* in $L^2([0\,;T] \times \mathcal{B}_x(0, R) \times \mathcal{B}_v(0, R))$.

We will actually show that the convergence is strong almost everywhere in every $W_\varepsilon \times \mathcal{B}_v(0, R)$ where $W_\varepsilon = \left\{(t, x) \in [0\,;T] \times B_R^N, \int F^2 \mathrm{d}v > \varepsilon\right\}$ for $\varepsilon > 0$.

So, for fixed $\varepsilon > 0$, we have, by convexity of $f \to f^2$,

$$\varepsilon < \int g^2 \mathrm{d}v \leq \varliminf_{n\to\infty} \int \sqrt{f^n}^2 \mathrm{d}v = \varliminf_{n\to\infty} \int f^n \mathrm{d}v = \int f \mathrm{d}v$$

using the strong convergence of $\int f^n \mathrm{d}v$ to $\int f \mathrm{d}v$ obtained previously in part 7.1.

Remind *Egorov's theorem*:

**Theorem 7.2** (Egorov)**.** *Let $(M, d)$ denote a separable metric space. Given a sequence $(f^n)$ of $M$-valued measurable functions on some measure space $(X, \Sigma, \mu)$, and a measurable subset $A$ of finite $\mu$-measure such that $(f^n)$ converges $\mu$-almost everywhere on $A$ to a limit function $f$, the following result holds: for every $\delta > 0$, there exists a measurable subset $B$ of $A$ such that $\mu(B) < \delta$, and $(f^n)$ converges to $f$ uniformly on the relative complement $A\backslash B$.*

In this case, we get for every $\delta > 0$ the existance of $U_\delta$ a Borel set such that $\mu(U_\delta) \leq \delta$ and $\int f^n \mathrm{d}v \to \int f \mathrm{d}v$ uniformly on $W_\varepsilon \backslash U_\delta$. As a consequence, for $n$ large enough, $\int f \mathrm{d}v \geq \frac{\varepsilon}{2}$ on $W_\varepsilon \backslash U_\delta$.

Now, they introduce $V_L^n(t, x) : H^n(t, x) > L$. Remind that $\int\limits_{0\leq t\leq T} H^n(t, x)\mathrm{d}x < +\infty$.

Let us prove that there exists $C$ independent of $n$ such that $L|V_L^n| \leq C$. First of all we know that $\sup\limits_n \sup\limits_t \int H^n(t, x)\mathrm{d}x \leq D < +\infty$. This, prove that $\exists R_L > 0, \forall n, \forall t, H^n(t, x) > L \implies x \in \mathcal{B}_x(0, R_L)$. Then, $|V_L^n| \leq C_0 T R_L^N$ with $C_0$ independant of $n$ and $T$. And we can choose $R_L^N \leq \frac{1}{L}$ otherwise the integrale would not converge, thanks to Riemann's criterion.

On $W_\varepsilon \backslash (U_\delta \cup V_L^n)$, we have:

- $\int f^n \mathrm{d}v \geq \frac{\varepsilon}{2}$ for $n$ large enough and
- $H^n(t, x) \leq L$.

So, we can apply theorem 7.1:

$$\int_{|\xi|\geq 1} |\mathcal{F}\sqrt{f_R^n}|^2 Z_n\left(\frac{1}{|\xi|}\right)\mathrm{d}\xi \leq C(R, \Phi_0, L, \varepsilon)(D^n(f^n) + \|f^n\|_{L^1_2}^2)$$

Then, $Z_n$ being decreasing, for $A \geq 1$, we have:

$$\begin{aligned} Z_n\left(\frac{1}{A}\right)\int_{|\xi|\geq A} |\mathcal{F}\sqrt{f_R^n}|^2 \mathrm{d}\xi &\leq \int_{|\xi|\geq A} |\mathcal{F}\sqrt{f_R^n}|^2 Z_n\left(\frac{1}{|\xi|}\right)\mathrm{d}\xi \\ &\leq \int_{|\xi|\geq 1} |\mathcal{F}\sqrt{f_R^n}|^2 Z_n\left(\frac{1}{|\xi|}\right)\mathrm{d}\xi \\ &\leq C(R, \Phi_0, L, \varepsilon)(D^n(f^n) + \|f^n\|_{L^1_2}^2) \end{aligned}$$

Then,

$$\begin{aligned}\int_{W_\varepsilon\setminus(U_\delta\cup V_L^n)}\int_{|\xi|\geq A}|\mathcal{F}\sqrt{f_R^n}|^2\mathrm{d}\xi\mathrm{d}x\mathrm{d}v &\leq \frac{C(R,\Phi_0,L,\varepsilon)}{Z_n(1/A)}\int_{[0;T]\times\mathbb{R}_x^N} D^n(f^n)+\|f^n\|_{L_2^1}^2\mathrm{d}x\mathrm{d}t\\ &\leq \frac{C(R,\Phi_0,L,\varepsilon)}{Z_n(1/A)}T\left(\sup_{t,n}\int f^n(1+|v|^2)\mathrm{d}v\mathrm{d}x+\sup_n H^n(f_0^n)\right)\\ &\leq \frac{C'}{Z_n(1/A)}\end{aligned}$$

where $C'$ does not depend on $n$. Here we have used the first and the second assumptions on the sequence $(f^n)$ in theorem 5.2.

Now, let us pass to the lim sup for $n\to\infty$.

$$\overline{\lim_{n\to\infty}}\int_{W_\varepsilon\setminus(U_\delta\cup V_L^n)}\int_{|\xi|\geq A}|\mathcal{F}\sqrt{f_R^n}|^2\mathrm{d}\xi\mathrm{d}x\mathrm{d}v\leq\frac{C'}{\underline{\lim}_{n\to\infty}Z_n(1/A)}=\frac{C'}{Z_\infty(1/A)}$$

with

$$\begin{aligned}Z_\infty(1/A)&=\underline{\lim_{n\to\infty}}|S^{N-2}|\int_{\frac{1}{A}}^{\frac{pi}{2}}b_0^n(\cos\theta)\sin^{N-2}\theta\mathrm{d}\theta\\ &\geq|S^{N-2}|\int_{\frac{1}{A}}^{\frac{pi}{2}}\underline{\lim_{n\to\infty}}b_0^n(\cos\theta)\sin^{N-2}\theta\mathrm{d}\theta\end{aligned}$$

Now, having the *overall singularity assumption* (part 5.3.2), we know that $Z_n(1/A)\xrightarrow{A\to+\infty}+\infty$. Thus,

$$\forall L,\varepsilon,\delta,\ \overline{\lim_{A\to\infty}}\overline{\lim_{n\to\infty}}\int_{W_\varepsilon\setminus(U_\delta\cup V_L^n)}\int_{|\xi|\geq A}|\mathcal{F}\sqrt{f_R^n}|^2\mathrm{d}\xi\mathrm{d}x\mathrm{d}v=0$$

Now, reminding that $|U_\delta\cup V_L^n|\leq\delta+\frac{C}{L}$, and $|\mathcal{F}\sqrt{f_R^n}|^2$ being uniformly equi-integrable, we have

$$\limsup_{\substack{\delta\to0\\L\to\infty}}\int_{U_\delta\cup V_L^n}\int|\mathcal{F}\sqrt{f_R^n}|^2\mathrm{d}\xi\mathrm{d}x\mathrm{d}v=0$$

Combining those two results, we get:

$$\boxed{\overline{\lim_{A\to\infty}}\overline{\lim_{n\to\infty}}\int_{W_\varepsilon}\int_{|\xi|\geq A}|\mathcal{F}\sqrt{f_R^n}|^2\mathrm{d}\xi\mathrm{d}x\mathrm{d}v=0}\tag{60}$$

### 7.2.2 Strong compactness of $\sqrt{f^n}$ in $L^2$

The idea consists in using the convolution with mollifiers. Let $\rho$ be a smooth, nonnegative, compactly supported fonction such that $\int\rho=1$. We define $\rho_\delta=\delta^{-N}\rho(\bullet/\delta)$. It is also smooth, nonnegative and compactly supported. Moreover, $\int\rho_\delta=1$. Its support it smaller and smaller when $\delta\to0$.

**Proposition 7.1.** $\overline{\lim_{\delta\to0}}\overline{\lim_{n\to\infty}}\left\|\sqrt{f^n}-\sqrt{f^n}\star_v\rho_\delta\right\|_{L^2(W_\varepsilon\times\mathcal{B}_v(0,R))}=0$

*Remark* 7.1. From proposition 7.1, we get that $(\sqrt{f^n})$ is strongly compact in $L^2$ using the results of section 7.1.

*Proof.* First of all, using the truncation results, we know that proposition 7.1 is the same as:

$$\overline{\lim_{\delta\to0}}\overline{\lim_{n\to\infty}}\left\|\sqrt{f_R^n}-\sqrt{f_R^n}\star_v\rho_\delta\right\|^2_{L^2(W_\varepsilon\times\mathcal{B}_v(0,R))}=0\tag{61}$$

First of all, using Parseval identity, we easily get:

$$\begin{aligned}\left\|\sqrt{f_R^n}-\sqrt{f_R^n}\star_v\rho_\delta\right\|^2_{L^2(W_\varepsilon\times\mathcal{B}_v(0,R))}&\leq\int_{W_\varepsilon}\int|\mathcal{F}(\sqrt{f_R^n})|^2|1-\mathcal{F}\rho_\delta|^2\mathrm{d}\xi\mathrm{d}x\mathrm{d}t\\ &\leq\int_{W_\varepsilon}\int|\mathcal{F}(\sqrt{f_R^n})|^2|\mathcal{F}(\delta_0-\rho_\delta)|^2\mathrm{d}\xi\mathrm{d}x\mathrm{d}t\end{aligned}$$

Then,

$$\begin{aligned}
\varlimsup_{\delta\to 0}\varlimsup_{n\to\infty}\left\|\sqrt{f_R^n}-\sqrt{f_R^n}\star_v\rho_\delta\right\|^2_{L^2(W_\varepsilon\times\mathcal{B}_v(0,R))} &\leq \varlimsup_{\delta\to 0}\varlimsup_{n\to\infty}\int_{W_\varepsilon}\int|\mathcal{F}(\sqrt{f_R^n})|^2|\mathcal{F}(\delta_0-\rho_\delta)|^2\mathrm{d}\xi\mathrm{d}x\mathrm{d}t\\
&\leq \lim_{\delta\to 0}\lim_{A\to\infty}\varlimsup_{n\to\infty}\int_{W_\varepsilon}\int_{|\xi|\geq A}|\mathcal{F}(\sqrt{f_R^n})|^2|\mathcal{F}(\delta_0-\rho_\delta)|^2\mathrm{d}\xi\mathrm{d}x\mathrm{d}t\\
&\quad+\lim_{\delta\to 0}\lim_{A\to\infty}\varlimsup_{n\to\infty}\int_{W_\varepsilon}\int_{|xi|\leq A}|\mathcal{F}(\sqrt{f_R^n})|^2|\mathcal{F}(\delta_0-\rho_\delta)|^2\mathrm{d}\xi\mathrm{d}x\mathrm{d}t\\
&\leq \lim_{\delta\to 0}\limsup_{A\to\infty}\sup_{|\xi|\geq A}|1-\mathcal{F}\rho_\delta|^2\varlimsup_{A\to\infty}\varlimsup_{n\to\infty}\int_{W_\varepsilon}\int_{|\xi|\geq A}|\mathcal{F}(\sqrt{f_R^n})|^2\mathrm{d}\xi\mathrm{d}x\mathrm{d}t\\
&\quad+\varlimsup_{A\to\infty}\varlimsup_{n\to\infty}\sup_{|\xi|\leq A}\int_{W_\varepsilon}|\mathcal{F}(\sqrt{f_R^n})|^2\mathrm{d}x\mathrm{d}t\lim_{\delta\to 0}\lim_{A\to\infty}\int_{|\xi|\leq A}|\mathcal{F}(\delta_0-\rho_\delta)|^2\mathrm{d}\xi
\end{aligned}$$

Using the fact that $\rho_\delta \longrightarrow \delta \to 0\delta_0$ and equation 60, and that the two sup factors are bounded, we can conclude that:

$$\varlimsup_{\delta\to 0}\varlimsup_{n\to\infty}\left\|\sqrt{f_R^n}-\sqrt{f_R^n}\star_v\rho_\delta\right\|^2_{L^2(W_\varepsilon\times\mathcal{B}_v(0,R))}=0$$

□

Now, they can conclude, given that from part 7.1 ($\sqrt{f^n}\star\rho_\delta$) is strongly convergent in $L^2$ (after extraction). Then, ($\sqrt{f^n}$) is strongly compact in $L^2$.

#### 7.2.3 Strong compactness of $f^n$ in $L^1$

Following the idea of DiPerna and Lions, they can get the strong convergence of $f^n$ in $L^1$.

They use the strong compactness of ($\sqrt{f^n}$) in $L^2$ to get the almost everywhere convergence of ($f^n$) in $L^1$ as it was done by Lions. Combining this and the weak convergence of ($f^n$) towards $f$ they get the $L^1$-convergence.

### 7.3 Use of previous decomposition: $f$ is a RSDM

In this section, I will present *Step 5* of the paper. The point is now to show that $f$ is a RSDM. We will here use the previous decomposition we did in part 6.

In the weak sense, it is clear that $\dfrac{\partial\beta(f^n)}{\partial t}\to\dfrac{\partial\beta(f)}{\partial t}$ and $v.\nabla_x\beta(f^n)\to v.\nabla_x\beta(f)$.

Now, we need to show that $\dfrac{\partial\beta(f^n)}{\partial t}+\nabla_x(f)\geq\beta'(f)Q(f,f)=R_1+R_2+R_3$ in the sense of distribution.

We denote $R_i^n$ the quantity $R_i$ associated to $f^n$ for $i\in\{1,2,3\}$. Let $\varphi$ be a test function.

We have $\int R_1^n\varphi\mathrm{d}v=\int(f^n\beta'(f^n)-\beta(f^n))f_*^nS^n(v-v_*)\varphi(v)\mathrm{d}v_*\mathrm{d}v$. The first factor being relatively compact in weak-$*$ $L^\infty$, we just need to use the srong convergence of $(f^n\star_v S^n)$ in $L^1$ to prove that $R_1^n\to R_1$ in the sense of distribution. Actually, the strong convergence of the convolution product comes from the bounds of $f^n$ and $S^n$ at $\infty$, the weak local convergence of $(S^n)$ and the strong convergence of $f^n$.

Because we made the assumption that $\mathcal{T}^n\to\mathcal{T}$ weakly, and because we know that $f_*^n\beta(f^n)\to f_*\beta(f)$ strongly in $L^1_{\mathrm{loc}}$, we can conclude easily that $R_2^n\to R_2$ in the sense of distributions.

$R_3\leq\underline{\lim}R_3^n$ in the weak sense is a consequence of Fatou's lemma and of the convergences $f^n\to f$ and $B_n\to B$ (and of the fact that $\Gamma$ is positive).

From those three results they can easily conclude.

### 7.4 Conclusion

In this section, I will present *Step 6* of the paper and conclude. Actually, in step 6, it is shown that they have all the conservation inequalities and equalities they expect. The use of Fatou's lemma when needed let them conclude easily.

So, here they proved that the Boltzmann equation with angular cutoff does have weak solutions, in the sense of RSDM.

# Part III
# About the section *Non-isotropic norm and estimates of linearized collision operator*

I'm presenting here a section of a more general paper: *The Boltzmann equation without angular cutoff in the whole space: I, Global existence for soft potential* by R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu and T. Yang. As suggested by the title of the paper, it gives a result of existence of classical solutions to the Boltzmann equation in the non-cutoff case. In particular, they use a new norm that they call "non-isotropic norm". I'll introduce this norm and present some useful results to use this norm in order to prove the existence of classical solutions.

## 8 Definitions and assumptions on the cross section

Here, the assumptions they make on cross-section are basically the same as in the two previous papers. Still, they did prove their results only in three dimensions. Following them, I will also focus on the $v \in \mathbb{R}^3$ and $x \in \mathbb{R}^3$ case.

### 8.1 Non-cutoff case for soft potential

As previously, we are in the non-cutoff case, so we assume:

$$B(z, \cos\theta) = \Phi(|z|)b(\cos\theta) \tag{62}$$

*Remark* 8.1. Here it is an equality, whereas for the two previous paper, it was an inequality.

Moreover, we will take $\Phi(|z|) = \Phi^\gamma(|z|) = |z|^\gamma$, and of course, for the cross-section to be non-cutoff, $b(\cos\theta) \underset{\theta\to 0^+}{\sim} K\theta^{-2-2s}$.

We are in the soft potential case when $\gamma + 2s \leq 0$.

*Remark* 8.2. The $\gamma + 2s > 0$ case is called hard potential.

We will especially focus on the case when:

$$0 < s < 1, \qquad \gamma > -3 \tag{63}$$

which is clearly soft potential.

### 8.2 Linearized Boltzmann operator

Remember the Boltzmann equation: $\dfrac{\partial f}{\partial t} + v.\nabla_x f = Q(f, f)$.

The static solution is the Maxwellian distribution: $\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}$. Actually, $Q(\mu, \mu) = 0$. We are looking for solutions "close to" $\mu$: $f = \mu + \sqrt{\mu}g$.

We rewrite $Q(f, f) = Q(\mu + \sqrt{\mu}g, \mu + \sqrt{\mu}g) = Q(\mu, \sqrt{\mu}g) + Q(\sqrt{\mu}g, \mu) + Q(\sqrt{\mu}g, \sqrt{\mu}g)$.

And, the Boltzmann equation is rewritten:

$$\sqrt{\mu}\frac{\partial g}{\partial t} + \sqrt{\mu}v.\nabla_x g = Q(\mu, \sqrt{\mu}g) + Q(\sqrt{\mu}g, \mu) + Q(\sqrt{\mu}g, \sqrt{\mu}g)$$

Let us set $\Gamma(g, h) = \frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}g, \sqrt{\mu}h)$. Then, $\mathcal{L}g = \mathcal{L}_1 g + \mathcal{L}_2 g = -\Gamma(\sqrt{\mu}, g) - \Gamma(g, \sqrt{\mu})$ is the *linearized Boltzmann operator*.

We can rewrite the Boltzmann equation w.r.t. $g$:

$$\begin{cases} \dfrac{\partial g}{\partial t} + v.\nabla_x g + \mathcal{L}g = \Gamma(g, g), t > 0 \\ g_{|t=0} = g_0 \end{cases} \tag{64}$$

## 8.3 "New norm"

In this part, I will introduce the norm they use to get existence of classical solution. This norm is the following:

$$|||g|||^2 = \int\int\int \Phi(|v-v_*|)b(\cos\theta)\mu_*(g'-g)^2 \mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma + \int\int\int \Phi(|v-v_*|)b(\cos\theta)g_*^2(\sqrt{\mu'}-\sqrt{\mu})^2\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma = J_1+J_2 \tag{65}$$

This norm is called by the authors *non-istropic*. We will just call it new norm.

## 8.4 Other tools

We call $\mathcal{N}$ the null space of $\mathcal{L}$ and $\mathbf{P}$ the orthogonal projection on $\mathcal{N}$.

We denote $W = \langle v\rangle = (1+|v|^2)^{\frac{1}{2}}$, $W^{\updownarrow} = \langle v\rangle^{\updownarrow}$.

Moreover, we denote $\tilde{W} = \langle v\rangle^{|2s+\gamma|}$ and $\tilde{W}_{\updownarrow} = \langle v\rangle^{|2s-\gamma|\updownarrow}$

Also, as we will work with smoothed versions of the cross section, let $\tilde{\Phi}(|z|) = (1+|z|^2)^{\frac{\gamma}{2}}$ be the smoothed version of $\Phi(|z|) = \Phi^\gamma(|z|) = |z|^\gamma$. When needed, we will denote $Q^\Phi$ the Boltzmann collision operator for $\Phi$ and $Q^{\tilde{\Phi}}$ the one for $\tilde{\Phi}$. Similarily, we denote $|||\bullet|||_\Phi$ the new norm for $\Phi$ and $|||\bullet|||_{\tilde{\Phi}}$ the one for $\tilde{\Phi}$.

We also will not $a \sim b$ when there exist two positive constant $c_1$ and $c_2$ such that $c_1 a \leq b \leq c_2 b$.

# 9 Working with smoothed versions of the cross section

One of the most important idea they use it to remplace $\Phi^\gamma(z) = |z|^\gamma$ by it's smoothed version $\tilde{\Phi}^\gamma(z) = (1+|z|^2)^{\frac{\gamma}{2}}$ so as to get rid of the problem in $z = 0$ (for $\gamma \leq 0$). They show some equivalence result for the "new norm" with $\Phi^\gamma$ and $\tilde{\Phi}^\gamma$. Actually, we can later work with both cross-sections, smoothed or not, which is very convenient.

## 9.1 Aim

We will prove here that under the assumptions of part 8.1, $|||g|||_\Phi \sim |||g|||_{\tilde{\Phi}}$.

## 9.2 Idea of the proof

First, we will show that $J_2^\Phi \sim \|g\|^2_{L^2_{s+\frac{\gamma}{2}}} \sim J_2^{\tilde{\Phi}}$. Then, we will show that $J_1^\Phi + \|g\|^2_{L^2_{s+\frac{\gamma}{2}}} \sim J_1^{\tilde{\Phi}} + \|g\|^2_{L^2_{s+\frac{\gamma}{2}}}$. Combining those two results, we can conclude.

### 9.2.1 $J_2$

Remember

$$J_2 = \int\int\int \Phi(|v-v_*|)b(\cos\theta)g_*^2(\sqrt{\mu'}-\sqrt{\mu})^2\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma \tag{66}$$

Let's first show that $J_2 \leq C\|g\|^2_{L^2_{s+\frac{\gamma}{2}}}$.

We set $A = (\mu'^{1/4}-\mu^{1/4})^2(\mu'^{1/2}+\mu^{1/2})$. Some basic comptutation gives $2A-(\mu'^{1/2}-\mu^{1/2})^2 = (\mu'^{1/4}-\mu^{1/4})^4 \geq 0$. Thus, we have $J_2 \leq F_1+F_2$ with $F_1 = \int\int\int \Phi(|v-v_*|)b(\cos\theta)g_*^2(\mu'^{1/4}-\mu^{1/4})^2\mu'^{1/2}\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma$ and $F_2 = \int\int\int \Phi(|v-v_*|)b(\cos\theta)g_*^2(\mu'^{1/4}-\mu^{1/4})^2\mu^{1/2}\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma$. We can see by direct computation that there is some constant $C_2$ for which $F_2 \leq C_2\|g\|^2_{L^2_{s+\frac{\gamma}{2}}}$. We have also $C_1$ such that $F_1 \leq C_1\|g\|^2_{L^2_{s+\frac{\gamma}{2}}}$. But, to get this, we need to use the $v \to v'$ change of variables that was introduced for the first paper.

This result is valid either with $\Phi$ or $\tilde{\Phi}$ as the computation can be done with both.

Now, they show that there exists some $C'$ such that $J_2 \geq C'\|g\|^2_{L^2_{s+\frac{\gamma}{2}}}$.

The proof given in the paper (proof of lemma 2.8) is well-developped. I shall not copy it here. Still, remember that various change of variables were used. and that, as before, the computations are valid both for $\Phi$ and $\tilde{\Phi}$.

#### 9.2.2 $J_1$

Remember $J_1 = \int\int\int \Phi(|v - v_*|)b(\cos\theta)\mu_*(g' - g)^2 \mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma$.

Here the fact that $\gamma > -3$ will play a role. The idea is also to use various change of variables and projections. One can look at the proof of proposition 2.10 in the paper, which is complete enough. Actually, one of the important that was used is the following:

**Proposition 9.1.** *In a N-dimension problem, let $\rho > 0$, $\delta \in \mathbb{R}$. Then, for all $\alpha > -N$, $\beta \in \mathbb{R}$,*

$$\int |w|^\alpha \langle w\rangle^\beta \langle w + u\rangle^\delta e^{-\rho|w+u|^2} \mathrm{d}w \sim \langle u\rangle^{\alpha+\beta} \tag{67}$$

# 10 "New norm" compared to wheighted Sobolev norms

So as to have the reader understand more the meaning of this new norm, they compare it to weighted Sobolev norms. Actually, one of the reason why this norm is said to be *non-isotropic* is because it cannot be compared to (regular) Sobolev norms, result I won't show.

## 10.1 Aim

Here, $B$ also satisfies assumptions of part 8.1. In this case, there exist two positives constants $D_1$ and $D_2$ such that:

$$D_1\left(\|g\|^2_{H^s_{\frac{\gamma}{2}}} + \|g\|^2_{L^2_{s+\frac{\gamma}{2}}}\right) \leq |||g|||^2 \leq D_2\|g\|^2_{H^s_{s+\frac{\gamma}{2}}} \tag{68}$$

as long as it makes sense for $g$.

## 10.2 Idea of the proof

We know from part 9 that we can work with the smoothed version of the cross section. We also know that

$$J_2 \sim \|g\|^2_{L^2_{s+\frac{\gamma}{2}}} \tag{69}$$

First, we need to show that $J_1 \geq A_1\|\langle v\rangle^{\gamma/2} g\|^2_{H^s} - A_2\|g\|^2_{L^2_{s+\frac{\gamma}{2}}}$ for some positive constants $A_1$ and $A_2$. To get this result they use the Fourier transform results from the first paper. They also can get the same result with $\mu^\rho$ instead of $\mu$ making the same computation. This is the proof of lemma 2.12 of the paper.

Actually, using 69, we get $|||g|||^2 \geq D_1\left(\|g\|^2_{H^s_{\frac{\gamma}{2}}} + \|g\|^2_{L^2_{s+\frac{\gamma}{2}}}\right)$.

They after show the second inequality by using a result they showed in a paper they wrote together: *Regularizing effect and local existence for non-cutoff Boltzmann equation* and which I didn't read.

# 11 The linearized collision operator is bounded by the "new norm"

The point of this section of the paper was to bounded the linearized collision operator with the "new norm". Here is what they did.

## 11.1 Aim

If the cross section $B$ satisfies the assumptions of part 8.1, then there exist two positive constants $C_1$ and $C_2$ such that:

$$C_1|||(\mathbf{I} - \mathbf{P})g|||^2 \leq (\mathcal{L}g, g)_{L^2(\mathbb{R}^3_v)} \leq 2(\mathcal{L}_1 g, g)_{L^2(\mathbb{R}^3_v)} \leq C_2|||g|||^2 \tag{70}$$

for any $g$ for which it makes sense.

## 11.2 Idea of the proof

Using the pre-post-collisional change of variables several times, they get

$$(\mathcal{L}_1 g, g)_{L^2} = \frac{1}{4}\int B((\sqrt{\mu'_*}g' - \sqrt{\mu_*}g)^2 + (\sqrt{\mu'}g'_* - sqrt\mu g_*)^2)\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma$$

and

$$(\mathcal{L} g, g)_{L^2} = \frac{1}{4}\int B((\sqrt{\mu'_*}g' - \sqrt{\mu_*}g) + (\sqrt{\mu'}g'_* - sqrt\mu g_*))^2\mathrm{d}v\mathrm{d}v_*\mathrm{d}\sigma$$

which allow them to conclude easily that

$$(\mathcal{L} g, g)_{L^2} \leq 2(\mathcal{L}_1 g, g)_{L^2}$$

Some basic computation (see proof of proposition 2.16 of the paper) give $|||g|||^2 \geq (\mathcal{L}_1 g, g)_{L^2}$.

Remains to prove that $C_1|||(\mathbf{I}-\mathbf{P})g|||^2 \leq (\mathcal{L} g, g)_{L^2(\mathbb{R}^3_v)}$. Actually, this a consequence of an other result proven by Mouhot in *Explicit coercivity estimates for the Boltzmann and Landau operators*. I didn't read his paper.

# Conclusion

This bound result will after be used to show the existence of classical solutions to the Boltzmann equation without angular cut-off in the soft potential case.

I think if I have to remember one thing about this section, it would be the use of the smoothed version of the cross-section.

# Conclusion

Those three papers present recent results on the Boltzmann equations without angular cut-off. As you have read, the two first papers are linked together. The third paper, which gives the existence of classical solutions, gave me a really hard time to understand. That is why I didn't go further in its reading.

As I said in the introduction, I knew nothing about the equation before my arriving in Beijing. After five months, I think I still know very little. Yet, if I have the opportunity, I may continue working on it.

To sum up, most of the work on the Boltzmann equation for the cutoff case had been done in the late 80's, especially by DiPerna and Lions. And some of their results and ideas were re-used for the non-cutoff case, for both classical and weak solutions.

To treat the non-cutoff case, the idea was to get rid of the irregulairities due to the cross section. For that matter, the *cancellation lemma* and the *truncation* were very useful, in particular for the existence of weak solutions. As for classical solutions, we should remember to use smoothed versions of the cross section instead of its original versions.

I also noticed that a big part of their work consisted in finding a suitable cutting out of the quantities they needed to bound. After cutting it out, they would deal with each term of the cutting and bound them separately, using different tools to get rid of the irregularities in 0 and $\infty$.

To my mind, the time I have spent in Tsinghua University working on this equation was too short. I couldn't understand *deeply* the problems and the solutions that were recently brought to solve them.

On a personal point of view, the opportunity to spend five months in Beijing taught me a lot.

One of the most important things I had to learn there was how to work by myself. It has been hard to keep the motivation. I also noticed that when I'm not feeling good, it is very hard for me to focus on my math. So the question was how to balance working-time and free-time so that I would feel happy enough and that my amount of work would be big enough.

I was feeling so lonely at the beginning of my internship. Living abroad alone is quite hard. As soon as I met some good friends, I could do maths again. This was amazing.

One of the best experience to me was attending the PRIMA 2013 Congress in june in Shanghai Jiatong University. I had the opportunity to exchange a few words with Cedric Villani and to attend various lectures on kinetic equations. I realised how far I was from the specialists in this field and I got motivated again. It became clear to me that I needed to work on my english so that one day I can give a good talk in english. Actually, we barely have the chance to practice our english by giving a math talk in English. That's the reason why I decided to give my oral defense in ENS de Cachan in English rather than in French.

Figure 7: Cedric Villani, a friend of mine and I during PRIMA 2013 Congress, Shanghai

In Shanghai, I had the chance to meet Andrea Bertozzi who works in University of California Los Angeles (UCLA). I had dinner with her and Christina, a friend of mine who knew her. Meeting her also helped me a lot with my motivation. I was kind of lost, maybe I wanted to stop studying math and to finish my chinese learning so as to become interpret in China. She made me change my mind back to math studies, and I thank her (and Christina) a lot for that.

Figure 8: Andrea Bertozzi, Christina Frederick and I during PRIMA 2013 Congress, Shanghai

From now on, I'm sure that I'm a "math girl" as Christina would say. Spending a lot of time with this texan girl in Shanghai and then in Beijing helped me a lot with my English in everyday life and math topics.

I also worked on my Chinese. Actually, I chose to do my internship in Tsinghua University so as to be in China again. But, this experience was not the same as the one I had before. I spent ten months in Nanjing in 2008 in order to learn chinese (mandarin). At the end of the programm, I didn't want to go back to France, I would have liked to stay there forever. Yet, I went back to France. Still, as soon as I had the chance to go abroad again, I chose to go wherever I could as long as it would be in China. Unfortunately, the 4 years and a half I've spent in France made me forget all my Chinese.

I had to re-learn it in Beijing. I was astonished how fast that was. Re-activating old knowledge is really faster than learning new knowledge. I'm proud to say that now my chinese is even better than it used to be in 2008, even though I didn't spend that much time working on it. Actually, all I did was speaking it whenever I would have the chance. My chinese reading and writing have improved too, as there are almost always chinese subtitles on chinese TV channels. I've espescially been watching "Xi Yangyang yu Hui Tailang" which is a comic for little kids where a wolf (Hui Tailang) tries to catch some sheep amongst Xi Yangyang who always makes the wolf's plans fail. I'm deeply convinced that the best way to learn a foreign language is to go where it is spoken and to live there, to get used to the local culture.


Figure 9: Xi Yangyang yu Hui Tailang

One would not expect it, but using three languages a day and toogling all the time between them is really tiring. Actually, when you master enough a language and speak it, you would use this very language to think. And, when you switch languages, you have to switch all your brain. I found that espescially hard when I was translating

between Chinese and English or French, because the way to make the sentences and think an idea through is not the same at all.

I'm in love with China again. I hope I can work there as a mathematician later. Still, I would never be a true beijinger or shanghaier, I belong in Nanjing.