

## Spontaneous emission

Due 11 March 2014

### Problem 1 - Weisskopf-Wigner theory of spontaneous emission; single-photon state

We want to calculate the quantum state of the field radiated after the spontaneous emission of a two-level system. The frequency of the transition between the ground state  $|g\rangle$  and the excited state  $|e\rangle$  is  $\omega_0$ .

1. Justify that the quantum state of the atom and field during the emission is of the form

$$|\Psi(t)\rangle = a(t)e^{-i\omega_0 t}|e, 0\rangle + \sum_k b_k(t)e^{-i\omega_k t}|g, 1_k\rangle, \quad (1)$$

with  $|0\rangle$  the state of the field with no photon and  $|1_k\rangle$  the state containing one photon in mode  $k$  with frequency  $\omega_k$ .

2. The coupling between the atom and the field is  $-\hat{\mathbf{D}} \cdot \boldsymbol{\epsilon}_k \hat{E}$ , with  $\hat{\mathbf{D}}$  and  $\hat{E}$  the dipole and the electric field operators, and  $\boldsymbol{\epsilon}_k$  the polarisation of mode  $k$ . We recall that

$$\langle 0 | \hat{E} | 1_k \rangle = \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}}. \quad (2)$$

*Handwritten notes:*  
 $\langle e | \hat{\mathbf{D}} | e \rangle = 0$   
 $\langle g | \hat{\mathbf{D}} | g \rangle = 0$

We call  $d = |\langle g | \hat{\mathbf{D}} | e \rangle|$ . Write the differential equations fulfilled by  $a(t)$  and  $b_k(t)$ .

3. Show that:

$$b_k(t) = -\frac{1}{i\hbar} \langle g, 1_k | \hat{\mathbf{D}} \cdot \boldsymbol{\epsilon}_k \hat{E} | e, 0 \rangle \int_0^t dt' e^{-i(\omega_0 - \omega_k)t'} a(t') \quad (3)$$

4. By replacing (3) in the differential equation of  $a(t)$ , show that

$$\dot{a}(t) = -\frac{1}{\hbar^2} \sum_k |\langle g, 1_k | \hat{\mathbf{D}} \cdot \boldsymbol{\epsilon}_k \hat{E} | e, 0 \rangle|^2 \int_0^t dt' e^{-i(\omega_0 - \omega_k)(t-t')} a(t'). \quad (4)$$

5. Show that

$$|\langle g, 1_k | \hat{\mathbf{D}} \cdot \boldsymbol{\epsilon}_k \hat{E} | e, 0 \rangle|^2 = d^2 \frac{\hbar\omega_k}{2\epsilon_0 V} \sin^2 \theta, \quad (5)$$

with  $\theta$  the angle between the axis of the dipole and the direction of the spontaneously emitted photon. We take  $\boldsymbol{\epsilon}_k$  in the plane defined by the direction of the photon and the axis of the dipole.

6. We replace the sum over all the modes  $k$  by an integral so that

$$\sum_k \rightarrow \frac{V}{4\pi^2} \int_0^\pi \sin \theta d\theta \int k^2 dk, \text{ avec } k = \frac{\omega}{c}. \quad (6)$$

Show that

$$\dot{a}(t) = -\frac{d^2}{6\pi^2\epsilon_0\hbar c^3} \int_0^\infty d\omega \omega^3 \int_0^t dt' e^{-i(\omega_0-\omega)(t'-t)} a(t'). \quad (7)$$

7. Justify that  $\omega^3$  can be taken out of the integrals and replaced by  $\omega_0^3$ . Simplify the equation on  $a(t)$  using the result (non rigorous)

$$\int_0^\infty d\omega e^{-i(\omega_0-\omega)(t'-t)} = \pi\delta(t-t'). \quad (8)$$

8. The spontaneous emission rate is given by (see lecture)

$$\Gamma = \frac{d^2}{3\pi\epsilon_0} \frac{\omega_0^3}{\hbar c^3} \quad (9)$$

Deduce that  $a(t) = e^{-\frac{\Gamma}{2}t}$ . What is the probability that the atom is still excited after a time  $t$ ?

9. Calculate  $b_k(t)$ .
10. Give the limit  $t \rightarrow \infty$  for  $b_k$  and give the expression of the final quantum state of the system.
11. Where is now the energy initially stored in the atom? Justify that this state is often called "single-photon" state. *???*  $\hookrightarrow$  *lorentzienne*  $\rightarrow$  *1 photon*.

## Problem 2 - Analysis of the article of R. G. Hulet *et al.*, "Inhibited spontaneous emission by a Rydberg Atom".

- Imagine we perform the experiment at 300 K instead of 6.5 K. Assume a thermodynamic equilibrium between the two plates. Estimate the number of thermal photons per mode at the resonant frequency of the atom. What is this number at 6.5 K?
- What is the stimulated emission rate due to this photons at 6.5 K expressed in  $s^{-1}$ ?
- What is the temperature necessary to render the stimulated emission negligible?
- Taking the lifetime of the cesium atom in vacuum (not between the plates), what is the ratio of the dipole moment matrix element of this transition to the dipole matrix element of the 1s - 2p transition of the hydrogen atom discussed in the lecture ( $\lambda = 121$  nm).
- Explain qualitatively the shape of the time-of-flight spectrum of the detected atoms shown in figures 1 and 2 of the paper.  $\rightarrow$  *debat peu d'atomes arrivés*  
 $\rightarrow$  *peu Boltzmann évasé*  
 $\rightarrow$  *peu plus d'atomes*