

# Mixed Nash Equilibria in Concurrent Terminal-Reward Games

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- 1 Presentation of the framework
- 2 Existence problem
- 3 Summary of the proof

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# Games with mixed strategies

Concurrent non-zero sum games allow

- To modelize heterogeneous systems
- Several events to occur simultaneously
- Agents' goals not to be necessarily antagonistic

whereas mixed strategies enable

- Synthesizing strategies for controllers
- with memory
- Breaking the symmetry (by randomization)
- Equilibrium more likely to occur

# Formal model

## Definition (Arena)

$$\mathcal{A} = \langle \text{States}, \text{Agt}, \text{Act}, \text{Tab}, (\text{Allow}_i)_{i \in \text{Agt}} \rangle$$

with

- $|\text{States}|, |\text{Agt}|, |\text{Act}| < +\infty$
- $\text{Tab} : \text{States} \times \text{Act}^{\text{Agt}} \rightarrow \text{States}$
- $\forall i \in \text{Agt} \quad \text{Allow}_i : \text{States} \rightarrow 2^{\text{Act} \setminus \{\emptyset\}}$

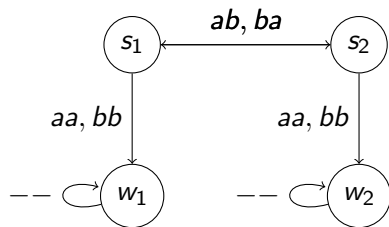
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## Definition (Game)

$$\mathcal{G} = \langle \mathcal{A}, s, \phi \rangle$$

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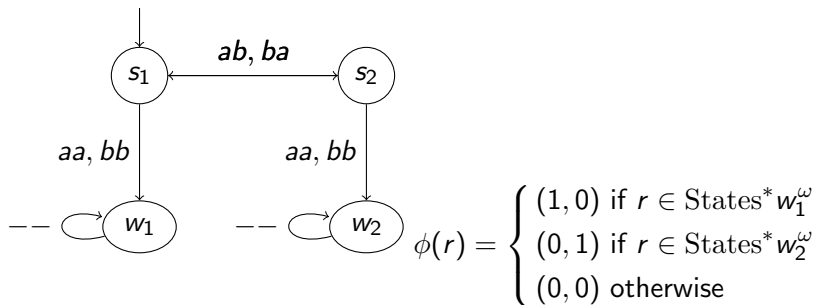
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- Safety condition
- Reachability
- Limit average
- Terminal reachability

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Let  $F$  denote the set of states that have no successor except themselves.  $\phi$  is a *terminal reachability utility function* if

$$\forall r \phi(r) \neq 0 \Leftrightarrow \exists h \in \text{States}^* \exists f \in F r = h \cdot f^\omega$$

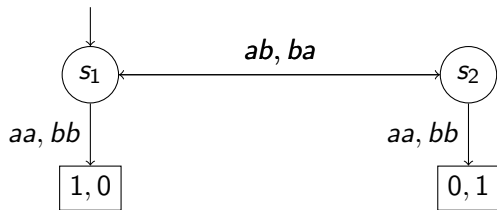
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## Definition (Strategies)

A *strategy* for player  $i$  in arena  $\mathcal{A}$  is given by  $\sigma_i$  such that for all  $h \in \text{States}^+$ ,

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## Definition (Expectation)

We consider a game  $\mathcal{G}$  and a strategy profile  $\sigma$ .  $X_0 = s$ ,

$X_{n+1} = \text{Tab}(X_n, A_n)$  with  $A_n \sim \prod_i \sigma_i(X_0 \dots X_n)$ .

Let  $r = \lim X_0 \dots X_n \in \text{States}^\omega$ .

Under some mesurability assumptions, the expectation of  $\phi(r)$  exists.

If  $\mathbb{P}(r \in h\text{States}^\omega) > 0$ , we write  $\mathbb{E}^\sigma(\phi \mid h)$  the conditionnal expectation.

# Nash Equilibrium

## Definition

Let  $\sigma$  a strategy profile and  $h$  an history, then  $(\sigma, h)$  is a *Nash Equilibrium* (NE) if for all agent  $i$  and any other strategy for  $i$  (*deviation*)  $\sigma'_i$ ,

$$\mathbb{E}^{\sigma^{[i/\sigma'_i]}}(\phi_i \mid h) \leq \mathbb{E}^{\sigma}(\phi_i \mid h)$$

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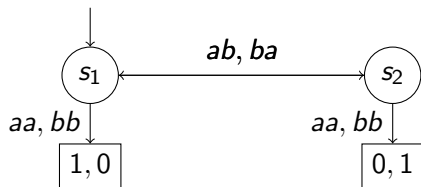
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The uniform strategy for both players is a NE (payoff  $(2/3, 1/3)$ ).



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# Does a mixed Nash Equilibrium always exist?

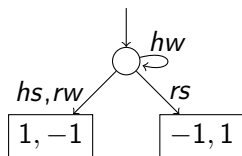
## Theorem (?)

*Every one-stage game has a Nash Equilibrium in mixed strategies.*

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Figure: Hide-or-Run game

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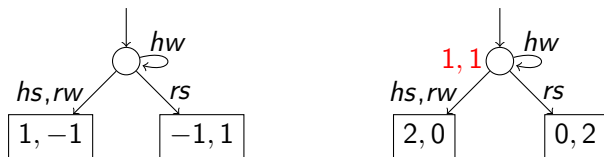


Figure: Hide-or-Run game

Value problem in a zero-sum game is *not* a special case of Nash Equilibrium problem with *positive* terminal rewards

# Constrained problem

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Given a terminal reward game and a linear constrain  $\Phi$  on  $\mathbb{R}^{\text{Agt}}$ , does there exists a NE with payoff satisfying  $\Phi$  ?

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- Constrained problem can be emulated with negative rewards (or winning loops)
- ? shows the existence of a constrained NE is undecidable for 14 players for turn-based games
- ? states that there always exists  $\epsilon$ - NE in concurrent reachability games (strategies are memoryless and  $\epsilon$ -optimal)

# Undecidability

## Theorem

*The constrained existence problem is undecidable for 3-player concurrent games with terminal rewards.*



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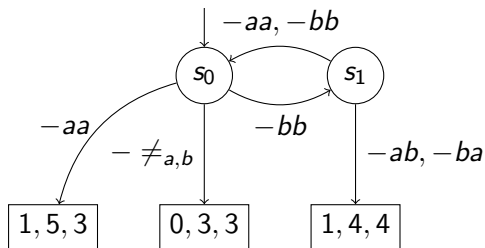
## Sketch of the proof

- Reduction from the *non*-halting problem for 2-counters machine.
- Three players 0, 1, 2
- $(x, y) \mapsto (1, 4 + \frac{1}{2^x 3^y}, 4 - \frac{1}{2^x 3^y})$
- 1 and 2 are antagonistic
- Constrain: 0 should earn payoff 1 (0-optimal).

## Counting module

There may be a countable number of equilibria.

$$(\neq_2 = \{(x, y) \in \{a, b\}^2 \mid x \neq y\})$$



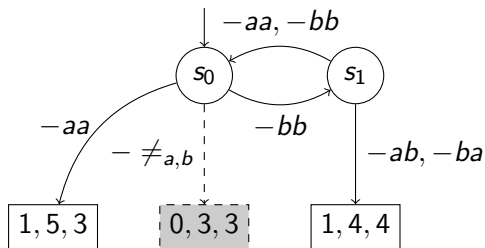
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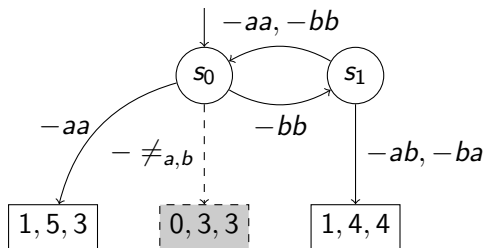
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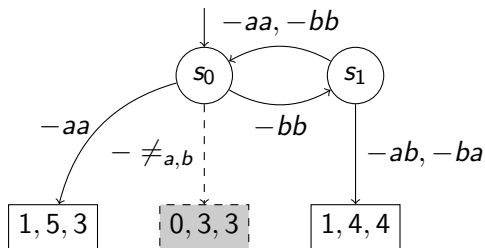


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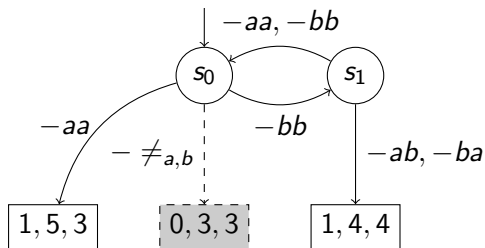


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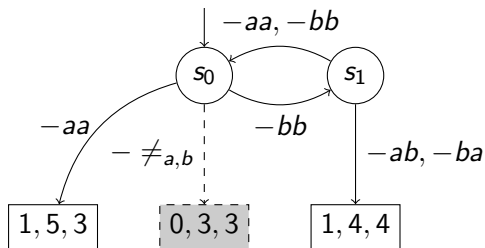


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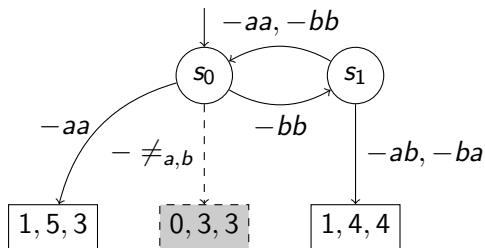
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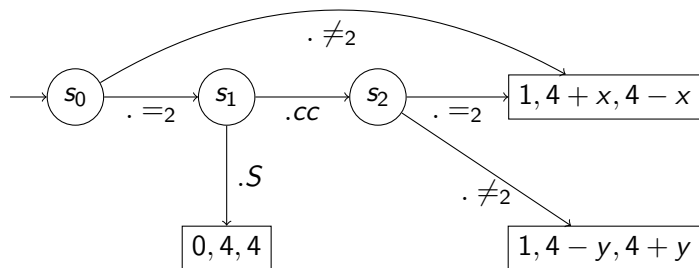
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$$= \left\{ \left( 1, 4 + \frac{1}{2^k}, 4 - \frac{1}{2^k} \right) \mid k \in \mathbb{N} \vee k = \infty \right\}$$

# Testing module

Assume  $x, y \geq 0$ .

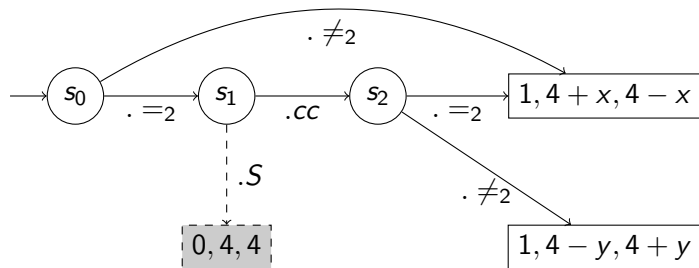
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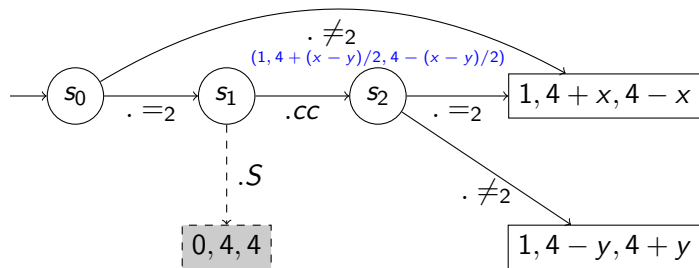
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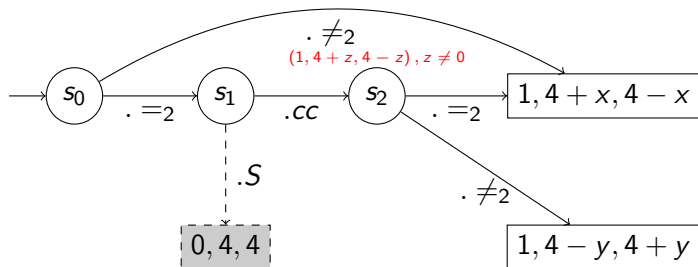
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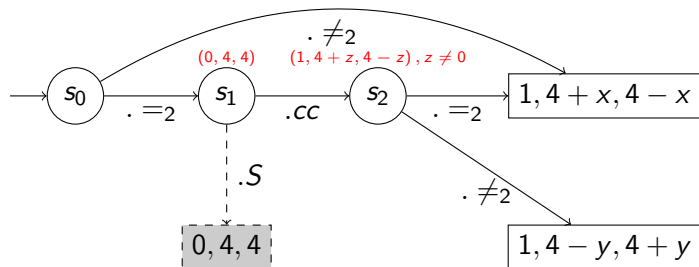


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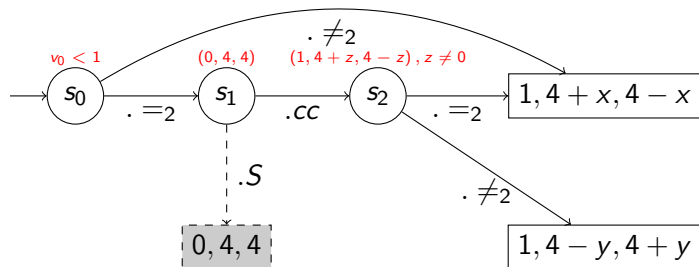


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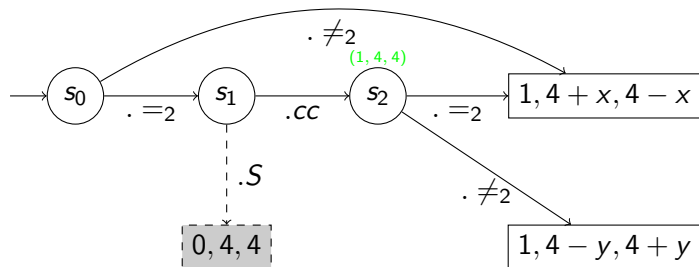


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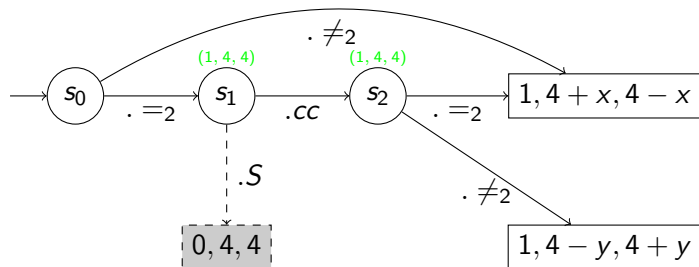
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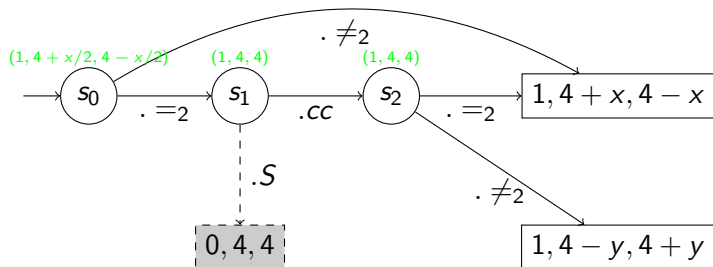


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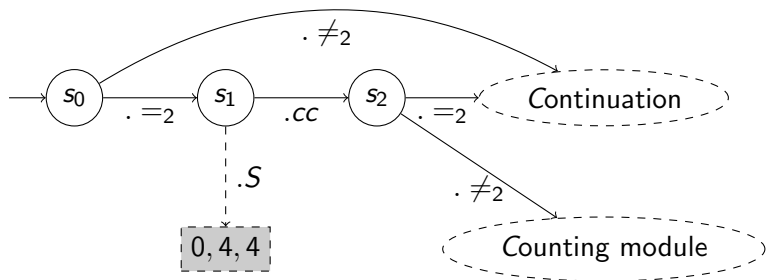


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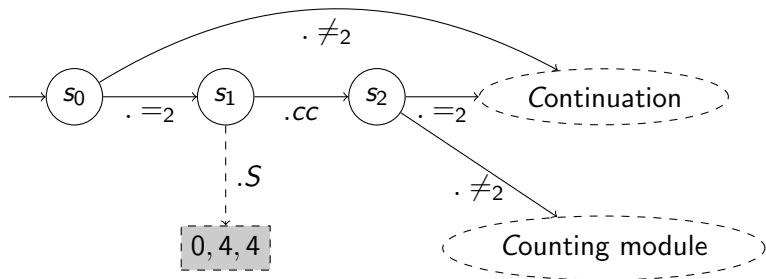
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- Other counting modules are necessary for the 2CM.
- A final rescale of the payoff is required.
- Strategies in the continuation shouldn't distinguish histories  $s_0$  and  $s_0s_1s_2$

## Qualitative decision problems

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- 0-optimality is here a case of Pareto-optimality
- One can consider only reachablage sets as winning conditions
- Constrained (qualitative) safe NE is also undecidable (despite there always exists an (un-constrained) NE [?])

# Overview

- Tighter bound on the number of players
- Several undecidable qualitative problems
- Heavily use of concurrent actions
- May be extended to  $n - \frac{1}{2}$ -games
- Still hope (decidability, existence ?) for the 2 players case



# Thank you for your attention

# Bibliography I