

# Recent progress on the Boltzmann equation without angular cutoff

Oral Defense at Cachan

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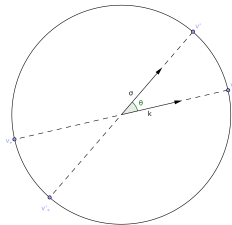
$$Q(g, f) = \int_{\mathbb{R}^N} \int_{S^{N-1}} B(g'_* f' - g_* f) d\sigma dv_*$$

with

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma$$

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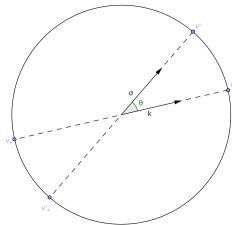
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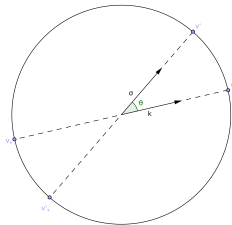
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$$B(z, \sigma) \geq \Phi(|z|) b(\cos \theta)$$

with  $\Phi(|z|) = |z|^\gamma$  and

$$\sin^{N-2} \theta b(\cos \theta) \underset{\theta \rightarrow 0^+}{\sim} \frac{K}{\theta^{1+\nu}}$$

## Three papers

- *Entropy dissipation and long-range interactions* by R. Alexandre, L. Desvillettes, C. Villani and B. Wennberg (2000)
- *On the Boltzmann equation for long-range interactions* by R. Alexandre and C. Villani (2002)
- *The Boltzmann equation without angular cutoff in the whole space: I, Global existence for soft potential, part 2*  
*Non-isotropic norm and estimates of linearized collision operators* by R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu and T. Yang (2011)

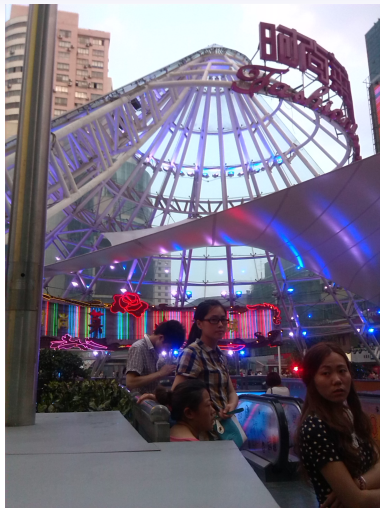


Figure: A GREAT shopping mall in Nanjing

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Main result

$$\|\sqrt{f}\|_{H^{\nu/2}(|v|<R)}^2 \leq C_{g,R} \left( D(g, f) + \|g\|_{L^1_2} \|f\|_{L^1_2} \right)$$



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- $\Lambda(|z|) + |z|\Lambda'(|z|) \leq C_0(1 + |z|)^2$   
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- generalize

## Cancellation lemma

$$D(g, f) = \int Bg_*(f - f')dv dv_* d\sigma + \int Bg_* \left( f \log \frac{f}{f'} - f + f' \right) dv dv_* d\sigma$$



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Theorem (Cancellation lemma)

$$S(z) = |S^{N-2}| \int_0^{\frac{\pi}{2}} \sin^{N-2} \theta \left( \frac{B\left(\frac{|z|}{\cos \frac{\theta}{2}}, \cos \theta\right)}{\cos^N \frac{\theta}{2}} - B(|z|, \cos \theta) \right) d\theta$$

We have:

1.  $|S(z)| \leq C_N (\Lambda(|z|) + |z|\Lambda'(|z|))$
2.  $\int B(f' - f)dv d\sigma = (f \star S)(v_*)$

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Corollary

There exists  $C$  depending only on  $N$  and  $B$  such that

$$\left| \int Bg_*(f' - f)dv dv_* d\sigma \right| \leq C \|g\|_{L_2^1} \|f\|_{L_2^1}$$

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$$A = B = B_R$$

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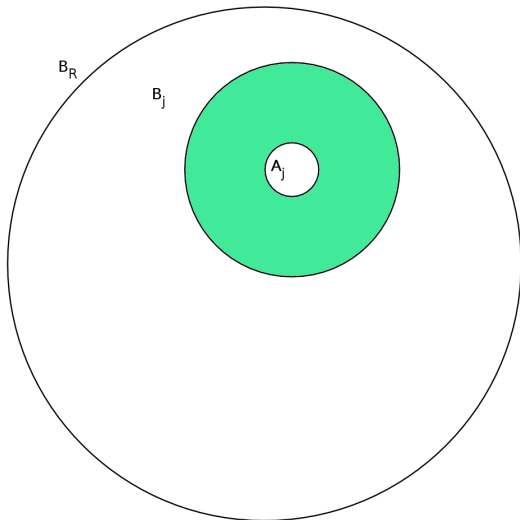




Figure: Shanghai Hongqiao train station

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## Definition (weak solution)

$f$ : renormalized solution to the Boltzmann equation with defect measure

$f \in C(\mathbb{R}^+; \mathcal{D}'(\mathbb{R}_x^N \times \mathbb{R}_v^N)) \cap L^\infty(\mathbb{R}^+; L^1((1 + |v|^2 + |x|^2)dx dv))$   
 $\forall \beta \in C^2(\mathbb{R}^+, \mathbb{R}^+)$  non-linear such that :

$$\beta(0) = 0 \qquad 0 < \beta'(f) \leq \frac{C}{1+f} \qquad \beta''(f) < 0$$

$$\implies \begin{cases} \frac{\partial \beta(f)}{\partial t} + v \cdot \nabla_x \beta(f) \geq \beta'(f) Q(f, f) \\ \forall t \geq 0, \int_{\mathbb{R}^N \times \mathbb{R}^N} f(t, x, v) dx dv = \int_{\mathbb{R}^N \times \mathbb{R}^N} f(0, x, v) dx dv \end{cases}$$

## Assumptions on $B$

AT MOST BORDERLINE

SINGULARITY.

$B(z, \sigma) = \frac{\beta_0(k.\sigma)}{|z|^N} + B_1(z, \sigma)$  with

- $\int_{S^{N-1}} \beta_0(k.\sigma)(1 - k.\sigma)d\sigma < +\infty$
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$M^\alpha(|z|) := \int_{S^{N-1}} B(z, \sigma)(1 - k.\sigma)^{\frac{\alpha}{2}} d\sigma$

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CONDITION

## Assumptions on $B$

AT MOST BORDERLINE

SINGULARITY.

$B(z, \sigma) = \frac{\beta_0(k, \sigma)}{|z|^N} + B_1(z, \sigma)$  with

- $\int_{S^{N-1}} \beta_0(k, \sigma)(1 - k \cdot \sigma) d\sigma < +\infty$
- $\Lambda_1(|z|), |z|\Lambda'_1(|z|) \in L^1_{\text{loc}}(\mathbb{R}^N)$

BEHAVIOUR AT INFINITY.

$M^\alpha(|z|) := \int_{S^{N-1}} B(z, \sigma)(1 - k \cdot \sigma)^{\frac{\alpha}{2}} d\sigma$

- $\exists \alpha \in [0; 2], M^\alpha(|z|) = \begin{matrix} o \\ |z| \rightarrow \infty \end{matrix} (|z|^{2-\alpha})$
- $|z|M'(|z|) = \begin{matrix} o \\ |z| \rightarrow \infty \end{matrix} (|z|^2)$

ANGULAR SINGULARITY

CONDITION.  $\int_{S^{N-1}} b(k, \sigma) d\sigma = +\infty$

$B^n$  APPROXIMATES  $B$ .

- $S^n \rightarrow S$  locally in the weak-measure sense
- For  $\mathcal{T} : \phi \mapsto \int_{S^{N-1}} B(\phi' - \phi) d\sigma$ ,  $\mathcal{T}^n \rightarrow \mathcal{T}$
- $B^n \rightarrow B$  a.e.
- BEHAVIOUR AT INFINITY *uniformly* in  $n$ .

OVERALL ANGULAR SINGULARITY  
CONDITION.

- $\forall n$ , ANGULAR SINGULARITY  
CONDITION
- $\int_{S^{N-1}} \lim_{n \rightarrow \infty} b^n(k, \sigma) d\sigma = \infty$

## Idea of the proof

$\beta'(f)Q(f, f)$  formally cut into three parts :

- something  $\times \int B(f'_* - f_*)dv_*d\sigma \implies$  use of the *cancellation lemma*
- $Q(f, \beta(f)) \implies$  apparition of  $\mathcal{T}$
- $-\int Bf'_*\Gamma(f, f')dv_*d\sigma$  which convergence is a consequence of the convergence of the first two terms

$$\Gamma(f, f') := \beta(f') - \beta(f) - \beta'(f)(f' - f) \leq 0$$

## Idea of the proof of strong compactity appearance

1. get strong compactity results in  $L^1_{(\text{loc})}$  for  $\int \beta(f^n)\phi(v)dv$ ,  
 $\beta(f^n) \star_v \phi$  and  $\sqrt{f^n} \star_v \phi. \implies$  AVERAGING LEMMAS  
strong convergence:  $\int f^n dv \rightarrow \int f dv$

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2. apply result of previous paper (*truncation*) and OVERALL  
ANGULAR SINGULARITY CONDITION  $\implies$  *velocity*  
*smoothness*:  $\lim_{A \rightarrow \infty} \overline{\lim}_{n \rightarrow \infty} \int_{W_\varepsilon} \int_{|\xi| \geq A} |\mathcal{F} \sqrt{f^n}|^2 d\xi dt dx$  where  
 $\sqrt{f^n} \rightarrow F$  and  $W_\varepsilon = \left\{ (t, x) \in [0; T] \times B_R^N, \int F^2 dv > \varepsilon \right\}$

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6. Conclude: check that  $f$  is a RSDM



Figure: The old Summer Palace, Beijing

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$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}$$

$$f = \mu + \sqrt{\mu}g$$

$$Q(f, f) = Q(\mu, \sqrt{\mu}g) +$$

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$$\mathcal{L}g := -\Omega(\sqrt{\mu}, g) - \Omega(g, \sqrt{\mu})$$

$$\text{linear: Ker } \mathcal{L} = \mathcal{N}$$

$\implies$  **P**  $L^2$ -orthogonal projection  
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**New norm**

$$\begin{aligned} |||g|||^2 &= \int \Phi b \mu_*(g' - g)^2 dv_* dv d\sigma + \int \Phi b g_*^2 (\sqrt{\mu'} - \sqrt{\mu})^2 dv_* dv d\sigma \\ &= J_1 + J_2 \end{aligned}$$

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## Theorem

$B(z, \sigma) = \Phi(|z|)b(\cos \theta)$  with  $\Phi(|z|) = |z|^\gamma$  and similar assumptions as before:  $\gamma > -3$  and  $\nu \in ]0; 2[$ .

$$C_1 |||(\mathbf{I} - \mathbf{P})g|||^2 \leq (\mathcal{L}g, g)_{L^2(\mathbb{R}_v^3)} \leq C_2 |||g|||^2$$

## About the new norm

$$C_3 \left( \|g\|_{H^{\frac{\nu}{2}}(\mathbb{R}^3)}^2 + \|g\|_{L^2(\mathbb{R}^3)}^2 \right) \leq \|g\|^2 \leq C_4 \|g\|_{H^{\frac{\nu+\gamma}{2}}(\mathbb{R}^3)}^2$$



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Show in particular that  $J_2^\Phi \sim \|g\|_{L^2(\mathbb{R}^3)}^2 \sim J_2^{\tilde{\Phi}}$  and

$$J_1^\Phi + \|g\|_{L^2(\mathbb{R}^3)}^2 \sim J_1^{\tilde{\Phi}} + \|g\|_{L^2(\mathbb{R}^3)}^2$$

## Conclusion



Figure: Me, C. Villani and a friend at PRIMA conference, Shanghai